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EXACT SOLUTIONS OF THE ZAKHAROV EQUATIONS BY USING THE FIRST INTEGRAL METHOD

ARZU ÖĞÜN ÜNAL

Abstract. In this paper some traveling wave solutions of the Zakharov equations are obtained by using the first integral method. The first integral method is a powerful an effective method for solving nonlinear partial differential equations.

1. Introduction

Nonlinear evolution equations (such as KdV, Burgers, Bousinesq, etc.) are widely used to describe nonlinear phenomena in physics fields like the fluid mechanics, plasma physics, optics. In recent years various techniques have been developed to obtain exact solutions of nonlinear evolution equations such as Bäcklund transformation method [13,10], Painlevé method [16,21], inverse scattering method [2,20], Hirota's bilinear method [9], tanh method [8,12] and the first integral method [1,3,4,6,7,11,14,17,19].

The first integral method used in the theory of commutative algebra was first proposed by Feng to solve the Burgers Korteweg-de Vries equation [6]. Recently, many authors has applied this method to various types of nonlinear problems [1,3,4,7,11,14,17,18,19]. In this paper, we use the first integral method to find the exact solutions of the Zakharov equations.

Zakharov equations are the coupled nonlinear partial differential equations as follow

$$iu_t + u_{xx} = u\nu \tag{1}$$

$$v_{tt} - v_{xx} = (|u|^2)_{xx}. \tag{2}$$

Here, u is the slow variation amplitude of the electric field intensity and v is the perturbed number density of the media or ions in media. The Zakharov equations has various applications in physics such as theory of deep-water waves, nonlinear pulse propogation in optical fibers and interaction of laser plasma [15].

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2. The First Integral Method

Consider the general second order autonomus partial differential equation

$$P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}) = 0 \tag{3}$$

Assume that the equation (3) has the travelling wave solutions in the form

$$u(x,t) = U(\xi),\ \xi = x - \omega t \tag{4}$$

where ω represent the wave speed; if $\omega > 0$ ($\omega < 0$), then $U(x - \omega t)$ represents a wave traveling to the right (left) [22]. Then the Eq. (3) is reduced to the autonomus ordinary differential equation

$$Q(U(\xi), U'(\xi), U''(\xi)) = 0 \tag{5}$$

Next, we introduce new dependent variables $X(\xi)$ and $Y(\xi)$ as

$$X(\xi) = U(\xi),\ Y(\xi) = U'(\xi) \tag{6}$$

which leads Eq. (5) to the system of ODE

$$\begin{aligned} X'(\xi) &= Y(\xi), \\ Y'(\xi) &= F(X(\xi), Y(\xi)). \end{aligned} \tag{7}$$

According to the qualitative theory of differential equations [5] if we can find two first independent integrals of system (7), then the general solutions of (7) can be expressed explicitly and so can all kind of travelling wave solutions of Eq. (3). However, it is generally difficult to find even one of the first integrals. Because there is not any systematic way to tell us how to find these integrals. So, our aim is to obtain at least one first integral of system (7). To do this, we will apply the Division Theorem which is based on the Hilbert-Nullsellensatz Theorem [6]. Now, we recall the Division Theorem for two variables in the complex domain $\mathbb{C}$.

Division Theorem. *Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $\mathbb{C}[w,z]$ and $P(w,z)$ is irreducible in $\mathbb{C}[w,z]$; if $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exist a polynomial $H(w,z)$ in $\mathbb{C}[w,z]$ such that,*

$$Q(w,z) = P(w,z)H(w,z)$$

3. Zakharov equation

In this section we study the (1)-(2) Zakharov equations. Applying the transformations

$$u(x,t) = e^{i\theta}U(\xi),\ v(x,t) = V(\xi),\ \theta = cx + t,\ \xi = x - 2ct \tag{8}$$

to the Eq. (1)-(2), we obtain the system of ordinary differential equations

$$U'' - (c^2+1)U = UV \tag{9}$$

$$(4c^2-1)V'' = \frac{\partial^2}{\partial \xi^2}(U^2) \tag{10}$$

Integrating Eq. (10) twice with respect to ξ, we have

$$(4c^2-1)V = U^2 + c_2 \tag{11}$$

where c_2 is second integration constant and first one is taken to zero. Inserting (11) into Eq (9), we have

$$U'' - (\frac{c_2}{4c^2-1} + c^2 + 1)U - \frac{1}{4c^2-1}U^3 = 0 \tag{12}$$

Using (6), we get the following system which is equivalent to (12)

$$X' = Y, \tag{13a}$$

$$Y' = (\frac{c_2}{4c^2-1} + c^2 + 1)X + \frac{1}{4c^2-1}X^3 \tag{13b}$$

According to the first integral method, we assume that $X(\xi)$, $Y(\xi)$ is a nontrivial solution of (13) and

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i \tag{14}$$

is an irreducible polynomial in the complex domain $\mathbb{C}$ such that

$$Q(X(\xi),Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y(\xi)^i = 0 \tag{15}$$

where $a_i(X)$ $(i = 0, 1, ..., m)$ are polynomials of X and $a_m(X) \neq 0$. Equation (14) is called the first integral of (13). According to the Division Theorem, there exist a polynomial $g(X) + h(X)Y$ in the complex domain $\mathbb{C}$ such that

$$\frac{dQ}{d\xi} = \frac{\partial Q}{\partial X}\frac{dX}{d\xi} + \frac{\partial Q}{\partial Y}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i. \tag{16}$$

We consider two different cases for (14) $m = 1$ and $m = 2$.

Case 1. $m = 1$

Equating the coefficients of Y^i on both sides of equation (16), we have

$$a_1'(X) = h(X)a_1(X), \tag{17a}$$

$$a_0'(X) = g(X)a_1(X) + h(X)a_0(X), \tag{17b}$$

$$a_1(X)[(\frac{c_2}{4c^2-1} + c^2 + 1)X + \frac{1}{4c^2-1}X^3] = g(X)a_0(X). \tag{17c}$$

Since $a_i(X)$ are polynomials, from (17a) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplification we take $a_1(X) = 1$. Hence (17) can be rewriten as

$$a_0'(X) = g(X), \tag{18a}$$

$$(\tfrac{c_2}{4c^2-1} + c^2 + 1)X + \tfrac{1}{4c^2-1}X^3 = g(X)a_0(X) \tag{18b}$$

Balancing the degrees of $a_0(X)$ and $g(x)$, we conclude that $\deg g(X) = 1$ only. Assume that

$$g(X) = AX + B \tag{19}$$

where $A,\ B \in \mathbb{C}$. Then, from (18a)

$$a_0(X) = \frac{A}{2}X^2 + BX + C \tag{20}$$

where C is an arbitrary integration constant. Substituting (19) and (20) into (18b) and setting all coefficients of X^i $(i = 0, 1, 2, 3)$ to be zero, we obtain the following two solutions

$$A = \pm\sqrt{\frac{2}{4c^2-1}}, \qquad B = 0, \qquad C = \pm\sqrt{\frac{4c^2-1}{2}}\left(\frac{c_2}{4c^2-1} + c^2 + 1\right) \tag{21}$$

Using the conditions (21) in equation (15), we have

$$Y \pm \left(\sqrt{\frac{1}{2(4c^2-1)}}X^2 + \sqrt{\frac{4c^2-1}{2}}\left(\frac{c_2}{4c^2-1} + c^2 + 1\right)\right) = 0 \tag{22}$$

Solving Eq (22) with subject to the Y and substituting them into Eq. (13a), we obtain the exact solution of (13) and then the exact solutions of Zakharov equations can be written as

$$u_1(x,t) = \pm e^{i\theta}\sqrt{c_2 + (c^2+1)(4c^2-1)}\tan\sqrt{\frac{c_2}{2(4c^2-1)} + \frac{c^2+1}{2}}(\xi + \xi_0) \tag{23}$$

$$v_1(x,t) = \frac{c_2}{4c^2-1} + \left(\frac{c_2}{4c^2-1} + c^2 + 1\right)\tan^2\sqrt{\frac{c_2}{2(4c^2-1)} + \frac{c^2+1}{2}}(\xi + \xi_0) \tag{24}$$

where $\theta = cx + t$, $\xi = x - 2ct$ and ξ_0 is an arbitrary constant.

Case 2. $m = 2$.

By equating the coefficients of Y^i on both sides of (16) we have

$$\begin{aligned}
&a_2'(X) = h(X)a_2(X) && (25a)\\
&a_1'(X) = g(X)a_2(X) + h(X)a_1(X) && (25b)\\
&a_0'(X) = -2a_2[(\tfrac{c_2}{4c^2-1} + c^2 + 1)X + \tfrac{1}{4c^2-1}X^3] && (25c)\\
&\qquad + g(X)a_1(X) + h(X)a_0(X) &&\\
&a_1(X)[(\tfrac{c_2}{4c^2-1} + c^2 + 1)X + \tfrac{1}{4c^2-1}X^3] = g(X)a_0(X) && (25d)
\end{aligned}$$

Since $a_i(X)$ are polynomials, from (25a), we deduce that $a_2(X)$ is constant and $h(X) = 0$. Again, let us take $a_2(X) = 1$. Thus the system can be rewriten as follow

$$\begin{aligned}
&a_1'(X) = g(X) && (26a)\\
&a_0'(X) = -2[(\tfrac{c_2}{4c^2-1} + c^2 + 1)X + \tfrac{1}{4c^2-1}X^3] + g(X)a_1(X) && (26b)\\
&a_1(X)[(\tfrac{c_2}{4c^2-1} + c^2 + 1)X + \tfrac{1}{4c^2-1}X^3] = g(X)a_0(X) && (26c)
\end{aligned}$$

Balancing the terms of $a_0(X),\ a_1(X)$ and $g(X)$, we conclude that either $\deg g(X) = 0$ or $\deg g(X) = 1$.

Let us consider the case of $\deg g(X) = 0$, that is,

$$g(x) = A \tag{27}$$

where $A \neq 0$. Then, from (26a-b), we get

$$a_1(X) = AX + B, \tag{28}$$

$$a_0(X) = -\frac{1}{2(4c^2-1)}X^4 + [\frac{A^2}{2} - (\frac{c_2}{4c^2-1} + c^2 + 1)]X^2 + ABX + C \tag{29}$$

where B and C are integration constants. Let us substitute $a_0(X)$, $a_1(X)$ and $g(X)$ into (26c) and equate the all coefficients of X^i $(i = 0, 1, 2, 3, 4)$ to the zero. Therefore, it follows

$$A = 0, \quad B = 0, \quad C = arbitrary. \tag{30}$$

Specialy if we choose $C = 0$, from (30), (14) and (13a) we find

$$X' = \pm X\sqrt{\frac{1}{2(4c^2-1)}X^2 + \frac{c_2}{4c^2-1} + c^2 + 1} \tag{31}$$

These equations have the following solutions;
if $|c| > \frac{1}{2},\ c_2 > -(c^2+1)(4c^2-1)$ then

$$X(\xi) = \mp\sqrt{2c_2 + 2(c^2+1)(4c^2-1)}\,\operatorname{csc}h\left[\sqrt{\frac{c_2}{4c^2-1} + c^2 + 1}(\xi+\xi_0)\right] \tag{32}$$

if $|c| > \frac{1}{2},\ c_2 < -(c^2+1)(4c^2-1)$ then

$$\begin{aligned} X(\xi) \quad &= \quad \pm\sqrt{-2c_2 - 2(c^2+1)(4c^2-1)}\,\cot[\sqrt{-\frac{c_2}{4c^2-1} - (c^2+1)}(\xi+\xi_0)] \\ &\quad \mathrm{x}\sqrt{1 - \tan[(\frac{c_2}{4c^2-1} + c^2 + 1)(\xi+\xi_0)^2]} \end{aligned} \tag{33}$$

if $|c| < \frac{1}{2},\ c_2 < -(c^2+1)(4c^2-1)$ then

$$X(\xi) = \sqrt{-2c_2 - 2(c^2+1)(4c^2-1)}\,\operatorname{sec}h\left[\sqrt{\frac{c_2}{4c^2-1} + c^2 + 1}(\xi+\xi_0)\right] \tag{34}$$

if $|c| < \frac{1}{2},\ c_2 \ > -(c^2+1)(4c^2-1)$ then

$$\begin{aligned} X(\xi) \quad &= \quad \pm\sqrt{2c_2 + 2(c^2+1)(4c^2-1)}\,\cot[\sqrt{-\frac{c_2}{4c^2-1} - (c^2+1)}(\xi+\xi_0)] \\ &\quad \mathrm{x}\sqrt{-1 + \tan[(\frac{c_2}{4c^2-1} + c^2 + 1)(\xi+\xi_0)^2]}. \end{aligned} \tag{35}$$

By combining (6), (8), (11) and above solutions, some exact solutions of Zakharov equations are obtained as follow;

if $|c| > \frac{1}{2},\ c_2 > -(c^2+1)(4c^2-1)$ then

$$\begin{aligned} u_2(\xi) &= \pm e^{i\theta}\sqrt{2[c_2+(c^2+1)(4c^2-1)]}\csc h\left[\sqrt{\frac{c_2+(c^2+1)(4c^2-1)}{4c^2-1}}(\xi+\xi_0)\right] \\ v_2(\xi) &= \frac{c_2}{4c^2-1}+2[\frac{c_2}{4c^2-1}+c^2+1]\csc h^2\left[\sqrt{\frac{c_2}{4c^2-1}+c^2+1}(\xi+\xi_0)\right] \end{aligned} \quad (36)$$

if $|c| > \frac{1}{2},\ c_2 < -(c^2+1)(4c^2-1)$ then

$$\begin{aligned} u_3(\xi) &= \pm e^{i\theta}\sqrt{-2c_2-2(c^2+1)(4c^2-1)}\cot\left[\sqrt{-\frac{c_2}{4c^2-1}-(c^2+1)}(\xi+\xi_0)\right] \\ &\quad \text{x}\sqrt{1-\tan[(\frac{c_2}{4c^2-1}+c^2+1)(\xi+\xi_0)^2]} \quad (37) \\ v_3(\xi) &= \frac{c_2}{(4c^2-1)}+[\frac{-2c_2}{(4c^2-1)}-2(c^2+1)]\cot^2\left[\sqrt{-\frac{c_2}{4c^2-1}-(c^2+1)}(\xi+\xi_0)\right] \\ &\quad \text{x}\left\{1-\tan[(\frac{c_2}{4c^2-1}+(c^2+1))(\xi+\xi_0)^2]\right\} \end{aligned}$$

if $|c| < \frac{1}{2},\ c_2 < -(c^2+1)(4c^2-1)$ then

$$\begin{aligned} u_4(\xi) &= e^{i\theta}\sqrt{-2[c_2+(c^2+1)(4c^2-1)]}\sec h\left[\sqrt{\frac{c_2+(c^2+1)(4c^2-1)}{4c^2-1}}(\xi+\xi_0)\right] \\ v_4(\xi) &= \frac{c_2}{4c^2-1}-2[\frac{c_2}{4c^2-1}+c^2+1]\sec h^2\left[\sqrt{\frac{c_2}{4c^2-1}+c^2+1}(\xi+\xi_0)\right] \end{aligned} \quad (38)$$

if $|c| < \frac{1}{2},\ c_2 > -(c^2+1)(4c^2-1)$ then

$$\begin{aligned} u_5(\xi) &= \pm e^{i\theta}\sqrt{2c_2+2(c^2+1)(4c^2-1)}\cot\left[\sqrt{-\frac{c_2}{4c^2-1}-(c^2+1)}(\xi+\xi_0)\right] \\ &\quad \text{x}\sqrt{-1+\tan[(\frac{c_2}{4c^2-1}+c^2+1)(\xi+\xi_0)^2]} \quad (39) \\ v_5(\xi) &= \frac{c_2}{4c^2-1}+2[\frac{c_2}{4c^2-1}+c^2+1]\cot^2\left[\sqrt{-\frac{c_2}{4c^2-1}-(c^2+1)}(\xi+\xi_0)\right] \\ &\quad \text{x}\left\{-1+\tan[(\frac{c_2}{4c^2-1}+c^2+1)(\xi+\xi_0)^2]\right\} \end{aligned}$$

where $\theta = cx+t$, $\xi = x-2ct$ and ξ_0 is an arbitrary constant.

Now we assume that $\deg g(X) = 1$; that is, $g(X) = AX+B$, where $A \neq 0$. Then, from (26a-b) we find

$$\begin{aligned} a_1 &= \frac{A}{2}X^2 + BX + C, \\ a_0 &= [\frac{A^2}{8} - \frac{1}{2(4c^2-1)}]X^4 + \frac{AB}{2}X^3 \\ &\quad +[\frac{AC}{2} + \frac{B^2}{2} - (\frac{c_2}{4c^2-1} + c^2+1)]X^2 + BCX + D \end{aligned}$$

where C, D are arbitrary integration constants. Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ into (26c) and setting all the coefficients of powers X to be zero, we obtain

$$\begin{aligned} A &= \pm\frac{2\sqrt{2}}{\sqrt{4c^2-1}}, \quad B=0, \quad C = \pm\frac{\sqrt{2}}{\sqrt{4c^2-1}}[c_2 + (4c^2-1)(c^2+1)], \\ D &= \frac{[c_2+(4c^2-1)(c^2+1)]^2}{2(4c^2-1)} \end{aligned} \tag{40}$$

Putting (40) into (15), we obtain the same equations as (22). So we have the same exact solutions as (23)-(24).

Özet: Bu çalışmada ilk integral metodu yardımıyla Zakharov denkleminin bazı hareketli dalga çözümleri elde edilmiştir. İlk integral metodu, lineer olmayan kısmi türevli denklemleri çözmek için oldukça güçlü ve etkili bir yöntemdir.

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Current address: Arzu Ünal, Department of Mathematics, Faculty of Science, Ankara University, 06100 Ankara, TURKEY

E-mail address: `aogun@science.ankara.edu.tr`

URL: `http://communications.science.ankara.edu.tr/index.php?series=A1`