Commun.Fac.Sci.Univ.Ank.Series A1
Volume 62, Number 1, Pages 11–20 (2013)
ISSN 1303–5991

SOME RESULTS IN SEMIPRIME RINGS WITH DERIVATION

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Abstract. Let R be a semiprime ring and S be a nonempty subset of R. A mapping F from R to R is called centralizing on S if $[F(x), x] \in Z$ for all $x \in S$. The mapping F is called strong commutativity preserving (SCP) on S if $[F(x), F(y)] = [x, y]$ for all $x, y \in S$. In the present paper, we investigate some relationships between centralizing derivations and SCP-derivations of semiprime rings. Also, we study centralizing properties derivation which acts homomorphism or anti-homomorphism in semiprime rin

1. Introduction

Throughout R will represent an assosiative ring with center Z. A ring R is said to be prime if $xRy = 0$ implies that either $x = 0$ or $y = 0$ and semiprime if $xRx = 0$ implies that $x = 0$, where $x, y \in R$. A prime ring is obviously semiprime. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol xoy stands for the commutator $xy + yx$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.

Let S be a nonempty subset of R. A mapping F from R to R is called centralizing on S if $[F(x), x] \in Z$, for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$, for all $x \in S$. Also, F is called strong commutativity preserving (simply, SCP) on S if $[x, y] = [F(x), F(y)]$, for all $x, y \in S$. The study of centralizing mappings was initiated by E. C. Posner [2] which states that there existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (Posner's second theorem). There has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of R (see [5] for a partial bibliography). Derivations as well as SCP mappings have been extensively studied by researchers in the context of operator algebras, prime rings and semiprime rings too. For more information on SCP, we refere [3] , [8], [7] and references therein.

On the other hand, in [9] M.N. Daif and H.E. Bell showed that if a semiprime ring R has a derivation d satisfiying the following condition, then I is a central

Received by the editors October 31, 2012; Accepted: May 16, 2013.
2000 *Mathematics Subject Classification.* 16W25, 16W10, 16U80.
Key words and phrases. Semiprime rings, derivations, centralizing mappings, scp maps.

ideal;

$$\text{there exists a nonzero ideal } I \text{ of } R \text{ such that}$$
$$\text{either } d([x,y]) = [x,y] \text{ for all } x,y \in I \text{ or } d([x,y]) = -[x,y] \text{ for all } x,y \in I.$$

This result was extended for semiprime rings in [11].

In [4], H. E. Bell and L. C. Kappe have proved that d is a derivation of R which is either an homomorphism or anti-homomorphism in semiprime ring R or a nonzero right ideal of R then $d = 0$. Some recent results were shown on specific types of derivations of R. In [1], A. Ali, M. Yasen and M. Anwar showed that if R is a semiprime ring, f is an endomorphism which is a strong commutativity preserving map on a non-zero ideal U of R, then f is commuting on U. In [10], M. S. Samman proved that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing. The purpose of this paper is to investigate some relationships between derivations mentioned above in semiprime rings. Throughout the present paper, we shall make use of the following basic identities without any specific mention:

i) $[x, yz] = y[x, z] + [x, y]z$
ii) $[xy, z] = [x, z]y + x[y, z]$
iii) $xyoz = (xoz)y + x[y, z] = x(yoz) - [x, z]y$
iv) $xoyz = y(xoz) + [x, y]z = (xoy)z + y[z, x]$.

2. Results

Lemma 2.1. [6, Lemma 1.1.8] *Let R be a semiprime ring and suppose that $a \in R$ centralizes all commutators $xy - yx$, $x, y \in R$. Then $a \in Z$.*

Theorem 2.2. *Let R be a semiprime ring and d be a derivation of R. If d satisfies one of the following conditions, then d is centralizing.*
i) $d([x, y]) = [x, y]$, for all $x, y \in R$.
ii) $d([x, y]) = -[x, y]$, for all $x, y \in R$.
iii) For each $x, y \in R$, either $d([x, y]) = [x, y]$ or $d([x, y]) = -[x, y]$.

Proof. i) Assume that

$$d([x, y]) = [x, y], \text{ for all } x, y \in R.$$

Replacing y by yx, we get

$$d([x, y]x) = [x, y]x,$$

and so

$$d([x, y])x + [x, y]d(x) = [x, y]x.$$

Using the hypothesis, we obtain

$$[x, y]d(x) = 0, \text{ for all } x, y \in R. \tag{2.1}$$

Substituting $d(x)y$ for y in (2.1) and using (2.1), we have

$$[x, d(x)]yd(x) = 0, \text{ for all } x, y \in R. \tag{2.2}$$

Replacing y by yx in (2.2), we find that

$$[x, d(x)]yxd(x) = 0, \text{ for all } x, y \in R. \tag{2.3}$$

Multiplying (2.2) on the right by x, we have

$$[x, d(x)]yd(x)x = 0, \text{ for all } x, y \in R. \tag{2.4}$$

Subtracting (2.4) from (2.3), we arrive at

$$[x, d(x)]y[x, d(x)] = 0, \text{ for all } x, y \in R.$$

By the semiprimeness of R, we conclude that $[x, d(x)] = 0$, for all $x \in R$, and so $[x, d(x)] \in Z$.

ii) If d is a derivation satisfying the property $d([x, y]) = -[x, y]$, for all $x, y \in R$, then $(-d)$ satisfies the condition $(-d)([x, y]) = -[x, y]$, for all $x, y \in R$. Hence d is centralizing by (i).

iii) For each $x \in R$, we put $R_x = \{y \in R \mid d([x, y]) = [x, y]\}$ and $R_x^* = \{y \in R \mid d([x, y]) = -[x, y]\}$. Then $(R, +) = R_x \cup R_x^*$, but a group cannot be the union of proper subgroups, hence $R = R_x$ or $R = R_x^*$. By the same method in (i) or (ii), we complete the proof. □

We can give the following useful corollaries by the preceding theorem.

Corollary 1. *Let R be a prime ring and d be a derivation of R. If d satisfies one of the following conditions, then R is a commutative integral domain.*
i) $d([x, y]) = [x, y]$, for all $x, y \in R$.
ii) $d([x, y]) = -[x, y]$, for all $x, y \in R$.
iii) For each $x, y \in R$, either $d([x, y]) = [x, y]$ or $d([x, y]) = -[x, y]$.

Corollary 2. *Let R be a semiprime ring and d be a derivation of R. If d satisfies one of the following conditions, then d is centralizing.*
i) $d(xy) = xy$, for all $x, y \in R$.
ii)$d(xy) = -xy$, for all $x, y \in R$.
iii) For each $x, y \in R$, either $d(xy) = xy$ or $d(xy) = -xy$.

Proof. i) By the hypothesis, we get $d(xy) = xy$, *for all* $x, y \in R$. Then, we obtain that

$$d(xy - yx) = d(xy) - d(yx) = xy - yx.$$

Therefore, $d([x, y]) = [x, y]$, for all $x, y \in R$. By Theorem 2.2 (i), we conclude that d is centralizing.

ii) Using the same arguments in the proof of (i), we find the required result.

iii) It can be proved by using the similar arguments in Theorem 2.2 (iii). □

Theorem 2.3. *Let R be a semiprime ring with $\operatorname{char} R \neq 2$ and d be a derivation of R. If d is strong commutativity preserving, then d is centralizing.*

Proof. For all $x, y \in R$, we get $[d(x), d(y)] = [x, y]$. Replacing y by yz, $z \in R$, we obtain

$$[d(x), d(y)z + yd(z)] = [x, yz].$$

By the hypothesis, we have

$$d(y)[d(x), z] + [d(x), y]d(z) = 0.$$

Taking $d(x)$ instead of z in the above equation, we find that

$$[d(x), y]d^2(x) = 0, \text{ for all } x, y \in R.$$

Again replacing y by $d(y)$, we get

$$[d(x), d(y)]d^2(x) = 0, \text{ for all } x, y \in R.$$

Using the hypothesis, we see that

$$[x, y]d^2(x) = 0, \text{ for all } x, y \in R. \tag{2.5}$$

Substituting yr for y in (2.5) and using (2.5), we have

$$[x, y]rd^2(x) = 0 \text{ for all } x, y, r \in R. \tag{2.6}$$

Multiplying (2.6) on the right by $[x, y]$ and the left by $d^2(x)$, we get

$$d^2(x)[x, y]Rd^2(x)[x, y] = 0, \text{ for all } x, y \in R.$$

By the semiprimeness of R, we obtain

$$d^2(x)[x, y] = 0, \text{ for all } x, y \in R.$$

Replacing y by ry in the last equation, we see that

$$d^2(x)r[x, y] = 0, \text{ for all } x, y, r \in R. \tag{2.7}$$

Writing $x + z$ by x in (2.5) and using (2.5), we have

$$[x, y]d^2(z) + [z, y]d^2(x) = 0$$

and so

$$[x, y]d^2(z) = -[z, y]d^2(x), \text{ for all } x, y, z \in R. \tag{2.8}$$

Moreover, equation (2.8) implies that, we arrive at

$$[x, y]d^2(z)r[x, y]d^2(z) = -[x, y]d^2(z)r[z, y]d^2(x)$$

Using (2.7), we find that

$$[x, y]d^2(z)r[x, y]d^2(z) = 0, \text{ for all } x, y, z, r \in R.$$

By the semiprimeness of R, we get

$$[x, y]d^2(z) = 0, \text{ for all } x, y, z \in R. \tag{2.9}$$

Taking yr instead of y in (2.9) and using (2.9), we have

$$[x, y]rd^2(z) = 0, \text{ for all } x, y, z, r \in R. \tag{2.10}$$

Multiplying (2.10) on the right by $[x,y]$ and the left by $d^2(z)$, we obtain that

$$d^2(z)[x,y]rd^2(z)[x,y]=0, \text{ for all } x,y,z,r\in R.$$

Since R is semiprime ring, we have

$$d^2(z)[x,y]=0, \text{ for all } x,y,z\in R. \tag{2.11}$$

Using the equations (2.9) and (2.11), we get

$$d^2(z)[x,y]=[x,y]d^2(z), \text{ for all } x,y,z\in R.$$

By Lemma 2.1, we have $d^2(z)\in Z$, for all $z\in R$. Hence we conclude that $d^2([x,y])\in Z$, for all $x,y\in R$. That is

$$\left[d^2(x),y\right]+2\left[d(x),d(y)\right]+\left[x,d^2(y)\right]\in Z, \text{ for all } x,y\in R.$$

Using $d^2(z)\in Z$ for all $z\in R$ and $charR\neq 2$ in this equation, we obtain that

$$[d(x),d(y)]\in Z, \text{ for all } x,y\in R.$$

By the hypothesis, we find that

$$[x,y]\in Z, \text{ for all } x,y\in R.$$

Commuting this term with $d(z)-z\in R$, we arrive at

$$[d(z)-z,[x,y]]=0, \text{ for all } x,y,z\in R.$$

Again using Lemma 2.1, we have $d(z)-z\in Z$ for all $z\in R$. This implies that

$$[d(z)-z,z]=0, \text{ for all } z\in R,$$

and so $[d(z),z]=0$. Thus d is commuting, and so d is centralizing. This completes proof. □

Corollary 3. *Let R be a prime ring and d be a derivation of R. If d is SCP on R, then R is a commutative integral domain.*

Theorem 2.4. *Let R be a semiprime ring and d be a derivation of R. If d acts as a homomorphism on R, then d is centralizing.*

Proof. Assume that d acts as an anti-homomorphism on R. Now we have

$$d(xy)=d(x)y+xd(y)=d(x)d(y), \text{ for all } x,y\in R.$$

Replacing y by yz, $z\in R$ in above equation, we get

$$d(x)yz+xd(y)z+xyd(z)=d(x)d(y)z+d(x)yd(z).$$

Using the hypothesis and d is a derivation of R in the last relation gives

$$xyd(z)=d(x)yd(z)$$

and so

$$(d(x)-x)yd(z)=0, \text{ for all } x,y,z\in R. \tag{2.12}$$

Writing y by $d(y)$ in (2.12), we get

$$(d(x)-x)d(y)d(z)=0, \text{ for all } x,y,z\in R.$$

By the hypothesis, we obtain

$$(d(x)-x)\,d(yz)=(d(x)-x)\,d(y)\,z+(d(x)-x)\,yd(z)=0.$$

Using (2.12), we have

$$(d(x)-x)\,d(y)\,z=0$$

and so

$$\begin{aligned}d(x)\,d(y)\,z&=xd(y)\,z\\ d(xy)\,z&=d(x)\,yz+xd(y)\,z=xd(y)\,z.\end{aligned}$$

That is $d(x)\,yz=0$ for all $x,y,z\in R$. Explain to this part of the, we can shown that $[x,d(x)]y[x,d(x)]=0$, for all $x,y\in R$. Since R is semiprime, we get $[x,d(x)]=0$, for all $x\in R$. Hence d is commuting, and so d is centralizing. □

Corollary 4. *Let R be a prime ring and d be a derivation of R. If d acts as a homomorphism on R, then R is a commutative integral domain.*

Theorem 2.5. *Let R be a semiprime ring and d be a derivation of R. If d acts as an anti-homomorphism on R, then d is centralizing.*

Proof. By the hypothesis, we have

$$d(xy)=d(x)\,y+xd(y)=d(y)\,d(x)$$

Replacing y by xy in the last relation and using d is a derivation of R, we arrive at

$$d(x)\,xy+xd(x)\,y+xxd(y)=d(x)\,yd(x)+xd(y)\,d(x).$$

By the hypothesis, we get

$$d(x)\,xy+xd(x)\,y+xxd(y)=d(x)\,yd(x)+xd(xy)$$

and so

$$d(x)\,xy+xd(x)\,y+xxd(y)=d(x)\,yd(x)+xd(x)\,y+xxd(y).$$

That is

$$d(x)\,xy=d(x)\,yd(x)\,,\ \text{for all } x,y\in R. \tag{2.13}$$

Writing yx by y in (2.13), we have

$$d(x)\,xyx=d(x)\,yxd(x)\,.$$

Using (2.13), we arrive at

$$d(x)\,yd(x)\,x=d(x)\,yxd(x)$$

and so $d(x)\,y\,[d(x)\,,x]=0$, for all $x,y\in R$. Using the same arguments in the proof Theorem 2.2 (i), we find that $[d(x)\,,x]=0$. Hence d is commuting, and so d is centralizing. □

Corollary 5. *Let R be a prime ring and d be a derivation of R. If d acts as an anti-homomorphism on R, then R is a commutative integral domain.*

Theorem 2.6. *Let R be a semiprime ring. If R admits a derivation d such that $d(x)d(y)-xy\in Z$ for all $x,y\in R$, then d is centralizing.*

Proof. Replacing x by xz in the hypothesis, we get

$$d(x)zd(y)+x(d(z)d(y)-zy)\in Z,\ \text{for all } x,y,z\in R. \tag{2.14}$$

Commuting (2.14) with x, we have

$$[d(x)zd(y),x]=0,\ \text{for all } x,y,z\in R$$

and so

$$[d(x)z,x]d(y)+d(x)z[d(y),x]=0,\ \text{for all } x,y,z\in R.$$

Writing z by $zd(t)$, $t\in R$ in this equation and using this equation yields that

$$[d(x)zd(t),x]d(y)+d(x)zd(t)[d(y),x]=0,\ \text{for all } t,x,y,z\in R.$$

That is,

$$d(x)zd(t)[d(y),x]=0,\ \text{for all } t,x,y,z\in R.$$

Taking x instead of y in the above equation, we find that

$$d(x)zd(t)[d(x),x]=0,\ \text{for all } t,x,z\in R. \tag{2.15}$$

Multiplying (2.15) on the left by x, we have

$$xd(x)zd(t)[d(x),x]=0,\ \text{for all } t,x,z\in R. \tag{2.16}$$

Again replacing z by xz in (2.15), we obtain that

$$d(x)xzd(t)[d(x),x]=0,\ \text{for all } t,x,z\in R. \tag{2.17}$$

Subtracting (2.16) from (2.17), we see that

$$[d(x),x]zd(t)[d(x),x]=0 \text{ for all } t,x,z\in R.$$

Again multiplying this equation on the left by $d(t)$, we have

$$d(t)[d(x),x]zd(t)[d(x),x]=0,\ \text{for all } t,x,z\in R.$$

Since R is semiprime ring, we get

$$d(t)[d(x),x]=0,\ \text{for all } t,x\in R.$$

Substituting xt for t in the last equation and using the last equation, we obtain

$$d(x)t[d(x),x]=0 \text{ for all } t,x\in R.$$

Using the same arguments in the proof Theorem 2.2 (i), we conclude that

$$[d(x),x]t[d(x),x]=0,\ \text{for all } t,x\in R.$$

Again using the semiprimenessly of R, we get $[d(x),x]=0$, for all $x\in R$. This yields that d is commuting, and so d is centralizing. □

Corollary 6. *Let R be a prime ring. If R admits a derivation d such that $d(x)d(y)-xy\in Z$ for all $x,y\in R$, then R is a commutative integral domain.*

Application of similar arguments in Theorem 2.6 yields the following.

Theorem 2.7. *Let R be a semiprime ring. If R admits a derivation d such that $d(x)d(y)+xy \in Z$ for all $x, y \in R$, then d is centralizing.*

Corollary 7. *Let R be a prime ring. If R admits a derivation d such that $d(x)d(y)+xy \in Z$ for all $x, y \in R$, then R is a commutative integral domain.*

Theorem 2.8. *Let R be a semiprime ring and d be a derivation of R. If d satisfies one of the following conditions, then d is centralizing.*
i) $d(xoy) = xoy$, for all $x, y \in R$.
ii) $d(xoy) = -xoy$, for all $x, y \in R$.
iii) For each $x, y \in R$, either $d(xoy) = xoy$ or $d(xoy) = -xoy$.

Proof. i) Assume that

$$d(xoy) = xoy, \text{ for all } x, y \in R.$$

Writing y by xy in this equation yields that

$$d(x)(xoy) + xd(xoy) = x(xoy), \text{ for all } x, y \in R.$$

Using the hypothesis, we get

$$d(x)(xoy) = 0, \text{ for all } x, y \in R.$$

Replacing y by yz in the above equation and using this equation, we find that

$$d(x)(xoy)z + d(x)y[z, x] = 0, \text{ for all } x, y, z \in R.$$

That is

$$d(x)y[z, x] = 0, \text{ for all } x, y, z \in R.$$

Again replacing z by $d(x)$ in the last equation, we obtain that

$$d(x)y[d(x), x] = 0, \text{ for all } x, y \in R.$$

Using the same techniques in the proof of Theorem 2.2 (i), we can prove that d is centralizing.

iii) It can be proved similarly.

iii) It can be proved by using the similar arguments in Theorem 2.2 (iii). □

Corollary 8. *Let R be a prime ring and d be a derivation of R. If d satisfies one of the following conditions, then R is a commutative integral domain.*
i) $d(xoy) = xoy$, for all $x, y \in R$.
ii) $d(xoy) = -xoy$, for all $x, y \in R$.
iii) For each $x, y \in R$, either $d(xoy) = xoy$ or $d(xoy) = -xoy$.

Theorem 2.9. *Let R be a semiprime ring with $\operatorname{char} R \neq 2$. If R admits a derivation d such that $d(x)od(y) = xoy$, for all $x, y \in R$, then d is centralizing.*

Proof. By the hyphothesis, we get

$$d(x)\,od(y) = xoy, \text{ for all } x, y \in R.$$

Replacing x by xz, $z \in R$ in the hypothesis, we obtain

$$(d(x)\,od(y))\,z + d(x)\,[z, d(y)] + x\,(d(z)\,od(y)) - [x, d(y)]\,d(z) = (xoy)\,z + x\,[z, y]\,.$$

Using the hypothesis, we have

$$d(x)\,[z, d(y)] + x\,(zoy) - [x, d(y)]\,d(z) = x\,[z, y]\,.$$

This implies that

$$d(x)\,[z, d(y)] + xzy + xyz - [x, d(y)]\,d(z) = xzy - xyz$$

and so

$$d(x)\,[z, d(y)] - [x, d(y)]\,d(z) + 2xyz = 0. \tag{2.18}$$

Substituting zx for z in (2.18) and using (2.18), we have

$$d(x)\,z\,[x, d(y)] = [x, d(y)]\,zd(x)\,, \text{ for all } x, y, z \in R.$$

Writing z by $z\,[x, d(y)]$ in this equation and using this equation, we find that

$$[x, d(y)]\,zd(x)\,[x, d(y)] = [x, d(y)]\,z\,[x, d(y)]\,d(x) \text{ for all } x, y, z \in R$$

and so

$$[x, d(y)]\,z\,[d(x)\,, [x, d(y)]] = 0, \text{ for all } x, y, z \in R. \tag{2.19}$$

Multiplying (2.19) on the left by $d(x)$, we have

$$d(x)\,[x, d(y)]\,z\,[d(x)\,, [x, d(y)]] = 0, \text{ for all } x, y, z \in R. \tag{2.20}$$

Taking $d(x)\,z$ instead of z in (2.19), we find that

$$[x, d(y)]\,d(x)\,z\,[d(x)\,, [x, d(y)]] = 0, \text{ for all } x, y, z \in R. \tag{2.21}$$

Subtracting (2.21) from (2.20), we see that

$$[d(x)\,, [x, d(y)]]\,z\,[d(x)\,, [x, d(y)]] = 0, \text{ for all } x, y, z \in R.$$

By the semiprimeness of R, we arrive at

$$[d(x)\,, [x, d(y)]] = 0, \text{ for all } x, y \in R.$$

Moreover, replacing z by x in (2.18) and using the last equation, we see that

$$d(x)\,[x, d(y)] - [x, d(y)]\,d(x) + 2xyx = 0$$

That is $2xyx = 0$, for all $x, y \in R$. Since $char R \neq 2$, we obtain $xyx = 0$, for all $x, y \in R$. By the semiprimeness of R, we conclude that $x = 0$. Hence, d is commuting, and so d is centralizing. We complate the proof. □

Corollary 9. *Let R be a prime ring with $char R \neq 2$. If R admits a derivation d such that $d(x)\,od(y) = xoy$, for all $x, y \in R$, then R is a commutative integral domain.*

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