Commun.Fac.Sci.Univ.Ank.Series A1
Volume 61, Number 2, Pages 51–66(2012)
ISSN 1303–5991

ON COMPLEX $q-$SZÁSZ-MIRAKJAN OPERATORS

DIDEM AYDIN

Abstract. In this paper, we introduce and study complex $q-$Szász-Mirakjan operators attached to analytic functions satisfying a suitable exponential type growth condition. We give a Voronovskaja-type theorem in compact disks for these new operators. Note that our results are different from the results given for other type complex $q-$Szász-Mirakjan operators in [8].

1. Introduction

In 1996, Phillips defined a generalization of the Bernstein operators called $q-$Bernstein operators by using the $q-$binomial coefficients and the $q-$binomial theorem [9]. In 2008, Aral introduced $q-$Szász-Mirakjan operators and studied some approximation properties of them [1]. In 2008, Gal studied some approximation results of the complex Favard-Szász-Mirakjan operators on compact disks [3].

In this work, we consider complex version of $q-$Szász-Mirakjan operators introduced by Aral in [1].

Now, we give some notations on $q-$analysis given in [2],[5] and [9] .
The $q-$integer $[n]$ is defined by

$$[n] := [n]_q = \begin{cases} \frac{1-q^n}{1-q}, \ q \neq 1 \\ n, \ q = 1 \end{cases}$$

for $q > 0$ and the $q-$factorial $[n]!$ by

$$[n]! := \begin{cases} [1]_q [2]_q ... [n]_q \, , \ n = 1, 2, ... \\ 1, \ n = 0. \end{cases}$$

We give the following two $q-$analogues of the exponential function e^x which is appeared in the definition of the operator :

Received by the editors Aug 16, 2012, Accepted: Dec. 23, 2012.
2000 *Mathematics Subject Classification.* 30E10.
Key words and phrases. $q-$Szász-Mirakjan operator, q-derivative, Voronovskaja-type theorem.

$$\varepsilon_q(x)=\sum_{n=0}^{\infty}\frac{1}{[n]_q!}x^n=\frac{1}{((1-q)x;\ q)_\infty},\ |x|<\frac{1}{1-q},\ |q|<1, \tag{1.1}$$

$$E_q(x)=\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}}{[n]_q!}x^n=\ (-(1-q)x\ ;\ q)_\infty\ ,\ x\in R,\ |q|<1, \tag{1.2}$$

where $(x;\ q)_\infty=\prod_{k=1}^{\infty}(1-xq^{k-1})$ (see [5]).

It is clear from (1.1) and (1.2) that $\varepsilon_q(x)E_q(-x)=1$ and

$$\lim_{q\to1^-}\varepsilon_q(x)=\lim_{q\to1^-}E_q(x)=e^x.$$

Suppose that $R_{n,q}:=\frac{b_n}{[n](1-q)}$, where (b_n) is a sequence of positive numbers such that $\lim_{n\to\infty}b_n=\infty$ and that $D_R=\{z\in\mathbb{C}:|z|<R\}$, $1<R<R_{n,q}$. The complex Szász-Mirakjan operator based on $q-$integers is obtained directly from the real version (see [1]) by taking z in place of x, namely

$$\begin{aligned} S_n^q(f;z) &= S_n(f;q;z) \\ &=: E_q\left(-[n]\frac{z}{b_n}\right)\sum_{k=0}^{\infty}f\left(\frac{[k]}{[n]}b_n\right)\frac{([n]\,z)^k}{[k]!\,(b_n)^k}, \end{aligned} \tag{1.3}$$

where $n\in\mathbb{N}$, $0<q<1$, and $f:[R,\infty)\cup\overline{D_R}\to\mathbb{C}$ has exponential growth and it has an analytical continuation into an open disk centered at the origin. (see [1]). Note that in the real case the $q-$Szász-Mirakjan operators are actually a $q-$extension of the Szász-Chlodovsky operators constructed by Stypinsky in [10]. A different type complex $q-$Szász-Mirakjan operator was introduced by Mahmudov in [8] for $q>1$ as

$$M_{n,q}(f;z)=\sum_{k=0}^{\infty}f\left(\frac{[k]}{[n]}\right)\frac{1}{q^{k(k-1)/2}}\frac{[n]^k z^k}{[k]!}\varepsilon_q\left(-[n]\,q^{-k}z\right) \tag{1.4}$$

for the functions which are continuous and bounded on $[0,\infty)$. In [8], the author studied quantitative estimates for the convergence, Voronovskaja's theorem and saturation for convergence of the operators attached to analytic functions in suitable compact disks. Moreover the rate of convergence is given.

In the present work, we study some approximation properties of complex $q-$Szász-Mirakjan operators. Also, by using $q-$derivative, we give a Voronovskaja type result with quantative estimate in the sense of Gal [4].

Notice that, the operator defined by (1.3) and the obtained results are completely different from to that of studied in [8] by Mahmudov. In this paper, we give some estimates on rate of convergence and Voronovskaja-type results with quantitative estimates for the operators (1.3) by means of $q-$derivative. Note also that similar results for complex Favard-Szász-Mirakjan operators was firstly studied by Gal [3] using classical derivative.

Throughout the paper we call the operator (1.3) as complex $q-$Szász-Mirakjan operator.

It is clear that by using divided differences $S_n^q(f;z)$ can be expressed as

$$S_n^q(f;z) = S_n(f,q,z) = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} f\left[0, \frac{b_n\,[1]}{[n]}, ..., \frac{b_n\,[j]}{[n]}\right] z^j, \tag{1.5}$$

similar to the real version of the $q-$Szász-Mirakjan operators (see [1]), where $f\left[0, \frac{b_n[1]}{[n]}, ..., \frac{b_n[j]}{[n]}\right]$ denotes the divided difference of f on the knots $0, \frac{b_n[1]}{[n]}, ..., \frac{b_n[j]}{[n]}$.

2. Convergence of $S_n^{q_n}(f;z)$

Let $q \in (0,1) \cup (1,\infty)$. The $q-$derivative of a function $f(x)$ is defined as

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x} \text{ for}, x \neq 0.$$

$$D_q f(0) = \lim_{x\to 0} D_q f(x), \text{ where } D_q^0 f := f,\ D_q^n f := D_q(D_q^{n-1} f),\ n = 1, 2, ...$$

As a consequence of the definition of $D_q f$, we find

$$D_q x^n = [n]_q\, x^{n-1},$$

$$D_q \varepsilon_q(ax) = a\varepsilon_q(ax),$$

$$D_q E_q(ax) = aE_q(qax).$$

Also, the formula for the $q-$differential of a product is

$$D_q(u(x)v(x)) = D_q(u(x))v(x) + u(qx)D_q(v(x)).$$

We know that

$$\left(D_q(t;x)_q^n\right)(t) = [n]_q\,(t;x)_q^{n-1},$$

where $(t;x)_q^n = \prod_{k=0}^{n-1}(t - xq^k)$ (see [2]).

Now, we give remark and lemma which we use in the proof of Theorem 2.3.

Remark 2.1. It is known that for a fixed value of q with $0 < q < 1$, since $\frac{1}{[n]} \to 1-q$ as $n \to \infty$. To ensure the convergence properties of $S_n^q(f;z)$, we will assume $q = q_n$ as a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$ so that $[n]_{q_n} \to \infty$ as $n \to \infty$. On the other hand, for the sequence (b_n) is of positive numbers satisfying $\lim_{n\to\infty} b_n = \infty$, $R_{n,q_n} = \frac{b_n}{[n]_{q_n}(1-q_n)} \to \infty$ as $n \to \infty$. Indeed, for example, if we choose a sequence q_n such that $q_n = \frac{n}{n+1}$, then we have $q_n^n \to \frac{1}{e}$ as $n \to \infty$, which gives that $R_{n,q_n} = \frac{b_n}{[n]_{q_n}(1-q_n)} = \frac{b_n}{1-q_n^n} \to \infty$ as $n \to \infty$.

Lemma 2.2. *Let* $D_R = \{z \in \mathbb{C} : |z| < R\}$, $1 < R < R_{n,q}$, *where* $R_{n,q} = \frac{b_n}{[n]_q(1-q)}$ *and*

$$f : [R, \infty) \cup \overline{D_R} \to \mathbb{C}$$

be continuous in $[R, \infty) \cup \overline{D_R}$, *analytic in* D_R, *namely* $f(z) = \sum_{k=0}^{\infty} c_k z^k$ *for all* $z \in D_R$ *and there exist* $M,\ C,\ B > 0$ *and* $A \in \left(\frac{1}{R}, 1\right)$, *with the property* $|c_k| \leq \frac{MA^k}{k!}$ *for all* $k = 0, 1, ...$*(which implies* $|f(z)| \leq Me^{A|z|}$ *for all* $z \in D_R$ *and* $|f(x)| \leq Ce^{Bx}$ *for all* $x \in [R, \infty)$*). Then* $S_n^q(f; z)$ *is well defined and analytic as function of* z *in* D_R.

Proof. Passing to modulus we have from (1.3)

$$\begin{aligned} |S_n^q(f;z)| &\leq \left|E_q\left(-[n]_q \frac{z}{b_n}\right)\right| \sum_{k=0}^{\infty} \frac{\left([n]_q\right)^k |z|^k}{[k]_q! \,(b_n)^k} \left| f\left(\frac{[k]}{[n]_q} b_n\right)\right| \\ &\leq C\left|E_q\left(-[n]_q \frac{z}{b_n}\right)\right| \sum_{k=0}^{\infty} \frac{\left([n]_q\right)^k |z|^k}{[k]_q! \,(b_n)^k} e^{B\left(\frac{[k]_q}{[n]_q} b_n\right)}. \end{aligned}$$

By using the ratio test, we obtain

$$\begin{aligned} \lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \frac{[n]_{q_n} |z|}{b_n\,[k+1]_q} e^{B_m\left(\frac{[k+1]_q}{[n]_q} b_n - \frac{[k]_q}{[n]_q} b_n\right)} \\ &= \frac{[n]_q}{b_n} \lim_{k\to\infty} \frac{e^{B_m q^k \frac{b_n}{[n]_q}} |z|}{[k+1]_q} \\ &= \frac{[n]_q}{b_n} \lim_{k\to\infty} \frac{re^{B_m q^k \frac{b_n}{[n]_q}} |z|}{\frac{1-q^{k+1}}{1-q}} \\ &= \frac{[n]_q}{b_n}(1-q)\,|z|\,, \end{aligned}$$

which shows that the series is convergent for $|z| < R$, by the hypothesis $R < R_{n,q} := \frac{b_n}{[n](1-q)}$, and therefore, $S_n^q(f; z)$ is well defined and analytic as function of z. □

We note here that from the hypothesis on f, the analyticity of $S_n^q(f; z)$ can be seen also from [6].

Theorem 2.3. *Suppose that the conditions of Lemma 2.2 are satisfied. Suppose also that* $q = q_n$ *is a sequence such that* $0 < q_n < 1$ *and* $q_n \to 1$ *as* $n \to \infty$ *and* $\frac{b_n}{[n]_{q_n}} \to 0$ *as* $n \to \infty$.

(i) *Let* $1 \leq r < \frac{1}{A}$ *be arbitrary fixed. There exist* n_0 *such that for all* $n > n_0$ *and all* $|z| \leq r$*, we have*

$$|S_n^{q_n}(f;z) - f(z)| \leq C_{r,A}$$

where

$$C_{r,A} = \left(1 - q_n + \frac{b_n}{[n]_{q_n}}\right) \frac{MA}{2} \sum_{k=2}^{\infty} (k+1)(rA)^k < \infty.$$

(ii) *For the simultaneous approximation by complex* $q-$*Szász-Mirakjan operator, we have*

$$\left| D_{q_n}^{(p)}\left(S_n^{q_n}(f;z)\right) - D_{q_n}^{(p)} f(z) \right| \leq \frac{C_{r_1,A} b_n}{[n]_{q_n}} \frac{p! r_1}{(r_1 - r)^{p+1}},$$

where $C_{r_1,A}$ *is given as in the case* (i).

Proof. (i) By taking $e_k(z) = z^k$, it is clear that $T_{n,k}(z) := S_n^{q_n}(e_k;z)$ is a polynomial of degree$\leq k$, $k = 0,1,2,...$ and

$$T_{n,0}(z) = 1, T_{n,1}(z) = z \text{ for all } \; z \in \mathbb{C}$$

Also, using $q-$derivative of $T_{n,k}(z)$ for $z \neq 0$, we get

$$\begin{aligned}
& D_q T_{n,k}(z) \\
= \; & -\frac{[n]_{q_n}}{b_n} E_q\left(-[n]_{q_n} \; q_n \frac{z}{b_n}\right) \sum_{j=0}^{\infty} \left(\frac{[j]_{q_n}}{[n]_{q_n}} b_n\right)^k \frac{\left([n]_{q_n} q_n z\right)^j}{[j]_{q_n}! \, (b_n)^j} \\
& + E_q\left(-[n]_{q_n} \frac{z}{b_n}\right) \sum_{j=0}^{\infty} \left(\frac{[j]_{q_n}}{[n]_{q_n}} b_n\right)^k \frac{\left([n]_{q_n}\right)^j \; [j]_{q_n} \; z^{j-1}}{[j]_{q_n}! \, (b_n)^j} \frac{z}{z} \frac{[n]_{q_n}}{b_n} \frac{b_n}{[n]_{q_n}} \\
= \; & \frac{[n]_{q_n}}{z b_n} T_{n,k+1}(z) \\
& -\frac{[n]_{q_n}}{b_n} E_q\left(-[n]_{q_n} \; q_n \frac{z}{b_n}\right) \sum_{j=0}^{\infty} \left(\frac{[j]_{q_n}}{[n]_{q_n}} b_n\right)^k \frac{\left([n]_{q_n} q_n z\right)^j}{[j]_{q_n}! \, (b_n)^j} \qquad (2.1)
\end{aligned}$$

for all $z \in \mathbb{C}$, $k = 0,1,2,....$ Therefore, we obtain

$$T_{n,k}(z) = z T_{n,k-1}(q_n z) + \frac{z b_n}{[n]_{q_n}} D_q\left(T_{n,k-1}(z)\right).$$

The last equality implies that

$$
\begin{aligned}
T_{n,k}(z)-z^{k} &= \frac{zb_n}{[n]_{q_n}} D_q\left(T_{n,k-1}(z)-z^{k-1}\right)+z\left[T_{n,k-1}(q_n z)-(q_n z)^{k-1}\right] \\
&\quad +\frac{zb_n}{[n]_{q_n}}[k-1]_{q_n} z^{k-2}+z^{k-1}q_n^{k-1}z-z^{k} \\
&= \frac{zb_n}{[n]_{q_n}} D_q\left(T_{n,k-1}(z)-z^{k-1}\right)+z\left[T_{n,k-1}(q_n z)-(q_n z)^{k-1}\right] \\
&\quad +\frac{[k-1]_{q_n}}{[n]_{q_n}} b_n z^{k-1}+z^{k}(q_n^{k-1}-1) \\
&= \frac{zb_n}{[n]_{q_n}} D_q\left(T_{n,k-1}(z)-z^{k-1}\right)+z\left[T_{n,k-1}(q_n z)-(q_n z)^{k-1}\right] \\
&\quad +\frac{[k-1]_{q_n}}{[n]_{q_n}} b_n z^{k-1}-z^{k}(1-q_n)[k-1]_{q_n}. \qquad (2.2)
\end{aligned}
$$

From the Bernstein inequality in $\overline{D_r}=\{z\in\mathbb{C}:\ |z|\leq r\}$, we have

$$
|D_q(P_k(z)|\leq\|P_k'\|\leq\frac{k}{r}\|P_k\|_r, \qquad (2.3)
$$

where $\|.\|_r=\max_{z\in\overline{D_r}}|f(z)|$ (see [4, p. 55]). From (2.2) and (2.3), we obtain that

$$
\begin{aligned}
&\left|T_{n,k}(z)-z^{k}\right| \\
\leq\ & \frac{rb_n}{[n]_{q_n}}\left|D_q\left(T_{n,k-1}(z)-z^{k-1}\right)\right|+r\left|T_{n,k-1}(q_n z)-(q_n z)^{k-1}\right| \\
&+\frac{[k-1]_{q_n}}{[n]_{q_n}} b_n r^{k-1}+r^{k}[k-1]_{q_n}|1-q_n| \\
\leq\ & \frac{rb_n}{[n]_{q_n}}\left\|T_{n,k-1}(z)-z^{k-1}\right\|_r\frac{k-1}{r} \\
&+r\left|T_{n,k-1}(q_n z)-(q_n z)^{k-1}\right|+r^{k-1}\frac{[k-1]_{q_n}}{[n]_{q_n}} b_n+r^{k}[k-1]_{q_n}|1-q_n|.
\end{aligned}
$$

By passing to norm we reach to

$$\begin{aligned}
&\left\|T_{n,k}(z)-z^{k}\right\|_{r}\\
\leq\ & \frac{(k-1)b_n}{[n]_{q_n}}\left\|T_{n,k-1}(z)-z^{k-1}\right\|_r + r\left\|T_{n,k-1}(q_nz)-(q_nz)^{k-1}\right\|_r\\
&+\frac{[k-1]_{q_n}}{[n]_{q_n}}b_nr^{k-1}+r^k\,[k-1]_{q_n}\,|1-q_n|\\
\leq\ & \frac{(k-1)b_n}{[n]_{q_n}}\left\|T_{n,k-1}(z)-z^{k-1}\right\|_r + r\left\|T_{n,k-1}(z)-z^{k-1}\right\|_r +\\
&+r^k\,[k-1]_{q_n}\left(1-q_n+\frac{b_n}{[n]_{q_n}}\right)\\
=\ & \left(\frac{(k-1)b_n}{[n]_{q_n}}+r\right)\ \left\|T_{n,k-1}(z)-z^{k-1}\right\|_r + r^kk\left(1-q_n+\frac{b_n}{[n]_{q_n}}\right).
\end{aligned}$$

By using mathematical induction with respect to k, the above recurrence formula gives that

$$\left\|T_{n,k}(z)-z^{k}\right\|_{r}\leq\frac{(k+1)!r^k}{2}\left(1-q_n+\frac{b_n}{[n]_{q_n}}\right)$$

for all $k\geq 2$ and fixed an arbitrary $n\geq n_0$. There exists an n_0 such that for all $n>n_0$, then $\frac{b_n}{[n]_{q_n}}<1$. Assume that it is true for k. Since $[k]_{q_n}\leq (k+1)$ is satisfied for all $0<q_n<1$, the recurrence formula reduces to

$$\begin{aligned}
&\left\|T_{n,k+1}(z)-z^{k+1}\right\|_{r}\\
\leq\ & \left(r+\frac{k}{[n]_{q_n}}b_n\right)\left\|T_{n,k}(z)-z^{k}\right\|_r + r^{k+1}\,[k]_{q_n}\left(1-q_n+\frac{b_n}{[n]_{q_n}}\right)\\
\leq\ & \left(r+\frac{k}{[n]_{q_n}}b_n\right)\frac{(k+1)!r^k}{2}\left(1-q_n+\frac{b_n}{[n]_{q_n}}\right)+r^{k+1}(k+1)\left(1-q_n+\frac{b_n}{[n]_{q_n}}\right)\\
\leq\ & \left(1-q_n+\frac{b_n}{[n]_{q_n}}\right)\frac{r^{k+1}}{2}\left\{(k+1)!k\frac{b_n}{[n]_{q_n}}+(k+1)!+2(k+1)\right\}
\end{aligned}$$

for all $k\geq 2$ and for all $n>n_0$. By this inequality, it follows

$$\left\|T_{n,k+1}(z)-z^{k+1}\right\|_{r}\leq\frac{(k+2)!}{2}r^{k+1}\left(1-q_n+\frac{b_n}{[n]_{q_n}}\right).$$

for $k\geq 2$ and for all $n>n_0$.

Now, we show that

$$S_n^{q_n}(f;z)=\sum_{k=0}^{\infty}c_kS_n^{q_n}(e_k;z)=\sum_{k=0}^{\infty}c_kT_{n,k}(z) \tag{2.4}$$

for all $z \in D_R$. For any $m \in \mathbb{N}$, let us define

$$f_m(z) = \sum_{j=0}^{m} c_j z^j \text{ if } |z| \leq r < R \text{ and } f_m(x) = f(x) \text{ if } x \in (r, \infty).$$

From the hypthesis on f, it is clear that for any $m \in \mathbb{N}$, $|f_m(x)| \leq C_m e^{B_m x}$ for all $x \in [0, \infty)$. Ratio test implies that for each fixed $m,\ n \in \mathbb{N}$ and z,

$$|S_n^{q_n}(f_m; z)| \leq C_m \left| E_q \left(-[n]_{q_n} \frac{z}{b_n} \right) \right| \sum_{k=0}^{\infty} \frac{\left([n]_{q_n}\right)^k |z|^k}{[k]_{q_n}! \, (b_n)^k} e^{B_m \left(\frac{[k]_{q_n}}{[n]_{q_n}} b_n \right)} < \infty.$$

Therefore, $S_n^{q_n}(f_m; z)$ is well defined. Now, we set

$$f_{m,k}(z) = c_k e_k(z) \text{ if } |z| \leq r \text{ and } f_{m,k}(x) = \frac{f(x)}{m+1} \text{ if } x \in (r, \infty).$$

It is clear that each $f_{m,k}$ is of exponential growth on $[0, \infty)$ and that

$$f_m(z) = \sum_{k=0}^{m} f_{m,k}(z).$$

Since $S_n^{q_n}$ is linear, we have

$$S_n^{q_n}(f_m; z) = \sum_{k=0}^{m} c_k S_n^{q_n}(e_k; z) \text{ for all } |z| \leq r,$$

which proves that

$$\lim_{m \to \infty} S_n^{q_n}(f_m; z) = S_n^{q_n}(f; z)$$

for any fixed $n \in \mathbb{N}$ and $|z| \leq r$. But this is immediate from

$$\lim_{m \to \infty} \|f_m - f\|_r = 0$$

and from the inequality

$$\begin{aligned} |S_n^{q_n}(f_m) - S_n^{q_n}(f)| &\leq \left| E_q \left(-[n]_{q_n} \frac{z}{b_n} \right) \right| \varepsilon_q \left([n]_{q_n} \frac{|z|}{b_n} \right) \|f_m - f\|_r \\ &\leq M_{r,n} \|f_m - f\|_r , \end{aligned}$$

for all $|z| \leq r$. Consequently the statement (2.4) is satisfied.

In this way, from the hypothesis on c_k, this implies for all $|z| \leq r$

$$\begin{aligned}
& |S_n^{q_n}(f;z) - f(z)| \\
\leq & \sum_{k=2}^{\infty} |c_k| \left|T_{n,k}(z) - z^k\right| \leq \sum_{k=2}^{\infty} \frac{MA^k}{k!} \frac{(k+1)!}{2} r^k \left(1 - q_n + \frac{b_n}{[n]_{q_n}}\right) \\
= & \left(1 - q_n + \frac{b_n}{[n]_{q_n}}\right) \frac{MA}{2} \sum_{k=2}^{\infty} (k+1)(rA)^k \\
= & \left(1 - q_n + \frac{b_n}{[n]_{q_n}}\right) C_{r,A},
\end{aligned}$$

where

$$C_{r,A} = \frac{MA}{2} \sum_{k=2}^{\infty} (k+1)(rA)^{k-1}$$

is finite for all $1 \leq r < \frac{1}{A}$. Note that the series $\sum_{k=2}^{\infty} u^{k+1}$ and its derivative $\sum_{k=2}^{\infty} (k+1) u^k$ are uniformly and absolutely convergent in any compact disk included in the open unit disk.

(**ii**) Let γ be the circle of radius $r_1 > r$ with centered 0, since for any $|z| \leq r$ and $v \in \gamma$, we have $|v - z| \geq r_1 - r$, by Cauchy's formulas it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$

$$\begin{aligned}
\left|D_{q_n}^{(p)}\left(S_n^{q_n}(f;z)\right) - D_{q_n}^{(p)} f(z)\right| & \leq \left|S_n^{q_n^{(p)}}(f;z) - f^{(p)}(z)\right| \\
& = \frac{p!}{2\pi} \left| \int_{\gamma} \frac{S_n^{q_n}(f;v) - f(v)}{(v-z)^{p+1}} dv \right| \\
& \leq \frac{C_{r_1,A} b_n}{[n]_{q_n}} \frac{p!}{2\pi} \frac{2\pi r_1}{(r_1 - r)^{p+1}} \\
& = \frac{C_{r_1,A} b_n}{[n]_{q_n}} \frac{p! r_1}{(r_1 - r)^{p+1}},
\end{aligned}$$

which gives (ii). The proof is completed. □

Note that in case of $q_n = 1$, the similar result for complex Favard-Szász-Mirakjan operators has been obtained by Gal in ([3, Theorem 2.1]).

In what follows, we give a Voronovskaja-type result for the complex $q-$Szász-Mirakjan operators. A similar result for the real $q-$Szász-Mirakjan operators has been given in [1].

Theorem 2.4. *Under the conditions of Theorem 2.3, suppose that* $1 \leq q_n r < \frac{1}{A}$ *be arbitrary fixed. Then the following Voronovskaja-type result holds.*

$$\begin{aligned} & \left| S_n^{q_n}(f;z) - f(z) - \frac{z b_n}{2\,[n]_{q_n}} D_{q_n}^2(f(z)) \right| \\ \leq & \left\{ \frac{r b_n}{[n]_{q_n}} \left(1 - q_n + \frac{b_n}{[n]_{q_n}} + 2 \right) + 2\frac{b_n^2}{[n]_{q_n}^2} \right\} \frac{M A^2 |z|}{r} \sum_{k=2}^{\infty} (k+1)(q_n A r)^{k-2}, \end{aligned}$$

for all $|z| \leq r$ *and* n *is large enough.*

Proof. Set $e_k(z) = z^k,\ k = 0, 1, ...$ and $T_{n,k}(z) = S_n^{q_n}(e_k; z)$, by the proof of Theorem 2.3 (i), we can write $S_n^{q_n}(f;z) = \sum_{k=0}^{\infty} c_k T_{n,k}(z)$, and obtain that

$$\begin{aligned} & \left| S_n^{q_n}(f;z) - f(z) - \frac{z b_n}{2\,[n]_{q_n}} D_q^2(f(z)) \right| \\ \leq & \sum_{k=0}^{\infty} |c_k| \left| T_{n,k}(z) - e_k(z) - \frac{z^{k-1}\,[k]_{q_n}\,[k-1]_{q_n}\,b_n}{2\,[n]_{q_n}} \right|, \end{aligned}$$

for all $z \in D_R$. By the recurrence relationship in the proof of Theorem 2.3 (i), satisfied by $T_{n,k}(z)$, denoting

$$E_{k,n}(z) = T_{n,k}(z) - e_k(z) - \frac{z^{k-1}\,[k]_{q_n}\,[k-1]_{q_n}\,b_n}{2\,[n]_{q_n}}, \tag{2.5}$$

we get that

$$E_{k-1,n}(z) = T_{n,k-1}(z) - e_{k-1}(z) - \frac{z^{k-2}\,[k-2]_{q_n}\,[k-1]_{q_n}\,b_n}{2\,[n]_{q_n}}$$

for all $k \geq 2,\ z \in D_R$. Using (2.5), we obtain the following recurrence for all $k \geq 2$ and $z \in D_R$. If we take the $q-$derivative of $E_{k-1,n}(z)$, we have

$$
\begin{aligned}
& D_q(E_{k-1,n}(z)) \\
= \; & D_q(T_{n,k-1}(z)) - [k-1]_{q_n} z^{k-2} - \frac{z^{k-3}[k-2]_{q_n}^2 [k-1]_{q_n} b_n}{2[n]_{q_n}} \\
= \; & \frac{[n]_{q_n}}{zb_n} T_{n,k}(z) - \frac{[n]_{q_n}}{b_n} T_{n,k-1}(q_n z) - [k-1]_{q_n} z^{k-2} - \frac{z^{k-3}[k-2]_{q_n}^2 [k-1]_{q_n} b_n}{2[n]_{q_n}} \\
= \; & \frac{[n]_{q_n}}{zb_n} \Bigg\{ T_{n,k}(z) - zT_{n,k-1}(q_n z) - \frac{z^{k-1}[k-1]_{q_n} b_n}{[n]_{q_n}} + \frac{z^{k-1}[k]_{q_n}[k-1]_{q_n} b_n}{2[n]_{q_n}} \\
& - \frac{z^{k-2}[k-2]_{q_n}^2 [k-1]_{q_n} b_n^2}{2[n]_{q_n}^2} - \frac{z^{k-1}[k]_{q_n}[k-1]_{q_n} b_n}{2[n]_{q_n}} + q_n^{k-1} z^k - q_n^{k-1} z^k + \\
& + z^k - z^k + \frac{q_n^{k-2} z^{k-1}[k-2]_{q_n}[k-1]_{q_n} b_n}{2[n]_{q_n}} \\
& - \frac{q_n^{k-2} z^{k-1}[k-2]_{q_n}[k-1]_{q_n} b_n}{2[n]_{q_n}} \Bigg\} \\
= \; & \frac{[n]_{q_n}}{zb_n} \Bigg\{ T_{n,k}(z) - e_k(z) - \frac{z^{k-1}[k]_{q_n}[k-1]_{q_n} b_n}{2[n]_{q_n}} - zT_{n,k-1}(q_n z) \\
& + \frac{q_n^{k-1} z^{k-2}[k-1]_{q_n}[k]_{q_n} b_n}{2[n]_{q_n}} + q_n^{k-1} z^{k-2} + \frac{z^{k-1}[k]_{q_n}[k-1]_{q_n} b_n}{[n]_{q_n}} - q_n^{k-1} z^{k-2} \\
& - \frac{q_n^{k-1} z^{k-2}[k-1]_{q_n}[k]_{q_n} b_n}{2[n]_{q_n}} - \frac{z^{k-1}[k-1]_{q_n} b_n}{[n]_{q_n}} \\
& - \frac{z^{k-2}[k-2]_{q_n}^2 [k-1]_{q_n} b_n^2}{2[n]_{q_n}^2} \Bigg\}
\end{aligned}
$$

If we make necessary arrangements, we reach to

$$
\begin{aligned}
& D_q(E_{k-1,n}(z)) \\
= \ & \frac{[n]_{q_n}}{zb_n}\{E_{k,n}(z)+\frac{z^{k-1}[k]_{q_n}[k-1]_{q_n}b_n}{2[n]_{q_n}}-q_n^{k-1}z^k \\
& -z\left(T_{n,k-1}(q_nz)-(q_nz)^{k-1}-\frac{(q_nz)^{k-2}[k-2]_{q_n}[k-1]_{q_n}b_n}{2[n]_{q_n}}\right) \\
& -\frac{q_n^{k-2}z^{k-1}[k-2]_{q_n}[k-1]_{q_n}b_n}{2[n]_{q_n}} \\
& -\frac{z^{k-1}[k-1]_{q_n}b_n}{2[n]_{q_n}}-\frac{z^{k-2}[k-2]_{q_n}^2[k-1]_{q_n}b_n^2}{2[n]_{q_n}^2}\Bigg\} \\
= \ & \frac{[n]_{q_n}}{zb_n}\left\{E_{k,n}(z)-zE_{k-1,n}(q_nz)+z^k\left(1-q_n^{k-1}\right)+\frac{2z^{k-1}[k-1]_{q_n}b_n}{2[n]_{q_n}}\right. \\
& \left.+\frac{z^{k-1}[k-1]_{q_n}b_n}{2[n]_{q_n}}\left([k]_{q_n}-q_n^{k-2}[k-2]_{q_n}\right)-\frac{z^{k-2}[k-2]_{q_n}^2[k-1]_{q_n}b_n^2}{2[n]_{q_n}^2}\right\} \\
= \ & \frac{[n]_{q_n}}{zb_n}\left\{E_{k,n}(z)-zE_{k-1,n}(q_nz)+z^k(1-q_n)[k-1]_{q_n}+\right. \\
& \left.+\frac{z^{k-1}[k-1]_{q_n}b_n}{2[n]_{q_n}}\left([k]_{q_n}-q_n^{k-2}[k-2]_{q_n}-2\right)-\frac{z^{k-2}[k-2]_{q_n}^2[k-1]_{q_n}b_n^2}{2[n]_{q_n}^2}\right\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{zb_n}{[n]_{q_n}}D_q(E_{k-1,n}(z)) &= E_{k,n}(z)-zE_{k-1,n}(q_nz)+z^k(1-q_n)[k-1]_{q_n} \\
&\quad -\frac{z^{k-2}[k-2]_{q_n}^2[k-1]_{q_n}b_n^2}{2[n]_{q_n}^2} \\
&\quad +\frac{z^{k-1}[k-1]_{q_n}b_n}{2[n]_{q_n}}\left([k]_{q_n}-2-q_n^{k-2}[k-2]_{q_n}\right).
\end{aligned}
$$

From the last equality, we can write

$$\begin{aligned} E_{k,n}(z) &= \frac{zb_n}{[n]_{q_n}} D_q(E_{k-1,n}(z)) + zE_{k-1,n}(q_n z) - z^k (1-q_n)[k-1]_{q_n} \\ &+ \frac{z^{k-2}[k-2]_{q_n}^2 [k-1]_{q_n} b_n^2}{2[n]_{q_n}^2} \\ &- \frac{z^{k-1}[k-1]_{q_n} b_n}{2[n]_{q_n}} \left([k]_{q_n} - 2 - q_n^{k-2}[k-2]_{q_n}\right). \end{aligned}$$

By passing to modulus, it follows that

$$\begin{aligned} |E_{k,n}(z)| &\leq \frac{|z| b_n}{2[n]_{q_n}} 2 \|D_q(E_{k-1,n}(z))\|_r \\ &+ |z| |E_{k-1,n}(q_n z)| + \frac{|z| b_n}{2[n]_{q_n}} |z|^{k-2} \frac{[k-2]_{q_n}^2 [k-1]_{q_n} b_n}{2[n]_{q_n}} \\ &+ |z|^k |1-q_n| [k-1]_{q_n} \\ &+ \frac{|z| b_n}{2[n]_{q_n}} |z|^{k-2} [k-1]_{q_n} \left([k-2]_{q_n} q_n^{k-2} - ([k]_{q_n} - 2)\right) \\ &\leq r |E_{k-1,n}(q_n z)| + \frac{|z| b_n}{2[n]_{q_n}} \left\{ 2\left(\frac{k-1}{r}\right) \|E_{k-1,n})\|_r \right. \\ &+ r^{k-2}[k-1]_{q_n} \left| [k-2]_{q_n} q_n^{k-2} - ([k]_{q_n} - 2) \right| \\ &\left. + \frac{r^{k-3}[k-2]_{q_n}^2 [k-1]_{q_n} b_n}{2[n]_{q_n}} + r^k (1-q_n)[k-1]_{q_n} \right\} \end{aligned}$$

for all $k \geq 2$ and for all $|z| \leq r$. This gives that

$$\begin{aligned}
|E_{k,n}(z)| &\leq r\,|E_{k-1,n}(q_n z)| + \frac{|z|\,b_n}{2\,[n]_{q_n}}\left\{2\tfrac{k-1}{r}\,\|T_{k-1,n}(z)-e_{k-1}\|_r\right.\\
&\quad +2\tfrac{k-1}{r}\tfrac{r^{k-2}[k-2]_{q_n}[k-1]_{q_n}b_n}{2[n]_{q_n}} + r^{k-3}\tfrac{[k-2]_{q_n}^2[k-1]_{q_n}b_n}{[n]_{q_n}}\\
&\quad \left. + r^{k-2}\,[k-1]_{q_n}\left(\left|[k-2]_{q_n}\,q_n^{k-2}\right| + \left|[k]_{q_n}-2)\right|\right)\right\}\\
&\leq r\,|E_{k-1,n}(q_n z)| + \tfrac{|z|b_n}{2[n]_{q_n}}\left\{2\tfrac{k-1}{r}\left(\frac{k!r^{k-1}}{2}\left(1-q_n+\frac{b_n}{[n]_{q_n}}\right)\right)\right.\\
&\quad +\tfrac{(k-1)r^{k-3}[k-2]_{q_n}[k-1]_{q_n}b_n}{2[n]_{q_n}} + r^{k-3}\tfrac{[k-2]_{q_n}^2[k-1]_{q_n}b_n}{[n]_{q_n}}\\
&\quad \left.+r^{k-2}\,[k-1]_{q_n}\,2\,[k-2]_{q_n}\right\}\\
&\leq r\,|E_{k-1,n}(q_n z)| + \tfrac{|z|b_n}{2[n]_{q_n}}\left\{(k+1)!2r^{k-2}\left(1-q_n+\frac{b_n}{[n]_{q_n}}\right)\right.\\
&\quad +\frac{(k-1)r^{k-3}\,[k-2]_{q_n}\,[k-1]_{q_n}\,b_n}{2\,[n]_{q_n}} + r^{k-3}\frac{[k-2]_{q_n}^2\,[k-1]_{q_n}\,b_n}{[n]_{q_n}}\\
&\quad \left.+\,2r^{k-2}\,[k-1]_{q_n}\,[k-2]_{q_n}\right\}\\
&\leq r\,|E_{k-1,n}(q_n z)| + \frac{|z|\,b_n}{2\,[n]_{q_n}}\left\{(k+1)!2r^{k-2}\left(1-q_n+\frac{b_n}{[n]_{q_n}}\right)\right\}\\
&\quad +\frac{|z|\,b_n}{2\,[n]_{q_n}}\frac{2r^{k-3(k+1)!}b_n}{[n]_{q_n}} + \tfrac{|z|b_n}{2[n]_{q_n}}(k+1)!2r^{k-2}
\end{aligned}$$

$$= r\left|E_{k-1,n}(q_nz)\right| + \tfrac{|z|b_n}{2[n]_{q_n}} r^{k-2}(k+1)!\left\{1-q_n+\frac{b_n}{[n]_{q_n}}+2\right\}$$
$$+\tfrac{|z|b_n^2}{[n]_{q_n}^2} r^{k-3}(k+1)!$$
$$= r\left|E_{k-1,n}(q_nz)\right|$$
$$+\frac{|z|\,(k+1)!r^{k-3}}{2}\left(\frac{rb_n}{[n]_{q_n}}\left(1-q_n+\frac{b_n}{[n]_{q_n}}+2\right)+2\tfrac{b_n^2}{[n]_{q_n}^2}\right)$$

for all $|z| \leq r$.

Taking $k = 2, 3, ...$ step by step, we find

$$\left|E_{2,n}(z)\right| \leq r\left|E_{1,n}(q_nz)\right| + \frac{|z|\,r^{-1}3!}{2}\left\{\frac{rb_n}{[n]_{q_n}}\left(1-q_n+\frac{b_n}{[n]_{q_n}}+2\right)+2\tfrac{b_n^2}{[n]_{q_n}^2}\right\},$$
$$\left|E_{2,n}(q_nz)\right| \leq \frac{q_n\,|z|\,r^{-1}3!}{2}\left\{\frac{rb_n}{[n]_{q_n}}\left(1-q_n+\frac{b_n}{[n]_{q_n}}+2\right)+2\tfrac{b_n^2}{[n]_{q_n}^2}\right\},$$
$$\ldots$$
$$\left|E_{k,n}(z)\right| \leq \frac{q_n^{k-2}\,|z|\,r^{k-3}}{2}\left\{\frac{rb_n}{[n]_{q_n}}\left(1-q_n+\frac{b_n}{[n]_{q_n}}+2\right)+2\tfrac{b_n^2}{[n]_{q_n}^2}\right\}\sum_{j=3}^{k+1} j!$$
$$= q_n^{k-2}\,|z|\,r^{k-3}(k+1)!\left\{\frac{rb_n}{[n]_{q_n}}\left(1-q_n+\frac{b_n}{[n]_{q_n}}+2\right)+2\tfrac{b_n^2}{[n]_{q_n}^2}\right\}$$

for $k \geq 2$. The last inequality gives that

$$\left|S_n^{q_n}(f;z)-f(z)-\frac{zb_n}{2\,[n]_{q_n}}D_q^2(f(z))\right|$$
$$\leq \sum_{k=0}^{\infty}|c_k|\left|E_{k,n}(z)\right|$$
$$\leq \left\{\frac{rb_n}{[n]_{q_n}}\left(1-q_n+\frac{b_n}{[n]_{q_n}}+2\right)+2\tfrac{b_n^2}{[n]_{q_n}^2}\right\}\sum_{k=2}^{\infty}\frac{q_n^{k-2}MA^k\,|z|\,r^{k-3}\,(k+1)!}{k!}$$
$$\leq \left\{\frac{rb_n}{[n]_{q_n}}\left(1-q_n+\frac{b_n}{[n]_{q_n}}+2\right)+2\tfrac{b_n^2}{[n]_{q_n}^2}\right\}\frac{MA^2\,|z|}{r}\sum_{k=2}^{\infty}(k+1)(q_nAr)^{k-2};$$

for all $|z| \leq r$, where $q_nrA < 1$ we have $\sum_{k=2}^{\infty}(k+1)(q_nrA)^{k-2} < \infty$, which completes the proof. □

Özet: Bu çalışmada, uygun üstel tipten büyüme koşulunu sağlayan analitik fonksiyonlar için kompleks $q-$Szász-Mirakjan operatörleri çalışılmıştır. Bu operatörler için kompakt disklerde bir Voronovskaja-tipi teorem verilmiştir. Ayrıca, burada elde edilen sonuçlar [8] nolu referanstaki farklı tipten kompleks $q-$Szász-Mirakjan operatörleri için verilen sonuçlardan farklıdır.

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Current address: Didem AYDIN :Ankara University, Faculty of Sciences,, Dept. of Mathematics, Ankara, TURKEY

E-mail address: `daydin@ankara.edu.tr`

URL: `http://communications.science.ankara.edu.tr/index.php?series=A1`