Commun.Fac.Sci.Univ.Ank.Series A1
Volume 62, Number 1, Pages 33–43 (2013)
ISSN 1303–5991

COMPOSITE DUAL SUMMABILITY METHODS OF THE NEW SORT*

MEDİNE YEŞİLKAYAGİL AND FEYZİ BAŞAR

Abstract. Following Altay and Başar [1], we define the duality relation of the new sort between a pair of infinite matrices. Our focus is to study the composite dual summability methods of the new sort and to give some inclusion theorems.

1. Introduction

We denote the space of all sequences with complex entries by ω. Any vector subspaces of ω is called a *sequence space*. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. A sequence space X is called an $FK-$space if it is a complete linear metric space with continuous coordinates $p_n\colon X \to \mathbb{C}$ for all $n \in \mathbb{N}$ with $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A normed $FK-$spaces is called a $BK-$space, that is, a $BK - space$ is a Banach space with continuous coordinates. The sequence spaces ℓ_∞, c and c_0 are $BK-$spaces with the usual sup-norm defined by $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$.

Let λ and μ be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A\colon \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k \quad \text{for each} \quad n \in \mathbb{N}.$$

Received by the editors Nov. 01, 2012; Accepted: April 30, 2013.
2010 *Mathematics Subject Classification.* 40C05.
Key words and phrases. Dual summability methods, sequence spaces, matrix transformations, composition of summability methods, inclusion theorems.

The main results of this paper were presented in part at the conference Algerian-Turkish International Days on Mathematics 2012 (ATIM' 2012) to be held October 9–11, 2012 in Annaba, Algeria at the Badji Mokhtar Annaba University.

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A\colon \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and each $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called the A-limit of x. Also by $(\lambda : \mu; p)$, we denote the subset of $(\lambda : \mu)$ for which the limits or sums are preserved whenever there is a limit or sum on the spaces λ and μ. The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in \omega \mid Ax \in \lambda\}$$

which is a sequence space.

Let $t = (t_k)$ be a sequence of non-negative numbers which are not all zero and write $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then the matrix $R^t = (r^t_{nk})$ of the Riesz mean (R, t_n) with respect to the sequence $t = (t_k)$ is given by

$$r^t_{nk} = \begin{cases} \frac{t_k}{T_n} & , \quad 0 \le k \le n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. It is well-known that the Riesz mean (R, t_n) is regular if and only if $T_n \to \infty$ as $n \to \infty$ (see [11, Theorem 1.4.4]). Let us define the sequence $y = (y_k)$, which will be used throughout, as the R^t- transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \frac{1}{T_k} \sum_{j=0}^{k} t_j x_j \quad \text{for all} \quad k \in \mathbb{N}. \tag{1.1}$$

2. The Dual Summability Methods of the New Sort

Lorentz introduced the concept of the dual summability methods for the limitation methods dependent on a Stieltjes integral and passed to the discontinuous matrix methods by means of a suitable step function,in [6]. After, several authors, such as Lorentz and Zeller [8], Kuttner [5], Öztürk [10], Orhan and Öztürk [9], Başar and Çolak [4], and the others, worked on the dual summability methods. Başar [3] recently introduced the dual summability methods of the new type which is based on the relation between the C_1-transform of a sequence and itself; where C_1 denotes the Cesàro mean of order 1. Following Kuttner [5] and Lorentz and Zeller [8] who defined the dual summability methods by using the relation between an infinite series and its sequence of partial sums, we desire to base the similar relation on (1.1) and call it as the duality of the new sort.

Let us suppose that the infinite matrix $A = (a_{nk})$ and $B = (b_{nk})$ map the sequences $x = (x_k)$ and $y = (y_k)$ which are connected with the relation (1.1) to the sequences (u_n) and (v_n), respectively, i.e.,

$$u_n = (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \quad \text{for all} \quad n \in \mathbb{N}, \tag{2.1}$$

$$v_n = (By)_n = \sum_{k=0}^{\infty} b_{nk} y_k \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.2}$$

It is clear here that the method B is applied to the R^t-transform of the sequence $x = (x_k)$, while the method A is directly applied to the entries of the sequence $x = (x_k)$. So, the methods A and B are essentially different.

Let us assume that the usual matrix product BR^t exists which is a much weaker assumption than the conditions on the matrix B belonging to any class of matrices, in general. We shall say in this situation that the matrices A and B in (2.1) and (2.2) are the *dual matrices of the new sort* if u_n reduces to v_n (or v_n reduces to u_n) under the application of the formal summation by parts. This leads us to the fact that BR^t exists and is equal to A and $Ax = (BR^t)x = B(R^t x) = By$ formally holds, if one side exists. This statement is equivalent to the relation between the entries of the matrices $A = (a_{nk})$ and $B = (b_{nk})$:

$$a_{nk} := \sum_{j=k}^{\infty} \frac{t_k}{T_j} b_{nj} \quad \text{or} \quad b_{nk} := \left(\frac{a_{nk}}{t_k} - \frac{a_{n,k+1}}{t_{k+1}} \right) T_k = \Delta \left(\frac{a_{nk}}{t_k} \right) T_k. \tag{2.3}$$

for all $k, n \in \mathbb{N}$. Now, we may give a short analysis on the dual summability methods of the new sort. One can see that v_n reduces to u_n, as follows: Since the equality

$$\sum_{k=0}^{m} b_{nk} y_k = \sum_{k=0}^{m} b_{nk} \sum_{j=0}^{k} \frac{t_j}{T_k} x_j = \sum_{j=0}^{m} \sum_{k=j}^{m} \frac{t_j}{T_k} b_{nk} x_j$$

holds for all $m, n \in \mathbb{N}$ one can obtain by letting $m \to \infty$ that

$$v_n = \sum_{k=0}^{\infty} b_{nk} y_k = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \frac{t_j}{T_k} b_{nk} x_j = \sum_{j=0}^{\infty} a_{nj} x_j = u_n \quad \text{for all} \quad n \in \mathbb{N}.$$

But the order of summation may not be reversed. So, the matrices A and B are not necessarily equivalent.

Let us suppose that the entries of the matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation (2.3) and $C = (c_{nk})$ be a strongly regular lower triangle matrix. Suppose also that the $C-$transforms of $u = (u_n)$ and $u = (v_n)$ be $t = (t_n)$ and $z = (z_n)$, respectively, i.e.,

$$t_n = (Cu)_n = \sum_{k=0}^{n} c_{nk} u_k \quad \text{for all} \quad n \in \mathbb{N}, \tag{2.4}$$

$$z_n = (Cv)_n = \sum_{k=0}^{n} c_{nk} v_k \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.5}$$

Define the matrices $D = (d_{nk})$ and $E = (e_{nk})$ by

$$d_{nk} := \sum_{j=0}^{n} c_{nj} a_{jk} \quad \text{and} \quad e_{nk} := \sum_{j=0}^{n} c_{nj} b_{jk} \quad \text{for all} \quad n \in \mathbb{N}.$$

For short, here and after, we call the methods A and B as "*original methods*" and call the methods D and E as "*composite methods*". Now, we can give the first theorem:

Theorem 2.1. *The original methods are dual of the new sort if and only if the composite methods are dual of the new sort.*

Proof. Suppose that the relation (2.3) exists between the elements of the original matrices $A = (a_{nk})$ and $B = (b_{nk})$. This means that $A = BR^t$ or equivalently $B = A(R^t)^{-1}$. Therefore, by applying the strongly regular triangle matrix $C = (c_{nk})$ to (u_n) and (v_n) in (2.1) and (2.2), we obtain that

$$Cu = C(Ax) = (CA)x = Dx$$
$$Cv = C(By) = (CB)y = Ey$$

Then, we have $Cu = Cv$ whenever $u = v$ which gives that $Ey = Dx$. Therefore, we derive that

$$Ey = E(R^t x) = (ER^t)x = Dx.$$

This shows that the composite methods D and E are dual of the new sort.

Conversely, suppose that the duality relation of the new sort exists between the elements of $D = (d_{nk})$ and $E = (e_{nk})$, i.e., $D = ER^t$ or equivalently $E = D(R^t)^{-1}$. Then, by applying the inverse matrix C^{-1} to the sequences $t = (t_n)$ and $z = (z_n)$ in (2.4) and (2.5), we observe that

$$C^{-1}z = C^{-1}(Dx) = (C^{-1}D)x = Ax,$$
$$C^{-1}v = C^{-1}(Ey) = (C^{-1}E)y = By.$$

Hence, $By = Ax$. Therefore, we get $B(R^t x) = (BR^t)x = Ax$ which means that the original matrices A and B are dual of the new sort. $\square$

Theorem 2.2. *Every $A-$summable sequence is $D-$summable. However, the converse of this fact does not hold, in general.*

Proof. Suppose that $x = (x_k)$ is $A-$summable to $a \in \mathbb{C}$, i.e.,

$$\lim_{n\to\infty} (Ax)_n = \lim_{n\to\infty} u_n = a.$$

Since $C = (c_{nk})$ is a strongly regular triangle matrix, we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (Cu)_n = a.$$

That is to say that

$$\lim_{n\to\infty} (Cu)_n = \lim_{n\to\infty} \{C(Ax)\}_n = \lim_{n\to\infty} \{(CA)x\}_n = \lim_{n\to\infty} (Dx)_n = a.$$

This shows that the sequence $x=(x_k)$ is summable D to the same point. Hence, the inclusion $c_A \subset c_D$ holds.

Let us choose the matrix $C=(c_{nk})$ defined by

$$c_{nk}=\begin{cases} \frac{2(k+1)}{(n+1)(n+2)} & , \quad 0\le k\le n, \\ 0 & , \quad k>n \end{cases}$$

for all $k,n\in\mathbb{N}$. A short calculation gives the inverse matrix $C^{-1}=(c_{nk}^{-1})$ as

$$c_{nk}^{-1}=\begin{cases} \frac{1+(-1)^{n-k}(n+1)}{2} & , \quad n-1\le k\le n, \\ 0 & , \quad 0\le k<n-1 \ \text{ or } \ k>n \end{cases}$$

for all $k,n\in\mathbb{N}$. Let us also choose the matrix $D=(d_{nk})$

$$d_{nk}=\begin{cases} \frac{1}{2^n} & , \quad 0\le k<n-1, \\ \frac{-1}{2^n} & , \quad k=n-1, \\ 1 & , \quad k=n, \\ 0 & , \quad k>n \end{cases}$$

for all $k,n\in\mathbb{N}$. Then, the matrix $A=(a_{nk})$ satisfying the equality $D=CA$ is obtained by a straightforward calculation as

$$a_{nk}=\begin{cases} \frac{2-n}{2^{n+1}} & , \quad 0\le k<n-2, \\ \frac{3n+2}{2^{n+1}} & , \quad k=n-2, \\ -\frac{n2^n+n+2}{2^{n+1}} & , \quad k=n-1, \\ \frac{n+2}{2} & , \quad k=n, \\ 0 & , \quad k>n \end{cases}$$

for all $k,n\in\mathbb{N}$. Therefore, $\|A\|=\sup_{n\in\mathbb{N}}\sum_{k=0}^{\infty}|a_{nk}|=\infty$. Hence, A does not even apply to the points belonging to the space ℓ_∞. This shows that the inclusion $c_A\subset c_D$ is strict. $\square$

Theorem 2.3. *Every $B-$summable sequence is $E-$summable. However, the converse of this fact does not hold, in general.*

Proof. Suppose that $y=(y_k)$ is $B-$summable to $b\in\mathbb{C}$, i.e.,

$$\lim_{n\to\infty}(By)_n=\lim_{n\to\infty}v_n=b.$$

Since $C=(c_{nk})$ is a strongly regular triangle matrix, then we have

$$\lim_{n\to\infty}v_n=\lim_{n\to\infty}(Cv)_n=b$$

and this yields that

$$\lim_{n\to\infty}(Cv)_n=\lim_{n\to\infty}\{C(By)\}_n=\lim_{n\to\infty}\{(CB)y\}_n=\lim_{n\to\infty}(Ey)_n=b.$$

Hence, the sequence $y=(y_k)$ is $E-$summable to the value b which means that the inclusion $c_B\subset c_E$ holds.

We choose the matrix $C = (c_{nk})$ as in Theorem 2.2. Let us also choose the matrix $E = (e_{nk})$ defined by

$$e_{nk} = \begin{cases} (\frac{-1}{2})^{n-k} & , \quad n-1 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then, the matrix $B = (b_{nk})$ satisfying the matrix equality $E = CB$ is found by a routine calculation as

$$b_{nk} = \begin{cases} \frac{n}{4} & , \quad k = n-2, \\ -(\frac{3n+2}{4}) & , \quad k = n-1, \\ \frac{n+2}{2} & , \quad k = n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Therefore, $\|B\| = \sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} |b_{nk}| = \infty$. Hence, B does not apply to the sequences in the space ℓ_∞. This shows that the composite method E is stronger than the original method B and this step completes the proof. □

Definition 2.4. A continuous linear functional ϕ on ℓ_∞ is called a Banach limit (see Banach [2]) if the following statements hold:

(i) $\phi(x) \geq 0$, where $x = (x_k)$ with $x_k \geq 0$ for every $k \in \mathbb{N}$,
(ii) $\phi(x_{k+1}) = \phi(x_k)$,
(iii) $\phi(e) = 1$, where $e = (1, 1, 1, \ldots)$.

A sequence $x \in \ell_\infty$ is said to be almost convergent to the generalized limit L if all of its Banach limits equal to L (see Lorentz, [7]). We denote the set of all almost convergent sequences by f, i.e.,

$$f := \left\{ x = (x_k) \in \omega \mid \exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} t_{mn}(x) = \alpha \ \text{ uniformly in } \ n \right\},$$

where

$$t_{mn}(x) = \sum_{k=0}^{m} \frac{x_{k+n}}{m+1} \ \text{ with } \ t_{-1,n} = 0 \ \text{ and } \ \alpha = f - \lim \ x_k.$$

We use the following notation in Theorem 2.5 and Theorem 2.6:

$$t_{mn}(Ax) = \frac{1}{m+1} \sum_{j=0}^{m} A_{n+j}(x) = \sum_{k=0}^{\infty} a(n, k, m) x_k,$$

where

$$a(n, k, m) = \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j,k} \ \text{ for all } \ k, m, n \in \mathbb{N}.$$

Theorem 2.5. *The inclusion $f_D \supset f_A$ strictly holds.*

Proof. Suppose that the sequence $x = (x_k)$ is almost $A-$summable to $l \in \mathbb{C}$. That is, $f - \lim(Ax)_n = l$. Since $C = (c_{nk})$ is a strongly regular triangle matrix, we have

$$f - \lim(Ax)_n = f - \lim\{C(Ax)\}_n = f - \lim\{(CA)x\}_n = f - \lim(Dx)_n = l.$$

Then, the sequence $x = (x_k)$ is almost $D-$summable. This means that the composite method D is stronger than the original method A. Hence, the inclusion $f_D \supset f_A$ holds.

Let us choose the matrix $C = (c_{nk})$ as C_1, the Cesàro matrix of order one. Then, a short calculation gives us the inverse matrix $C^{-1} = (c_{nk}^{-1})$ as

$$c_{nk}^{-1} = \begin{cases} (-1)^{n-k}(k+1) & , \quad n-1 \le k \le n, \\ 0 & , \quad 0 \le k \le n-2 \text{ or } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Let us also choose the matrix $D = (d_{nk})$ defined by

$$d_{nk} = \begin{cases} \frac{1+(-1)^n}{(n+1)} & , \quad 0 \le k \le n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then, the matrix $A = (a_{nk})$ satisfying the matrix equality $D = CA$ is obtained as

$$a_{nk} = \begin{cases} 2 & , \quad \text{n is even and } 0 \le k \le n, \\ -2 & , \quad \text{n is odd and } 0 \le k \le n-1, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$. Now, take $x = (x_k) = \left\{\frac{k}{(k+1)!}\right\}$. Then,

$$\frac{1}{m+1}\sum_{i=0}^{m}(Dx)_{n+i} = \frac{1}{m+1}\sum_{i=0}^{m}\sum_{k=0}^{n+i} d_{n+i,k}x_k$$

if $n+i$ is odd, $d_{n+i,k}$ will be zero. Therefore,

$$\frac{1}{m+1}\sum_{i=0}^{m} D_{n+i}(x) = 0.$$

if $n+i$ is even, we'll have

$$\begin{aligned} \frac{1}{m+1}\sum_{i=0}^{m} D_{n+i}(x) &= \frac{1}{m+1}\sum_{i=0}^{m}\frac{1+(-1)^{n+i}}{n+i+1}\sum_{k=0}^{n+i}\frac{k}{(k+1)!} \\ &= \frac{1}{m+1}\sum_{i=0}^{m}\frac{2}{n+i+1}\left[1 - \frac{1}{(n+i+1)!}\right] \qquad (2.6) \\ &= \frac{2}{m+1}\sum_{i=0}^{m}\frac{1}{n+i+1} - \frac{2}{m+1}\sum_{i=0}^{m}\frac{1}{(n+i+1)(n+i+1)!} \end{aligned}$$

which tends to zero, as $m \to \infty$. Because we know that

$$\lim_{m\to\infty} \frac{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+m+1}}{m+1} = \lim_{m\to\infty} \frac{1}{n+m+1} = 0.$$

We can observe the second sum on the right hand side of (2.6) is zero, i.e., $x \in f_D$. Now, we derive for the matrix $A = (a_{nk})$ that

$$\frac{1}{m+1}\sum_{i=0}^{m}(Ax)_{n+i} = \frac{1}{m+1}\sum_{i=0}^{m}\sum_{k=0}^{n+i} a_{n+i,k}x_k \quad \text{for all} \quad m,n \in \mathbb{N}.$$

If $n+i$ is even, then we have

$$\begin{aligned}\frac{1}{m+1}\sum_{i=0}^{m}(Ax)_{n+i} &= \frac{1}{m+1}\sum_{i=0}^{m} 2\sum_{k=0}^{n+i}\frac{k}{(k+1)!}\\ &= \frac{2}{m+1}\sum_{i=0}^{m}\left[1 - \frac{1}{(n+i+1)!}\right]\\ &= 2 - \frac{2}{m+1}\sum_{i=0}^{m}\frac{1}{(n+i+1)!}\end{aligned}$$

which tends to 2, as $m \to \infty$. If $n+i$ is odd, then we have

$$\begin{aligned}\frac{1}{m+1}\sum_{i=0}^{m}(Ax)_{n+i} &= \frac{1}{m+1}\sum_{i=0}^{m}(-2)\sum_{k=0}^{n+i}\frac{k}{(k+1)!}\\ &= \frac{-2}{m+1}\sum_{i=0}^{m}\left[1 - \frac{1}{(n+i+1)!}\right]\\ &= -2 + \frac{2}{m+1}\sum_{i=0}^{m}\frac{1}{(n+i+1)!}\end{aligned}$$

which tends to -2, as $m \to \infty$. Therefore, we have

$$\lim_{m\to\infty}\frac{1}{m+1}\sum_{i=0}^{m}(Ax)_{n+i} = \begin{cases} 2 & , \quad \text{n+i is even,} \\ -2 & , \quad \text{n+i is odd,} \end{cases}$$

i.e., $x \notin f_A$, so the inclusion $f_D \supset f_A$ strictly holds and this completes the proof. $\square$

Theorem 2.6. *The inclusion $f_E \supset f_B$ strictly holds.*

Proof. Let $y = (y_k)$ be almost $B-$summable to $r \in \mathbb{C}$. That is, $f - \lim(By)_n = r$. Since $C = (c_{nk})$ is a strongly regular triangle matrix, we have

$$f - \lim(By)_n = f - \lim\{C(By)\}_n = f - \lim\{(CB)y\}_n = f - \lim(Ey)_n = r.$$

Therefore, $y = (y_k)$ is almost $E-$summable to r. Hence, the inclusion $f_E \supset f_B$ holds.

Let us choose the matrix $C = (c_{nk})$ as in Theorem 2.5 and define the matrix $B = (b_{nk})$ by

$$b_{nk} = \begin{cases} n+1 & , \quad k = n, \\ -(2n+1) & , \quad k = n-1, \\ n & , \quad k = n-2, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$. Then, the matrix $E = (e_{nk})$ such that $E = CB$ is obtained as

$$e_{nk} = \begin{cases} (-1)^{n-k} & , \quad n-1 \leq k \leq n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$. Now, take the sequence $y = (y_k) = \{(-1)^k\}$. Then, we have

$$\begin{aligned} \frac{1}{m+1} \sum_{i=0}^{m} (Ey)_{n+i} &= \frac{1}{m+1} \sum_{i=0}^{m} \sum_{k=0}^{n+i} e_{n+i,k} y_k \\ &= \frac{1}{m+1} \sum_{i=0}^{m} (e_{n+i,n+i-1} y_{n+i-1} + e_{n+i,n+i} y_{n+i}) \\ &= \frac{1}{m+1} \sum_{i=0}^{m} [-(-1)^{n+i-1} + (-1)^{n+i}] \\ &= \frac{2(-1)^n}{m+1} \sum_{i=0}^{m} (-1)^i \end{aligned}$$

which gives by letting $m \to \infty$ that the sequence y is almost $E-$summable to zero, that is, $y \in f_E$. On the other hand, we have

$$\begin{aligned}
&\frac{1}{m+1}\sum_{i=0}^{m}(By)_{n+i} \qquad (2.7)\\
&= \frac{1}{m+1}\sum_{i=0}^{m}\sum_{k=0}^{n+i} b_{n+i,k}y_k\\
&= \frac{1}{m+1}\sum_{i=0}^{m}(b_{n+i,n+i-2}y_{n+i-2} + b_{n+i,n+i-1}y_{n+i-1} + b_{n+i,n+i}y_{n+i})\\
&= \frac{1}{m+1}\sum_{i=0}^{m}[b_{n+i,n+i-2}(-1)^{n+i-2} + b_{n+i,n+i-1}(-1)^{n+i-1} + b_{n+i,n+i}(-1)^{n+i}]\\
&= \frac{1}{m+1}\sum_{i=0}^{m}(-1)^{n+i}[(n+i) + (2n+2i+1) + (n+i+1)]\\
&= \frac{(-1)^n}{m+1}\sum_{i=0}^{m}(-1)^{i}(4n+4i+2)\\
&= \frac{4n(-1)^n}{m+1}\sum_{i=0}^{m}(-1)^{i} + \frac{4(-1)^n}{m+1}\sum_{i=0}^{m}(-1)^{i}i + \frac{2(-1)^n}{m+1}\sum_{i=0}^{m}(-1)^{i}
\end{aligned}$$

It is not hard to see that the first and third sums on the right hand side of (2.7) tend to zero, as $m \to \infty$ and since the second sum on the right hand side of (2.7) is; if m is even,

$$\frac{4(-1)^n}{m+1}\sum_{i=0}^{m}(-1)^{i}i = \frac{4(-1)^n}{m+1}[-(1+3+\cdots+(m-1)) + (2+4+\cdots+m)] = \frac{m}{2}$$

and if m is odd,

$$\frac{4(-1)^n}{m+1}\sum_{i=0}^{m}(-1)^{i}i = \frac{4(-1)^n}{m+1}[-(1+3+\cdots+m) + (2+4+\cdots+(m-1))] = -\frac{m+1}{2}$$

which leads us by letting $m \to \infty$ that

$$\lim_{m\to\infty}\frac{4(-1)^n}{m+1}\sum_{i=0}^{m}(-1)^{i}i = \begin{cases} 2(-1)^n & , \quad m \text{ is even} \\ -2(-1)^n & , \quad m \text{ is odd} \end{cases}$$

This shows that $y \notin f_B$. Therefore, the inclusion $f_D \supset f_A$ strictly holds and this completes the proof. □

Theorem 2.7. *The duality relation of the new sort is not preserved under the usual inverse operation.*

Proof. Suppose that the relation (2.3) exists between the original matrices $A = (a_{nk})$ and $B = (b_{nk})$. Choose the matrix B as the identity matrix I. Then, the dual matrix of the new sort corresponding to the matrix $B = I$ is $A = R^t$. Nevertheless, the inverses $B^{-1} = I$ and $A^{-1} = (R^t)^{-1}$ are not dual of the new sort. This shows that there are dual matrices of the new sort while their usual inverses are not dual of the new sort. This completes the proof. □

Acknowledgement

The authors would like to express their pleasure to the anonymous referee for his/her many helpful suggestions and interesting comments on the main results of the earlier version of the manuscript.

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Current address: Medine Yeşilkayagil; Department of Mathematics, Uşak University, 1 Eylül Campus, 64200–Uşak, TURKEY

, Feyzi Başar; Department of Mathematics, Fatih University, The Hadımköy Campus, Büyükçekmece, 34500–İstanbul, TURKEY,

E-mail address: medine.yesilkayagil@usak.edu.tr, fbasar@fatih.edu.tr.

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