Available online: July 29, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 2, Pages 2198–2208 (2019)
DOI: 10.31801/cfsuasmas.598146
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

C^*-ALGEBRA-VALUED RECTANGULAR b-METRIC SPACES AND SOME FIXED POINT THEOREMS

G. KALPANA AND Z. SUMAIYA TASNEEM

ABSTRACT. The concept of C^*-algebra-valued rectangular b-metric spaces is introduced as a generalization of C^*-algebra-valued b-metric spaces. An analogue of Banach contraction principle and Kannan's fixed point theorem is proved in this space. As applications, existence and uniqueness results for a type of operator equation is given.

1. INTRODUCTION

It is well known that the Banach contraction mapping principle is a very useful, simple and classical tool in modern analysis and it has many applications in applied mathematics. In particular, it is an important tool for solving existence problems in many branches of mathematics and physics.

In 1989, Backhtin introduced the concept of b-metric spaces. Later, Czerwik [4] extended the results of b-metric spaces. Using this idea many researcher presented generalization of the renowned Banach fixed point theorem in the b-metric spaces. In 2000, Branciari [3] introduced the notion of a rectangular metric spaces where the triangle inequality of a metric spaces was replaced by another inequality, the so-called rectangular inequality. In [6], George. et. al established the concept of rectangular b-metric space which generalizes the concept of metric space, rectangular metric space and b-metric space.

Now, we recall some necessary concepts and results in C^*-algebra. Suppose that A is an unital algebra with the unit I. An involution on A is a conjugate-linear map $a \mapsto a^*$ on A such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. The pair $(A, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra A together with a complete sub multiplicative norm such that $\|a^*\| = \|a\| (\forall a \in A)$. A C^*-algebra

Received by the editors: February 05, 2018; Accepted: July 19, 2018.

2010 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. C^*-algebra; C^*-algebra-valued rectangular b-metric spaces, contractive mapping, fixed point theorem.

Submitted via International Conference on Current Scenario in Pure and Applied Mathematics [ICCSPAM 2018].

is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ [10]. Notice that the seeming mild requirement on a C^*-algebra above is in fact very strong. It is clear that under the norm topology, $L(H)$, the set of all bounded linear operators on a Hilbert space H is a C^*-algebra.

In [8], Ma established the notion of C^*-algebra-valued metric spaces and proved some fixed point theorems for self maps with contractive or expansive mappings. The main idea consists in using the set of all positive elements of a unital C^*-algebra instead of the set of real numbers. Further in [9], Ma introduced a concept of C^*-algebra-valued b-metric spaces which generalizes the concept of C^*-algebra valued metric spaces.

Batul and Kamran [2] generalized the notion of C^*-valued contraction mappings and established a fixed point theorem for such mappings. In [11], Caristi's fixed point theorem was given for C^*-algebra-valued metric spaces. Kamran et al. [7] proved the Banach contraction principle in C^*-algebra-valued b-metric spaces with application. Bai [1] presented coupled fixed point theorems in C^*-algebra-valued b-metric spaces. In [5], Meltem and Alaca presented the concept of C^*-algebra-valued S-metric spaces and proved Banach contraction principle in such spaces.

In this paper, we introduce the new type of metric spaces namely C^*-algebra-valued rectangular b-metric spaces and we give some fixed point theorems for self maps with contractive conditions. As applications, existence and uniqueness results for a type of operator equation is given.

2. Preliminaries

Throughout this paper, A denotes an unital C^*-algebra. Set $A_h = \{a \in A : a = a^*\}$. We call an element $a \in A$ a positive element, denote it by $\theta \preceq a$, if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$, where θ is a zero element in A and $\sigma(a)$ is the spectrum of a. There is a natural partial ordering on A_h given by $a \preceq b$ if and only if $\theta \preceq b-a$. From now on, A_+ and $A^{'}$ will denote the set $\{a \in A : \theta \preceq a\}$ and the set $\{a \in A : ab = ba, \ \forall b \in A\}$ and $|x| = (x^*x)^{\frac{1}{2}}$ respectively.

Lemma 2.1. [10] *Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I.*

(1) *For any $x \in \mathbb{A}_+$ we have $x \preceq I \Leftrightarrow \|x\| \leq 1$.*
(2) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(1-a)^{-1}\| < 1$.*
(3) *Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$.*
(4) *Let $a \in \mathbb{A}^{'}$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}_+^{'}$ is an invertible operator, then $(I-a)^{-1}b \succeq (I-a)^{-1}c$.*

Definition 2.2. [8] *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:*

(1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ iff $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,y) \preceq [d(x,z)+d(z,y)]$ for all $x,y,z \in X$.
Then d is called a C^*-algebra-valued metric on X and $(X,\mathbb{A},d)$ is called a C^*-algebra-valued metric space.

It is obvious that C^*-algebra-valued metric spaces generalize the concept of metric spaces, replacing the set of real numbers by A_+.

Definition 2.3. [9] *Let X be a nonempty set and $A \in \mathbb{A}'$ such that $A \succeq I$. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:*
(1) $\theta \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y)=\theta$ iff $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y \in X$;
(3) $d(x,y) \preceq A[d(x,z)+d(z,y)]$ for all $x,y,z \in X$.
Then d is called a C^*-algebra-valued b-metric on X and $(X,\mathbb{A},d)$ is called a C^*-algebra-valued b-metric space.

We now introduce the definition of a C^*-algebra-valued rectangular b-metric spaces and give an example to show that this concept is more general than that of C^*-algebra-valued b-metric spaces and C^*-algebra-valued rectangular metric spaces.

Definition 2.4. *Let X be a nonempty set and $A \in \mathbb{A}'$ such that $A \succeq I$. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:*
(1) $\theta \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y)=\theta$ iff $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y \in X$;
(3) $d(x,y) \preceq A[d(x,u)+d(u,v)+d(v,y)]$ for all $x,y,u,v \in X$ and for all distinct points $u,v \in X-\{x,y\}$.

Then d is called a C^*-algebra-valued rectangular b-metric on X and $(X,\mathbb{A},d)$ is called a C^*-algebra-valued rectangular b-metric space.

Remark 2.5. *If $A=I$, then the ordinary rectangular inequality condition in a C^*-algebra-valued rectangular metric space is satisfied. Thus a C^*-algebra-valued rectangular b-metric space is an ordinary C^*-algebra-valued rectangular metric space. The following example illustrates that, in general, a C^*-algebra-valued rectangular b-metric space is not necessarily a C^*-algebra-valued rectangular metric space.*

Example 2.6. *Let $X=A\cup B$ where $A=\{\frac{1}{n} : n \in N\}$ and B is the set of all positive integers. Let $\mathbb{A}=M_2(\mathbb{R})$ of all 2×2 matrices with the usual addition, scalar multiplication and multiplication. Define partial ordering on $\mathbb{A}$ as*

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$
$$\Leftrightarrow a_i \geq b_i \text{ for } i=1,2,3,4.$$

For any $A \in \mathbb{A}$ we define its norm as, $\|A\| = \max_{1 \leq \leq i \leq 4} |a_i|$. Define $d : X \times X \to \mathbb{A}$ such that $d(x,y) = d(y,x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{if } x = y \\ \begin{pmatrix} 2\alpha & 0 \\ 0 & 2\alpha \end{pmatrix}, \text{if } x, y \in A \\ \begin{pmatrix} \frac{\alpha}{2n} & 0 \\ 0 & \frac{\alpha}{2n} \end{pmatrix}, \text{if } x \in A, y \in \{2,3\} \\ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, Otherwise \end{cases}$$

where $\alpha > 0$ is a constant. Then $(X, \mathbb{A}, d)$ is a C^*-algebra-valued rectangular b-metric space with coefficient $A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and $\|A\| = 2 > 1$.

(1) *$(X, \mathbb{A}, d)$ is not a C^*-algebra-valued rectangular metric space. Since*

$$d(\frac{1}{2}, \frac{1}{3}) = \begin{pmatrix} 2\alpha & 0 \\ 0 & 2\alpha \end{pmatrix} \succ \begin{pmatrix} \frac{17\alpha}{12} & 0 \\ 0 & \frac{17\alpha}{12} \end{pmatrix}$$
$$= d(\frac{1}{2}, 2) + d(2,3) + d(3, \frac{1}{3})$$

(2) *There does not exist $A \succ I$ satisfying $d(x,y) \preceq A[d(x,z) + d(z,y)]$ for all $x, y, z \in X$. So $(X, \mathbb{A}, d)$ is not a C^*-algebra-valued b-metric space.*

Definition 2.7. *Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued rectangular b-metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If $d(x_n, x) \to \theta (n \to \infty)$, then it is said that $\{x_n\}$ converges to x with respect to $\mathbb{A}$, and we denote it by $\lim_{n \to \infty} x_n = x$.*

If for any $p \in \mathbb{N}$, $d(x_{n+p}, x_n) \to \theta (n \to \infty)$, then $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$.

We say $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued rectangular b-metric space if every Cauchy sequence is convergent in X with respect to $\mathbb{A}$.

Definition 2.8. *Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued rectangular b-metric spaces. We call a mapping $T : X \to X$ a C^*-algebra-valued contractive mapping on X, if there exists an element $B \in \mathbb{A}$ with $\|B\| < 1$ such that*

$$d(Tx, Ty) \preceq B^* d(x,y) B, \quad \forall x, y \in X.$$

3. Main Results

Theorem 3.1. *If $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued rectangular b-metric spaces and $T : X \to X$ is a contractive mapping, then there exists a unique fixed point in X.*

Proof. It is clear that if $B = \theta$, then T maps the X into a single point. Thus without loss of generality, one can suppose that $B \neq \theta$.

Choose an $x_0 \in X$ and set $x_{n+1} = Tx_n = \ldots = T^{n+1}x_0, n = 1, 2, \ldots$ For convenience, by B_0 we denote the element $d(x_1, x_0)$ in $\mathbb{A}$. Notice that in C^*-algebra, if $a, b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in A$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$ [10].

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \preceq B^* d(x_n, x_{n-1}) B \\ &\preceq (B^*)^2 d(x_{n-1}, x_{n-2}) B^2 \\ &\preceq \cdots \\ &\preceq (B^*)^n d(x_1, x_0) B^n \\ &= (B^n)^* B_0 B^n. \end{aligned} \tag{1}$$

For $m \geq 1$ and $p \geq 1$, it follows that

$$\begin{aligned} d(x_{m+p}, x_m) &\preceq A[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m)] \\ &\preceq A\, d(x_{m+p}, x_{m+p-1}) + A\, d(x_{m+p-1}, x_{m+p-2}) \\ &\quad + A\,[A\,[d(x_{m+p-2}, x_{m+p-3}) + d(x_{m+p-3}, x_{m+p-4}) + d(x_{m+p-4}, x_m)]] \\ &= A\, d(x_{m+p}, x_{m+p-1}) + A\, d(x_{m+p-1}, x_{m+p-2}) \\ &\quad + A^2\, d(x_{m+p-2}, x_{m+p-3}) + A^2\, d(x_{m+p-3}, x_{m+p-4}) + A^2\, d(x_{m+p-4}, x_m) \\ &\preceq A\, d(x_{m+p}, x_{m+p-1}) + A\, d(x_{m+p-1}, x_{m+p-2}) \\ &\quad + A^2\, d(x_{m+p-2}, x_{m+p-3}) + A^2\, d(x_{m+p-3}, x_{m+p-4}) + \cdots \\ &\quad + A^{\frac{p-1}{2}}\, d(x_{m+3}, x_{m+2}) + A^{\frac{p-1}{2}}\, d(x_{m+2}, x_{m+1}) + A^{\frac{p-1}{2}}\, d(x_{m+1}, x_m) \\ &\preceq A[(B^{m+p-1})^* B_0 (B^{m+p-1}) + (B^{m+p-2})^* B_0 (B^{m+p-2})] \\ &\qquad + A^2[(B^{m+p-3})^* B_0 (B^{m+p-3}) + (B^{m+p-4})^* B_0 (B^{m+p-4})] + \cdots \\ &\qquad + A^{\frac{p-1}{2}}[(B^{m+2})^* B_0 (B^{m+2}) + (B^{m+1})^* B_0 (B^{m+1})] + A^{\frac{p-1}{2}} (B^m)^* B_0 B^m \\ &= A[(B^*)^{m+p-1} B_0 (B^{m+p-1}) + (B^*)^{m+p-2} B_0 (B^{m+p-2})] \\ &\qquad + A^2[(B^*)^{m+p-3} B_0 (B^{m+p-3}) + (B^*)^{m+p-4} B_0 (B^{m+p-4})] + \cdots \\ &\qquad + A^{\frac{p-1}{2}}[(B^*)^{m+2} B_0 (B^{m+2}) + (B^*)^{m+1} B_0 (B^{m+1})] + A^{\frac{p-1}{2}} (B^*)^m B_0 B^m \end{aligned}$$

$$
\begin{aligned}
&= A(B^*)^{m+p-1}B_0(B^{m+p-1}) + A^2(B^*)^{m+p-3}B_0(B^{m+p-3}) + \cdots \\
&\quad + A^{\frac{p-1}{2}}(B^*)^{m+2}B_0(B^{m+2}) \\
&\quad + A(B^*)^{m+p-2}B_0(B^{m+p-2}) + A^2(B^*)^{m+p-4}B_0(B^{m+p-4}) + \cdots \\
&\quad + A^{\frac{p-1}{2}}(B^*)^{m+1}B_0(B^{m+1}) + A^{\frac{p-1}{2}}(B^*)^{m}B_0B^{m} \\
&= \sum_{k=1}^{\frac{p-1}{2}} A^k(B^*)^{m+p-(2k-1)}B_0B^{m+p-(2k-1)} + \sum_{k=1}^{\frac{p-1}{2}} A^k(B^*)^{m+p-2k}B_0B^{m+p-2k} \\
&\quad + A^{\frac{p-1}{2}}(B^*)^{m}B_0B^{m} \\
&= \sum_{k=1}^{\frac{p-1}{2}} \Big(B_0^{\frac{1}{2}}A^{\frac{k}{2}}B^{m+p-(2k-1)}\Big)^*\Big(B_0^{\frac{1}{2}}A^{\frac{k}{2}}B^{m+p-(2k-1)}\Big) + \\
&\quad \sum_{k=1}^{\frac{p-1}{2}} \Big(B_0^{\frac{1}{2}}A^{\frac{k}{2}}B^{m+p-2k}\Big)^*\Big(B_0^{\frac{1}{2}}A^{\frac{k}{2}}B^{m+p-2k}\Big) + \Big(B_0^{\frac{1}{2}}A^{\frac{p-1}{4}}B^{m}\Big)^*\Big(B_0^{\frac{1}{2}}A^{\frac{p-1}{4}}B^{m}\Big) \\
&= \sum_{k=1}^{\frac{p-1}{2}} |B_0^{\frac{1}{2}}A^{\frac{k}{2}}B^{m+p-(2k-1)}|^2 + \sum_{k=1}^{\frac{p-1}{2}} |B_0^{\frac{1}{2}}A^{\frac{k}{2}}B^{m+p-2k}|^2 \\
&\quad + |B_0^{\frac{1}{2}}A^{\frac{p-1}{4}}B^{m}|^2 \\
&\preceq \sum_{k=1}^{\frac{p-1}{2}} \|B_0^{\frac{1}{2}}A^{\frac{k}{2}}B^{m+p-(2k-1)}\|^2 I + \sum_{k=1}^{\frac{p-1}{2}} \|B_0^{\frac{1}{2}}A^{\frac{k}{2}}B^{m+p-2k}\|^2 I \\
&\quad + \|B_0^{\frac{1}{2}}A^{\frac{p-1}{4}}B^{m}\|^2 I \\
&\preceq \|B_0\| \sum_{k=1}^{\frac{p-1}{2}} \|A\|^{\frac{k}{2}}\|B\|^{2(m+p-(2k-1))} I + \|B_0\| \sum_{k=1}^{\frac{p-1}{2}} \|A\|^{\frac{k}{2}}\|B\|^{2(m+p-2k)} I \\
&\quad + \|B_0\|\|A\|^{\frac{p-1}{2}}\|B\|^{2m} I \\
&= \|B_0\|\|A\|\|B\|^{2(m+p-1)}\Big[\frac{\|A\|^{\frac{p-1}{2}}\|B\|^{2(-p+1)}-1}{\|A\|\|B\|^{-4}-1}\Big]I + \\
&\quad \|B_0\|\|A\|\|B\|^{2(m+p-2)}\Big[\frac{\|A\|^{\frac{p-1}{2}}\|B\|^{2(-p+1)}-1}{\|A\|\|B\|^{-4}-1}\Big]I + \|B_0\|\|A\|^{\frac{p-1}{2}}\|B\|^{2m} I \\
&\preceq \frac{\|B_0\|\|A\|^{\frac{p+1}{2}}\|B\|^{2(m+2)}}{\|A\|-\|B\|^4} + \frac{\|B_0\|\|A\|^{\frac{p+1}{2}}\|B\|^{2(m+1)}}{\|A\|-\|B\|^4} + \|B_0\|\|A\|^{\frac{p-1}{2}}\|B\|^{2m} I \\
&\to \theta \ (m \to \infty).
\end{aligned}
$$

Therefore $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$ there exists an $x \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x.$$

$$\begin{aligned}\theta \preceq d(Tx,x) &\preceq A[d(Tx,Tx_n) + d(Tx_n,Tx_{n+1}) + d(Tx_{n+1},x)]\\ &\preceq A[B^*\, d(x,x_n)\, B + B^*\, d(x_n,x_{n+1})B + d(x_{n+2},x)]\\ &\to \theta \ \text{ as n} \to \infty.\end{aligned}$$

Therefore $d(Tx,x) = 0$ which implies $Tx = x$, i.e. x is a fixed point of T. Now suppose that $y(\neq x)$ is another fixed point of T. Since $\theta \preceq d(x,y) = d(Tx,Ty) \preceq B^*d(x,y)B$, we have

$$\begin{aligned}0 \leq \|d(x,y)\| &= \|d(Tx,Ty)\|\\ &\leq \|B^*d(x,y)B\|\\ &\leq \|B^*\|\|d(x,y)\|\|B\|\\ &= \|B\|^2\|d(x,y)\|\\ &< \|d(x,y)\|, \ \text{ which is a contradiction.}\end{aligned}$$

Hence $d(x,y) = \theta$ and $x = y$, which implies that the fixed point is unique. □

Theorem 3.2. *(Kannan-type) Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued rectangular b-metric spaces. Suppose the mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \preceq B(d(Tx,x) + d(Ty,y))(\forall x,y \in X)$$

where $B \in \mathbb{A}'_+$ and $\|B\| < \frac{1}{2}$. Then there exists a unique fixed point in X.

Proof. Without loss of generality, one can suppose that $B \neq \theta$. Notice that $B \in \mathbb{A}'_+$, $B(d(Tx,x) + d(Ty,y))$ is also a positive element.

Choose an $x_0 \in X$ and set $x_{n+1} = Tx_n = \ldots = T^{n+1}x_0, n = 1,2,\ldots$, by B_0 we denote the element $d(x_1,x_0)$ in $\mathbb{A}$. Then

$$\begin{aligned}d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1})\\ &\preceq B(d(Tx_n,x_n) + d(Tx_{n-1},x_{n-1}))\\ &= B(d(x_{n+1},x_n) + d(x_n,x_{n-1}))\end{aligned}$$

Since $B \in \mathbb{A}'_+$ with $\|B\| < \frac{1}{2}$, using Lemma 2.1, $I - B$ is invertible. Thus

$$d(x_{n+1},x_n) \preceq \ (I - B)^{-1}Bd(x_n,x_{n-1}) = t\, d(x_n,x_{n-1})$$

where $t = (I - B)^{-1}B$. Therefore

$$\begin{aligned}d(x_{n+1},x_n) &\preceq td(x_n,x_{n-1})\\ &\preceq t^2d(x_{n-1},x_{n-2})\\ &\ \ \vdots\\ &\preceq t^nd(x_1,x_0) = t^nB_0\end{aligned} \tag{2}$$

For any $m \geq 1, p \geq 1$

$$\begin{aligned}
d(x_{m+p}, x_m) &\preceq A[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m)] \\
&\preceq A\, d(x_{m+p}, x_{m+p-1}) + A\, d(x_{m+p-1}, x_{m+p-2}) \\
&\quad + A\,[A\,[d(x_{m+p-2}, x_{m+p-3}) + d(x_{m+p-3}, x_{m+p-4}) + d(x_{m+p-4}, x_m)]] \\
&= A\, d(x_{m+p}, x_{m+p-1}) + A\, d(x_{m+p-1}, x_{m+p-2}) \\
&\quad + A^2\, d(x_{m+p-2}, x_{m+p-3}) + A^2\, d(x_{m+p-3}, x_{m+p-4}) + A^2\, d(x_{m+p-4}, x_m) \\
&\preceq A\, d(x_{m+p}, x_{m+p-1}) + A\, d(x_{m+p-1}, x_{m+p-2}) \\
&\quad + A^2\, d(x_{m+p-2}, x_{m+p-3}) + A^2\, d(x_{m+p-3}, x_{m+p-4}) + \cdots \\
&\quad + A^{\frac{p-1}{2}}\, d(x_{m+3}, x_{m+2}) + A^{\frac{p-1}{2}}\, d(x_{m+2}, x_{m+1}) + A^{\frac{p-1}{2}}\, d(x_{m+1}, x_m) \\
&\preceq A[t^{m+p-1}\, B_0 +\, t^{m+p-2} B_0] + A^2[t^{m+p-3}\, B_0 + t^{m+p-4} B_0] + \cdots \\
&\quad + A^{\frac{p-1}{2}}[t^{m+2}\, B_0 + t^{m+1}\, B_0] + A^{\frac{p-1}{2}}\, t^m\, B_0 \\
&= A\, t^{m+p-1}\, B_0 + A^2\, t^{m+p-3}\, B_0 + \ldots + A^{\frac{p-1}{2}}\, t^{m+2}\, B_0 + \\
&\quad A\, t^{m+p-2}\, B_0 + A^2\, t^{m+p-4}\, B_0 + \ldots + A^{\frac{p-1}{2}}\, t^{m+1}\, B_0 + \\
&\quad A^{\frac{p-1}{2}}\, t^m\, B_0 \\
&= \sum_{k=1}^{\frac{p-1}{2}} A^k t^{m+p-(2k-1)}\, B_0 + \sum_{k=1}^{\frac{p-1}{2}} A^k\, t^{m+p-2k}\, B_0 + A^{\frac{p-1}{2}}\, t^m\, B_0 \\
&= \sum_{k=1}^{\frac{p-1}{2}} |B_0^{\frac{1}{2}}\, t^{\frac{m+p-(2k-1)}{2}}\, A^{\frac{k}{2}}|^2 + \sum_{k=1}^{\frac{p-1}{2}} |B_0^{\frac{1}{2}}\, t^{\frac{m+p-2k}{2}}\, A^{\frac{k}{2}}|^2 + |B_0^{\frac{1}{2}}\, t^{\frac{m}{2}}\, A^{\frac{p-1}{4}}|^2 \\
&\preceq \sum_{k=1}^{\frac{p-1}{2}} \|B_0^{\frac{1}{2}}\, t^{\frac{m+p-(2k-1)}{2}}\, A^{\frac{k}{2}}\|^2\, I + \sum_{k=1}^{\frac{p-1}{2}} \|B_0^{\frac{1}{2}}\, t^{\frac{m+p-2k}{2}}\, A^{\frac{k}{2}}\|^2\, I + \|B_0^{\frac{1}{2}}\, t^{\frac{m}{2}}\, A^{\frac{p-1}{4}}\|^2 I \\
&\preceq \|B_0\| \sum_{k=1}^{\frac{p-1}{2}} \|A\|^k \|t\|^{m+p-(2k-1)} I + \|B_0\| \sum_{k=1}^{\frac{p-1}{2}} \|A\|^k \|t\|^{m+p-2k} I + \\
&\quad \|B_0\| \|A\|^{\frac{p-1}{2}} \|t\|^m I \\
&= \|B_0\| \|A\| \|t\|^{m+p-1} \Big[\frac{\|A\|^{\frac{p-1}{2}} \|t\|^{-p+1} - 1}{\|A\| \|t\|^{-2} - 1}\Big] I + \\
&\quad \|B_0\| \|A\| \|t\|^{m+p-2} \Big[\frac{\|A\|^{\frac{p-1}{2}} \|t\|^{-p+1} - 1}{\|A\| \|t\|^{-2} - 1}\Big] I + \|B_0\| \|A\|^{\frac{p-1}{2}} \|t\|^m I
\end{aligned}$$

$$
\begin{aligned}
&\preceq \|B_0\|\|A\|\|t\|^{m+p-1}\Big[\frac{\|A\|^{\frac{p-1}{2}}\|t\|^{-p+1}}{\|A\|-\|t\|^2}\Big]\|t\|^2 I+\\
&\quad \|B_0\|\|A\|\|t\|^{m+p-2}\Big[\frac{\|A\|^{\frac{p-1}{2}}\|t\|^{-p+1}}{\|A\|-\|t\|^2}\Big]\|t\|^2 I+\|B_0\|\|A\|^{\frac{p-1}{2}}\|t\|^m I\\
&= \frac{\|B_0\|\|A\|^{\frac{p+1}{2}}\|t\|^{m+2}}{\|A\|-\|t\|^2} I+\frac{\|B_0\|\|A\|^{\frac{p+1}{2}}\|t\|^{m+1}}{\|A\|-\|t\|^2} I+\\
&\quad \|B_0\|\|A\|^{\frac{p-1}{2}}\|t\|^m I\\
&\to \theta \ (m\to\infty).
\end{aligned}
$$

Therefore $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$ there exists an $x \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x.$$

Since

$$
\begin{aligned}
\theta \preceq d(Tx, x) &\preceq A[d(Tx, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x)]\\
&\preceq A[B\, d(Tx, x) + B\, d(Tx_n, x_n)) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x)]
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
d(Tx, x) \preceq{}& (I-AB)^{-1} AB\, d(Tx_n, Tx_{n-1}) + (I-AB)^{-1} A\, d(Tx_n, Tx_{n+1})+\\
&(I-AB)^{-1} A\, d(Tx_{n+1}, x)
\end{aligned}
$$

Then

$$
\begin{aligned}
\|d(Tx, x)\| \le{}& \|(I-AB)^{-1} AB\|\, \|d(Tx_n, Tx_{n-1})\|+\\
&\|(I-AB)^{-1} A\|\, \|d(Tx_n, Tx_{n+1})\|+\\
&\|(I-AB)^{-1} A\|\, \|d(Tx_{n+1}, x)\|\\
\to{}& 0 \ \text{ as n} \to \infty.
\end{aligned}
$$

This implies that $Tx = x$ that x is a fixed point of T. Now if $y (\neq x)$ is another fixed point of T, then

$$\theta \preceq d(x, y) = d(Tx, Ty) \preceq B\big(d(Tx, x) + d(Ty, y)) = \theta.$$

Hence $x = y$. Therefore the fixed point is unique and the proof is complete. □

4. Applications

As applications of contractive mapping theorem on complete C^*-algebra-valued rectangular b-metric spaces, existence and uniqueness results for a type of operator equation is given.

Example 4.1. *Suppose that H is a Hilbert space, $L(H)$ is the set of linear bounded operators on H. Let $A_1, A_2, \ldots A_n, \ldots \in L(H)$ which satisfy $\sum\limits_{n=1}^{\infty} \|A_n\| < 1$ and $Q \in L(H)_+$. Then the operator equation*

$$X - \sum_{n=1}^{\infty} A_n^* X A_n = Q$$

has a unique solution in $L(H)$.

Proof. Set $\alpha = \big(\sum\limits_{n=1}^{\infty} \|A_n\|\big)^p$ with $p \geq 1$, then $\|\alpha\| < 1$. Without loss of generality, one can suppose that $\alpha > 0$.

Choose a positive operator $T \in L(H)$. For $X, Y \in L(H)$ and $p \geq 1$, set

$$d(X, Y) = \|X - Y\|^p T.$$

Then $d(X, Y)$ is a C^*-algebra-valued rectangular b-metric and $(L(H), d)$ is complete since $L(H)$ is a Banach space. Indeed it suffices to check the third condition of Definition 2.4 as follows:

Suppose that $X, Y, U, V \in L(H)$ and set $L = X - U$, $M = U - V$, $N = V - Y$. Using the well known inequality

$$(a + b + c)^p \leq (3 \max\{a, b, c\})^p \leq 3^p(a^p + b^p + c^p) \text{ for all } a, b, c \geq 0,$$

we have

$$\begin{aligned}\|X - Y\|^p = \|L + M + N\|^p &\leq 3^p(\|L\|^p + \|M\|^p + \|N\|^p)\\ &= 3^p(\|X - U\|^p + \|U - V\|^p + \|V - Y\|^p)\end{aligned}$$

which implies that

$$d(X, Y) \leq A[d(X, U) + d(U, V) + d(V, Y)]$$

where $A = 3^p I$. Consider the map $T : L(H) \to L(H)$ defined by

$$F = \sum_{n=1}^{\infty} A_n^* X A_n + Q.$$

Then

$$\begin{aligned}d(F(X), F(Y)) &= \|F(X) - F(Y)\|^p T\\ &= \|\sum_{n=1}^{\infty} A_n^*(X - Y)A_n\|^p T\\ &\preceq \sum_{n=1}^{\infty} \|A_n\|^{2p} \|X - Y\|^p T\\ &\preceq \alpha^2\, d(X, Y)\\ &= (\alpha I)^* d(X, Y)(\alpha I).\end{aligned}$$

Using Theorem 3.1, there exists a unique fixed point X in $L(H)$. Furthermore, since $\sum_{n=1}^{\infty} A_n^* X A_n + Q$ is a positive operator, the solution is a Hermitian operator. $\square$

Acknowledgements. The authors thank the Principal and management of SSN institutions for their support to carry out this research work.

References

[1] Bai, C., Coupled fixed point theorems in C^*-algebra-valued b-metric spaces with application, *Fixed Point Theory and Appl,* 70(2016).

[2] Batul, S. and Kamran, T., C^*-valued contractive type mappings, *Fixed Point Theory Appl.*, 142(2015).

[3] Branciari, A., A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publicationes Mathematicae Debrecen*, 57(12),(2000) 31-37.

[4] Czerwik, S., Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Sem. Mat. Fis. Univ. Modena*, 46(2),(1998) 263-276.

[5] Erden, M. and Alaca, C., C^*-algebra-valued S-metric spaces, *Communications Faculty of Sciences University of Ankara-Series A1 Mathematics and Statistics,* 67(2),(2018) 165-177.

[6] George, R., Radenovic, S., Reshma, K.P. and Shukla, S., Rectangular b-metric space and contraction principles, *J. Nonlinear Sci. Appl.*, 8,(2015) 1005-1013.

[7] Kamran, T., Postolache, M., Ghiura, A., Batul, S. and Ali, R., The Banach contraction principle in C^*-algebra-valued b-metric spaces with application, *Fixed Point Theory and Appl,* 10(2016).

[8] Ma, ZH. and Jiang, L. Hongkai, C^*-algebra-valued metric spaces and related fixed point theorems: Fixed point Theory Appl., 206(2014).

[9] Ma, ZH. and Jiang, L., C^*-algebra-valued b-metric spaces and related fixed point theorems, *Fixed Point Theory Appl.*, 222(2015).

[10] Murphy, G.J., C^*-Algebras and Operator Theory, Academic Press, London, 1990.

[11] Shehwar Dur, E., Batul, S., Kamran, T. and Ghiura, A., Caristia's fixed point theorem on C*-algebra valued metric spaces, *J. Nonlinear Sci. Appl.*, 9,(2016) 584-588.

Current address: G. Kalpana: Department of Mathematics, SSN College of Engineering, Kalavakkam, Chennai-603 110, India.

E-mail address: `kalpanag@ssn.edu.in`

ORCID Address: `http://orcid.org/0000-0002-2426-0574`

Current address: Z. Sumaiya Tasneem: Department of Mathematics, SSN College of Engineering, Kalavakkam, Chennai-603 110, India.

E-mail address: `sumaiyatasneemz@ssn.edu.in`

ORCID Address: `http://orcid.org/0000-0001-7233-4876`