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SOME PROPERTIES OF RICKART MODULES

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Abstract. Let R be an arbitrary ring with identity and M a right R-module with $S = \mathrm{End}_R(M)$. Following [8], the module M is called *Rickart* if for any $f \in S$, $r_M(f) = eM$ for some $e^2 = e \in S$, equivalently, Ker$f$ is a direct summand of M. In this paper, we continue to investigate properties of Rickart modules. For a Rickart module M, we prove that M is S-rigid (resp., S-reduced, S-symmetric, S-semicommutative, S-Armendariz) if and only if its endomorphism ring S is rigid (resp., reduced, symmetric, semicommutative, Armendariz). We also prove that if $M[x]$ is a Rickart module with respect to $S[x]$, then M is Rickart, the converse holds if M is S-Armendariz. Among others it is also shown that M is a Rickart module if and only if every right R-module is M-principally projective.

1. Introduction

Throughout this paper R denotes an associative ring with identity and modules will be unitary right R-modules. For a module M, $S = \mathrm{End}_R(M)$ denotes the ring of right R-module endomorphisms of M. Then M is a left S-module, right R-module and (S, R)-bimodule. In this work, for any rings S and R and any (S, R)-bimodule M, $r_R(.)$ and $l_M(.)$ denote the right annihilator of a subset of M in R and the left annihilator of a subset of R in M, respectively. Similarly, $l_S(.)$ and $r_M(.)$ will be the left annihilator of a subset of M in S and the right annihilator of a subset of S in M, respectively. A ring R is *reduced* if it has no nonzero nilpotent elements. A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. The module M is called *S-semicommutative* [2], if for any $f \in S$ and $m \in M$, $fm = 0$ implies $fSm = 0$. *Baer rings* [3] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. According to Rizvi and Roman, an R-module M is called *Baer* [7]

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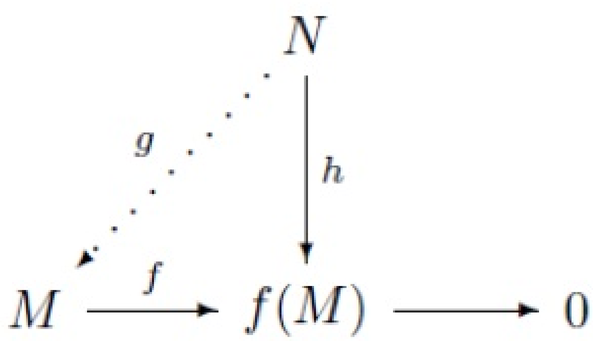
if for any R-submodule N of M, $l_S(N) = Se$ with $e^2 = e \in S$. Also, they defined Rickart modules in [8]. Recently Rickart modules are studied extensively by different authors (see [1] and [5]).

2. Rickart Modules

Let M be an R-module with $S = \mathrm{End}_R(M)$. The module M is called *Rickart* if for any $f \in S$, $r_M(f) = eM$ for some $e^2 = e \in S$, equivalently, $\mathrm{Ker} f$ is a direct summand of M. It is clear that every semisimple module, every Baer module is a Rickart module. We continue to investigate properties of Rickart modules.

Let M be an R-module. A right R-module N is called *M-principally projective* [9], if for any $f \in S$, and any $N \xrightarrow{h} f(M)$ there exists a $N \xrightarrow{g} M$ such that the following diagram is commutative.



By the following Theorem 2.1 we investigate the relations between this class of modules and Rickart modules.

Theorem 2.1. *Let M be an R-module. Then M is a Rickart module if and only if every right R-module is M-principally projective.*

Proof. Assume that M is a Rickart module and let $f \in S$. There exists $e^2 = e \in S$ such that $r_M(f) = eM$. Then $M = r_M(f) \oplus K$ for some $K \leq M$. For any right R-module N and any $N \xrightarrow{h} f(M)$, since $f(M) \cong M/r_M(f)$ for any $n \in N$ we may write $h(n) = k + r_M(f)$ for some $k \in K$ and we define $N \xrightarrow{g} M$ by $g(n) = k$. Then g is a well defined R-map and for $n \in N$, $h(n) = fg(n)$. Conversely, suppose that every right R-module N is M-principally projective and $f \in S$. In particular $M/r_M(f)$ is M-principally projective. So consider the identity map from $M/r_M(f)$ onto $M/r_M(f)$. By considering $f(M) \cong M/r_M(f)$ and supposition there exists a map g from $M/r_M(f)$ to M such that $1 = fg$. For any $m \in M$, $m - g(f(m)) \in r_M(f)$

and $g(f(m)) \in Img$, we have $M = r_M(f) \oplus Img$. Let e denote the projection of M onto $r_M(f)$. Then $r_M(f) = eM$. □

Let M be an R-module and consider the set

$$F(M) = \{m \in M \mid fm = 0 \text{ for some nonzero } f \in S\}$$

of all torsion elements of the module M with respect to S. The subset $F(M)$ of M need not be a submodule of the modules ${}_SM$ and M_R in general. If S is a commutative domain, then $F(M)$ is an (S, R)-submodule of M.

Proposition 2.2. *Let M be an R-module with a domain $S = End_R(M)$. If M is a Rickart module, then $F(M) = 0$ and every nonzero element of S is a monomorphism.*

Proof. Let M be a Rickart module and $0 \neq f \in S$. Then there exists an idempotent $e \in S$ such that $r_M(f) = eM$. Hence $feM = 0$. Thus $fe = 0$ in S. Since S is a domain and f is nonzero, $e = 0$ or every nonzero element of S is a monomorphism. If $m \in F(M)$, then there exists a nonzero $f \in S$ such that $fm = 0$. Since f is a monomorphism, we have $m = 0$, and so $F(M) = 0$. □

The following result is an immediate consequence of Proposition 2.2.

Corollary 2.3. *Let M be an R-module with a domain $S = End_R(M)$. If M is a Rickart module, then M is torsion-free.*

The next result can be obtained from Proposition 2.2 and [7, Theorem 2.23].

Corollary 2.4. *Let M be an R-module. Then the following are equivalent.*

(1) *M is an indecomposable Baer module.*
(2) *S is a domain and M is a Rickart module.*
(3) *Every nonzero element of S is a monomorphism.*

Our next endeavor is to investigate relationships among reduced, rigid, symmetric, semicommutative, Armendariz modules and their endomorphism rings by using Rickart modules.

Definition 2.5. Let M be an R-module. A module M is called *S-reduced* if $fm = 0$ implies $Imf \cap Sm = 0$ for each $f \in S$, $m \in M$.

It can be easily proved that M is an S-reduced module if and only if $f^2m = 0$ implies $fSm = 0$ for each $f \in S$, $m \in M$.

Lemma 2.6. *Let M be an R-module. If M is an S-reduced module, then S is a reduced ring. The converse holds if M is a Rickart module.*

Proof. The first statement is clear from [1, Lemma 2.11] and [1, Proposition 2.14]. Conversely, assume that M is a Rickart module and S is a reduced ring. Let $f \in S$ and $m \in M$ with $fm = 0$. Then $r_M(f) = eM$ for some $e^2 = e \in S$. Hence $fe = 0$ and $m = em$. Since e is central, we have $ef = 0$. Let $fm_1 = gm \in fM \cap Sm$, where $m_1 \in M$ and $g \in S$. Thus $0 = efm_1 = egm = gem = gm$, and so $fM \cap Sm = 0$. Therefore M is S-reduced. □

Let M be an R-module. Recall that M is called *an S-rigid module* [1] if for any $f \in S$ and $m \in M$, $f^2 m = 0$ implies $fm = 0$.

Lemma 2.7. *Let M be an R-module. If M is an S-rigid module, then S is a reduced ring. The converse holds if M is a Rickart module.*

Proof. The first statement is clear from [1, Lemma 2.20]. Conversely, assume that M is a Rickart module and S is a reduced ring. Let $f \in S$ and $m \in M$ with $f^2 m = 0$. Then $r_M(f) = eM$ for some $e^2 = e \in S$. Hence $fe = 0$ and $fm = efm$. Since e is central, we have $fm = efm = fem = 0$. Therefore M is S-rigid. □

According to Lambek [4], a ring R is called *symmetric* if whenever $a, b, c \in R$ satisfy $abc = 0$, we have $acb = 0$. For the module case, we have the following.

Definition 2.8. Let M be an R-module. A module M is called *S-symmetric* if for any $m \in M$ and f, $g \in S$, $fgm = 0$ implies $gfm = 0$.

Lemma 2.9. *Let M be an R-module. If M is an S-symmetric module, then S is a symmetric ring. The converse holds if M is a Rickart module.*

Proof. Let $f, g, h \in S$ and assume $fgh = 0$. Then $fg(h(m)) = 0$ and $g(fh)(m) = 0$ implies $fhg(m) = 0$ for all $m \in M$. Hence $fhg = 0$. Conversely, assume that M is a Rickart module and S is a symmetric ring. Let $f, g \in S$ and $m \in M$ with $fgm = 0$. Then $r_M(fg) = eM$ for some $e^2 = e \in S$. Hence $fge = 0$ and $m = em$. By assumption $gef = 0$. Since e is central, we have $gfm = gfem = gefm = 0$. Therefore M is S-symmetric. □

Lemma 2.10. *Let M be an R-module. If M is an S-semicommutative module, then S is a semicommutative ring. The converse holds if M is Rickart.*

Proof. The first statement is proved in [2, Lemma 2.1]. Conversely, assume that M is a Rickart module and S is a semicommutative ring. Let $f \in S$, $m \in M$ with $fm = 0$. Then $r_M(f) = eM$ for some $e^2 = e \in S$. Hence $fe = 0$ and $m = em$. Since e is central, $fgm = fgem = fegm = 0$ for any $g \in S$. Thus M is S-semicommutative. □

In [6], the ring R is called *Armendariz* if for any $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{s} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all i and j. Let M be an R-module. The module M is called *S-Armendariz* if the following condition (1) is satisfied, while M is said to be *S-Armendariz of power series type* if the following condition (2) is satisfied.

(1) For any $f(x) = \sum_{i=0}^{s} a_i x^i \in S[x]$ and $m(x) = \sum_{j=0}^{n} m_j x^j \in M[x]$, $f(x)m(x) = 0$ implies $a_i m_j = 0$ for all i and j.
(2) For any $f(x) = \sum_{i=0}^{\infty} a_i x^i \in S[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$, $f(x)m(x) = 0$ implies $a_i m_j = 0$ for all i and j.

Lemma 2.11. *Let M be an R-module. If M is an S-Armendariz module, then S is an Armendariz ring. The converse holds if M is a Rickart module.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{k} b_j x^j \in S[x]$ with $f(x)g(x) = 0$. For any $m \in M$, $g(x)m = \sum_{j=0}^{k} (b_j m)x^j \in M[x]$. Since $f(x)g(x) = 0$, we have $f(x)(g(x)m) = 0$. This implies that $a_i(b_j m) = (a_i b_j)m = 0$ for all $0 \le i \le n$ and $0 \le j \le k$, and so $a_i b_j = 0$ for all i and j. Therefore S is Armendariz. Conversely, assume that S is an Armendariz ring and M is a Rickart module. By [5, Proposition 3.2], S is a right Rickart ring. Since S is Armendariz, S is a reduced ring. By Lemma 2.6, M is S-reduced and so S-Armendariz. □

Corollary 2.12. *Let M be an R-module. If M is an Armendariz of power series type, then S is an S-Armendariz of power series type. The converse holds if M is a Rickart module.*

Proof. Similar to the proof of Lemma 2.11. □

We now summarize the relations between rigid, reduced, symmetric, semicommutative, Armendariz modules and their endomorphism rings by using Rickart modules.

Theorem 2.13. *Let M be an R-module. If M is a Rickart module, then*

(1) *M is S-rigid if and only if S is a reduced ring.*

(2) *M is S-reduced if and only if S is a reduced ring.*

(3) *M is S-symmetric if and only if S is a symmetric ring.*

(4) *M is S-semicommutative if and only if S is a semicommutative ring.*

(5) *M is S-Armendariz if and only if S is an Armendariz ring.*

(6) *M is S-Armendariz of power series type if and only if S is an Armendariz of power series type ring.*

Proof. (1) Lemma 2.6. (2) Lemma 2.7. (3) Lemma 2.9. (4) Lemma 2.10. (5) Lemma 2.11. (6) Corollary 2.12. □

The next result follows from Theorem 2.13 and [1, Theorem 2.25].

Corollary 2.14. *Let M be an R-module. If M is a Rickart module, then the following conditions are equivalent.*

(1) *S is a reduced ring.*

(2) *S is a symmetric ring.*

(3) *S is a semicommutative ring.*

(4) *S is an Armendariz ring.*

(5) *S is an Armendariz of power series type ring.*

In the sequel, we study the polynomial extension of Rickart modules. Let M be an R-module. It can be easily shown that $M[x] = \{\sum_{i=0}^{s} m_i x^i : s \geq 0, m_i \in M\}$ is an abelian group under an obvious addition operation and $M[x]$ becomes a module over $R[x]$ with

$$m(x) = \sum_{i=0}^{s} m_i x^i \in M[x] \quad , \quad f(x) = \sum_{i=0}^{t} a_i x^i \in R[x],$$

$$m(x)f(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} m_i a_j \right) x^k.$$

Similarly, $M[x]$ is a left $S[x]$-module with

$$f(x) = \sum_{i=0}^{t} f_i x^i \ \in S[x] \quad , \quad m(x) = \sum_{j=0}^{s} m_j x^j \ \in M[x],$$

$$f(x)m(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} f_i m_j \right) x^k.$$

The module $M[x]$ is called *Rickart with respect to* $S[x]$ if for any $f(x) \in S[x]$, there exists $e(x)^2 = e(x) \in S[x]$ such that $r_{M[x]}(f(x)) = e(x)M[x]$.

Theorem 2.15. *Let M be an R-module with $S = \mathrm{End}_R(M)$. If $M[x]$ is a Rickart module with respect to $S[x]$, then M is Rickart. The converse holds if M is S-Armendariz.*

Proof. Assume that $M[x]$ is a Rickart module and $f \in S$. Consider $\bar{f} \in S[x]$ defined by $\bar{f}(\sum m_i x^i) = \sum f(m_i)x^i$. Then $\mathrm{Ker}\bar{f}$ is a direct summand of $M[x]$, that is, $M[x] = \mathrm{Ker}\bar{f} \oplus K$. It is easy to show that $M = \mathrm{Ker}f \oplus K_0$, where K_0 is the set of elements in K evaluated in zero. Then M is a Rickart module. Conversely, assume that M is a Rickart module and $f(x) = \sum\limits_{i=0}^{k} f_i x^i \in S[x]$. By hypothesis, there exist $e_i^2 = e_i \in S$ $(i = 0, 1, 2, ..., k)$ such that $r_M(f_i) = e_i M$. Let $e = e_0 e_1 e_2 ... e_k$. We prove $r_{M[x]}(f(x)) = eM[x]$. For if $m(x) = \sum\limits_{j=0}^{t} m_j x^j \in r_{M[x]}(f(x))$, then $f(x)m(x) = 0$. Since M is S-Armendariz, $f_i m_j = 0$ for each $i = 0, 1, 2, ..., k$ and for each $j = 0, 1, 2, ..., t$. Then $m_j \in r_M(f_i) = e_i M$ and so $e_i m_j = m_j$, $em_j = m_j$ and $em(x) = m(x)$. Hence $m(x) \in eM[x]$ and so $r_{M[x]}(f(x)) \leq eM[x]$. On the other hand, $eM[x] \leq r_{M[x]}(f(x))$ and so $eM[x] = r_{M[x]}(f(x))$. □

Then we have the following result.

Corollary 2.16. *Let R be a ring. If $R[x]$ is a left Rickart ring, then R is a left Rickart ring. The converse holds if R is Armendariz.*

Özet: R birimli bir halka, M sağ R-modül ve M nin endomorfizma halkası $S = \mathrm{End}_R(M)$ olsun. Her $f \in S$ için $r_M(f) = eM$ olacak biçimde $e^2 = e \in S$ varsa (denk olarak $\mathrm{Ker}f$, M modülünün bir direkt toplananı ise) M ye *Rickart modül* adı verilmiştir [8]. Bu çalışmada Rickart modüllerin özellikleri incelenmeye devam edilmiştir. M bir Rickart modül olmak üzere, M nin S-katı (sırasıyla S-indirgenmiş, S-simetrik, S-yarı değişmeli, S-Armendariz) modül olması için gerek ve yeter şartın S nin katı (sırasıyla indirgenmiş, simetrik, yarı değişmeli, Armendariz) halka olduğu gösterilmiştir. $M[x]$, $S[x]$ halkasına

göre Rickart modül iken M nin de Rickart modül olduğu, tersinin M nin S-Armendariz olması durumunda doğru olduğu ispatlanmıştır. Ayrıca bir M modülünün Rickart olması için gerek ve yeter şartın her sağ modülün M-temel projektif olduğu elde edilmiştir.

References

[1] N. Agayev, S. Halicioglu and A. Harmanci, *On Rickart modules*, Bulletin of the Iranian Mathematical Society, 38(2) (2012), 433-445.

[2] N. Agayev, T. Ozen and A. Harmanci, *On a Class of Semicommutative Modules*, Proc. Indian Acad. Sci. 119(2009), 149-158.

[3] I. Kaplansky, Rings of Operators, Math. Lecture Note Series, Benjamin, New York, 1965.

[4] J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull. 14(1971), 359-368.

[5] G. Lee, S. T. Rizvi and C. S. Roman, *Rickart Modules*, Comm. Algebra 38(11)2010, 4005-4027.

[6] M. B. Rege and S. Chhawchharia, *Armendariz Rings*, Proc. Japan Acad. Ser. A Math. Sci. 73(1997), 14-17.

[7] S. T. Rizvi and C. S. Roman, *Baer and Quasi-Baer Modules*, Comm. Algebra 32(2004), 103-123.

[8] S. T. Rizvi and C. S. Roman, *On direct sums of Baer modules*, J. Algebra 321(2009), 682-696.

[9] H. Tansee and S. Wongwai, *A note on semi-projective modules*, Kyungpook Math. J. 42(2002), 369-380.

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