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A NEIGHBOURHOOD SYSTEM OF FUZZY NUMBERS AND ITS TOPOLOGY*

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Abstract. The neighbourhood system obtained by the neighbourhoods (whose radii are positive fuzzy numbers) in a fuzzy number-valued metric space is a basis of a topology for the set of all fuzzy numbers. In this paper, the convergence with respect to this topology is introduced and its basic properties are studied.

1. Introduction

In most of the situations in real world problems, the data obtained for decision making are only approximately known. To meet such problems, Zadeh [24] introduced the concept of fuzzy set in 1965. Later, Chang and Zadeh [3] defined the concept of a fuzzy number as a fuzzy subset of the real line. A fuzzy number is a quantity whose value is imprecise, rather than exact as in the case of crisp, single-valued numbers. Any fuzzy number can be thought of as a function whose domain is a specified set (usually the set of real numbers).

In fact, there is a wide range of possibilities to define a fuzzy number. However, many of these definitions are not particularly amenable to practical manipulations. In many cases, exact computations or comparisons of fuzzy numbers, and representation of ill-defined magnitudes are difficult by using those definitions of fuzzy numbers. With this in mind, in this paper, we adopt a widely accepted and practical definition of a fuzzy number encountered in the literature of fuzzy set theory.

Fuzzy numbers allow us to make the mathematical models of linguistic quantities and fuzzy environments. In many respects, fuzzy numbers depict the physical world more realistically than the real numbers do. Fuzzy numbers are used in statistics, computer programming, engineering (especially in communications), and

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experimental science. They are also important for the study of fuzzy integrals, fuzzy control problems and fuzzy optimization problems which are widely used in fuzzy information theory and fuzzy signal systems (see [5, 8, 23]). Therefore, a careful and scientific mathematical analysis of fuzzy numbers is very important for the theoretical background of applied studies. In this context, the distance between two fuzzy numbers and the convergence of a sequence of fuzzy numbers with respect to this distance plays a key role in the analysis of fuzzy numbers.

In 1991 Fuller [9] calculated the membership function of the product-sum of triangular fuzzy numbers. Later Hong and Hwang [13] determined the exact membership function of the t-norm-based sum of fuzzy numbers. In 1997 Hwang and Hong [14] have studied the membership function of the t-norm-based sum of fuzzy numbers on Banach spaces, which generalizes earlier results Fuller [9] and Hong and Hwang [13]. These papers are important ones related to the theory of convergence.

Recently, many authors have discussed the convergence of a sequence of fuzzy numbers and obtained many important results (see [1, 2, 7, 12, 22]). The first steps towards constructing such convergence theories go back to Matloka's [16] and Kaleva's [15] works. To this end, they used the supremum metric that gives a real (crisp) value for the distance between two fuzzy numbers. On the other hand, via positive fuzzy numbers, it is also possible to define a fuzzy (non-crisp) distance between two fuzzy numbers (as is exemplified by Guangquan [10]), because it is more natural that the distance between two fuzzy numbers is a fuzzy number rather than this distance is a real number. Nevertheless, although a fuzzy distance is used in Guangquan's studies [10, 11], the convergence of a sequence of fuzzy numbers discussed in these studies somehow depends on the supremum metric, since characteristic functions of positive numbers are used as radii of open neighborhoods of fuzzy numbers. In this case, the convergence with respect to the supremum metric and the convergence with respect to the fuzzy distance turn out to be equivalent.

We think that it will be a good step to examine the convergence of a sequence of fuzzy numbers from different perspectives to explore the boundaries of these convergence theories related to fuzzy numbers. In this context, we introduce a new type of convergence by using more positive fuzzy numbers, instead of just the positive characteristic functions used in Guangquan's [10, 11] definition of convergence. We note that this convergence should not be perceived as a generalization of ordinary convergence. Throughout the text, we also compare these different types of convergences of a sequence of fuzzy numbers.

2. Preliminaries

First we recall some of the basic concepts and notations in the theory of fuzzy numbers, and we refer to [4, 6, 11, 15, 16, 17, 18, 19, 20, 21] for more details.

A *fuzzy number* is a function X from $\mathbb{R}$ to $[0, 1]$, satisfying:

(i) X is *normal*, i.e., there exists an $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$;

(ii) X is *fuzzy convex*, i.e., for any $x, y \in \mathbb{R}$ and $\lambda \in [0,1], X(\lambda x + (1-\lambda)y) \geq \min\{X(x), X(y)\}$;
(iii) X is *upper semi-continuous*;
(iv) the *closure* of $\{x \in \mathbb{R} : X(x) > 0\}$, denoted by X^0, is *compact*.

These properties imply that, for each $\alpha \in (0,1]$, the $\alpha-$ *level set* $X^\alpha := \{x \in \mathbb{R} : X(x) \geq \alpha\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right]$ is a non-empty compact convex subset of $\mathbb{R}$, as is the support X^0. We denote the set of all fuzzy numbers by $\mathcal{F}(\mathbb{R})$. Note that the function a_1 defined by

$$a_1(x) := \begin{cases} 1 & , \quad \text{if } x = a, \\ 0 & , \quad \text{otherwise,} \end{cases}$$

where $a \in \mathbb{R}$, is a fuzzy number. By the *decomposition theorem* of fuzzy sets, we have

$$X = \sup_{\alpha \in [0,1]} \alpha \chi_{\left[\underline{X}^\alpha, \overline{X}^\alpha\right]}$$

for every $X \in \mathcal{F}(\mathbb{R})$, where each $\chi_{\left[\underline{X}^\alpha, \overline{X}^\alpha\right]}$ denotes the characteristic function of the subinterval $\left[\underline{X}^\alpha, \overline{X}^\alpha\right]$.

Now we recall the *partial order relation* on the set of fuzzy numbers. For $X, Y \in \mathcal{F}(\mathbb{R})$, we write $X \preceq Y$, if for every $\alpha \in [0,1]$, the inequalities

$$\underline{X}^\alpha \leq \underline{Y}^\alpha \quad \text{and} \quad \overline{X}^\alpha \leq \overline{Y}^\alpha$$

hold. We write $X \prec Y$, if $X \preceq Y$ and there exists an $\alpha_0 \in [0,1]$ such that

$$\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0} \quad \text{or} \quad \overline{X}^{\alpha_0} < \overline{Y}^{\alpha_0}.$$

If $X \preceq Y$ and $Y \preceq X$, then $X = Y$. Two fuzzy numbers X and Y are said to be *incomparable* and denoted by $X \nsim Y$, if neither $X \preceq Y$ nor $Y \preceq X$ holds. When $X \succeq Y$ or $X \nsim Y$, then we can write $X \not\prec Y$.

Now let us briefly review the operations of *summation* and *subtraction* on the set of fuzzy numbers. For $X, Y, Z \in \mathcal{F}(\mathbb{R})$, the fuzzy number Z is called the *sum of X and Y*, and we write $Z = X + Y$, if $Z^\alpha = \left[\underline{Z}^\alpha, \overline{Z}^\alpha\right] := X^\alpha + Y^\alpha$ for every $\alpha \in [0,1]$. Similarly, we write $Z = X - Y$, if $Z^\alpha = \left[\underline{Z}^\alpha, \overline{Z}^\alpha\right] := X^\alpha - Y^\alpha$ for every $\alpha \in [0,1]$.

We define the set of *positive fuzzy numbers* by

$$\mathcal{F}^+(\mathbb{R}) := \left\{X \in \mathcal{F}(\mathbb{R}) : X \succeq 0_1 \text{ and } \overline{X}^1 > 0\right\}.$$

A subset E of $\mathcal{F}(\mathbb{R})$ is said to be *bounded from above* if there exists a fuzzy number μ, called an *upper bound* of E, such that $X \preceq \mu$ for every $X \in E$. μ is called the *least upper bound* (sup) of E if μ is an upper bound and $\mu \preceq \mu'$ for all upper bounds μ'. A *lower bound* and the *greatest lower bound* (inf) are defined similarly. E is said to be *bounded* if it is both bounded from above and below. A sequence of fuzzy numbers (briefly, SFN henceforth) $X = \{X_n\}$ is said to be *bounded* if the set $\{X_n : n \in \mathbb{N}\}$ of fuzzy numbers is bounded.

If $X_n \preceq X_{n+1}$ for all $n \in \mathbb{N}$, then $X = \{X_n\}$ is said to be a *monotone increasing* SFN. A *monotone decreasing* SFN can be defined similarly.

Definition 2.1. *The map* $d_M : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathbb{R}^+ \cup \{0\}$ *defined as*

$$d_M(X,Y) := \sup_{\alpha\in[0,1]} \max\left\{\left|\underline{X}^\alpha - \underline{Y}^\alpha\right|, \left|\overline{X}^\alpha - \overline{Y}^\alpha\right|\right\}$$

is called the supremum metric *on* $\mathcal{F}(\mathbb{R})$.

An SFN $X = \{X_n\}$ is said to be $M-$*convergent* to the fuzzy number X_0, written as $M - \lim X_n = X_0$, if for every $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that

$$d_M(X_n, X_0) < \varepsilon \quad \text{for every } n > n_0.$$

A fuzzy number λ is called an $M-$*limit point* of the SFN $X = \{X_n\}$ provided that there is a subsequence of X that $M-$converges to λ. We will denote the set of all $M-$limit points of $X = \{X_n\}$ by L_X^M.

3. τ_F-CONVERGENCE OF A SEQUENCE OF FUZZY NUMBERS

Guangquan [10] introduced the concept of fuzzy distance between two fuzzy numbers as in Definition 3.1, and thus presented a concrete fuzzy metric in (3.1), which is very similar to an ordinary metric.

Definition 3.1. [10] A map $d : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ is called a fuzzy metric on $\mathcal{F}(\mathbb{R})$ provided that the conditions

(i) $d(X,Y) \succeq 0_1$,
(ii) $d(X,Y) = 0_1$ if and only if $X = Y$,
(iii) $d(X,Y) = d(Y,X)$,
(iv) $d(X,Y) \preceq d(X,Z) + d(Z,Y)$

are satisfied for all $X, Y, Z \in \mathcal{F}(\mathbb{R})$.

If d is a fuzzy metric on the set of fuzzy numbers, then we call the triple $(\mathbb{R}, \mathcal{F}(\mathbb{R}), d)$ a *fuzzy metric space*. Guangquan [10] presented an example of a fuzzy metric space via the function d_G defined by

$$d_G(X,Y) := \sup_{\alpha\in[0,1]} \alpha\chi_{\left[\left|\underline{X}^1-\underline{Y}^1\right|,\ \sup\limits_{\mu\in[\alpha,1]}\max\left\{\left|\underline{X}^\mu-\underline{Y}^\mu\right|,\left|\overline{X}^\mu-\overline{Y}^\mu\right|\right\}\right]}. \tag{3.1}$$

Here the map d_G satisfies the conditions (i)-(iv) in Definition 3.1.

Now we present a practical example for the fuzzy metric d_G. Define two fuzzy numbers X and Y by

$$X(x) := \begin{cases} x & , \ x \in [0,1] \\ 2-x & , \ x \in [1,2] \\ 0 & , \ \text{otherwise} \end{cases} \quad \text{and} \quad Y(x) := \begin{cases} \frac{x-3}{2} & , \ x \in [3,5] \\ \frac{5-x}{2}+1 & , \ x \in [5,7] \\ 0 & , \ \text{otherwise} \end{cases}.$$

Then the fuzzy distance between the fuzzy numbers X and Y is

$$d_G(X,Y)(x) = \begin{cases} 5-x & , \quad x \in [4,5] \\ 0 & , \quad \text{otherwise} \end{cases} .$$

Remark 3.2. *Let*

$$\mathcal{B}_F := \left\{ K(X,P) : X \in \mathcal{F}(\mathbb{R}),\ P \in \mathcal{F}^+(\mathbb{R}) \right\} \subset \mathcal{P}(\mathcal{F}(\mathbb{R})),$$

where $\mathcal{P}(\mathcal{F}(\mathbb{R}))$ is the power set of $\mathcal{F}(\mathbb{R})$ and

$$K(X,P) := \left\{ Z \in \mathcal{F}(\mathbb{R}) : d_G(X,Z) \prec P,\ P \in \mathcal{F}^+(\mathbb{R}) \right\}.$$

Then the set $\mathcal{B}_F$ forms a basis of a natural topology on $\mathcal{F}(\mathbb{R})$, denoted by τ_F. Thus, the pair $(\mathcal{F}(\mathbb{R}), \tau_F)$ is a topological space.

Now we investigate the properties of the convergence of a sequence in this topological space. Since this convergence is in the topology τ_F, we will denote it by τ_F-convergence.

Definition 3.3 (τ_F-Convergence)**.** *Let $X = \{X_n\} \subset \mathcal{F}(\mathbb{R})$ and $X_0 \in \mathcal{F}(\mathbb{R})$. Then $\{X_n\}$ is τ_F-*convergent *to X_0 and we denote this by*

$$\tau_F - \lim X_n = X_0 \text{ or } \{X_n\} \overset{\tau_F}{\to} X_0 \ (n \to \infty),$$

provided that for any $P \in \mathcal{F}^+(\mathbb{R})$ there exists an $n_0 = n_0(P) \in \mathbb{N}$ such that

$$d_G(X_n, X_0) \prec P \qquad \text{as } n > n_0.$$

Example 3.4. Define the sequence $\{X_n\}$ by

$$X_n(x) := \begin{cases} 1 - \frac{nx}{2n-1} & , \quad x \in \left[0, 2 - \frac{1}{n}\right] \\ 0 & , \quad \text{otherwise} \end{cases}$$

and the fuzzy number X_0 by

$$X_0(x) := \begin{cases} 1 - \frac{x}{2} & , \quad x \in [0,2] \\ 0 & , \quad \text{otherwise} \end{cases} .$$

It is easy to see that $d_G(X_n, X_0) = \sup\limits_{\alpha \in [0,1]} \alpha \chi_{\left[0, \frac{1}{n}\right]}$. Then we have $\underline{d_G(X_n, X_0)}^{\alpha} = 0$ and $\overline{d_G(X_n, X_0)}^{\alpha} = \frac{1}{n}$ for every $\alpha \in [0,1]$ and each $n \in \mathbb{N}$. Take $P \in \mathcal{F}^+(\mathbb{R})$. Then we have $\underline{P}^{\alpha} \geq 0$ and $\overline{P}^{\alpha} > 0$ for every $\alpha \in [0,1]$. Hence we get $\underline{d_G(X_n, X_0)}^{\alpha} = 0 \leq \underline{P}^{\alpha}$ and there exists an $n_0 = n_0(P) \in \mathbb{N}$ such that $\overline{d_G(X_n, X_0)}^{\alpha} = \frac{1}{n} < \overline{P}^{\alpha}$ for every $n > n_0$. Consequently, we get $d_G(X_n, X_0) \prec P$ for each $n > n_0$, which proves that $\tau_F - \lim X_n = X_0$.

Now our first step is to compare τ_F-convergence with $M-$ convergence.

Theorem 3.5. *Let $X = \{X_n\} \subset \mathcal{F}(\mathbb{R})$ and $X_0 \in \mathcal{F}(\mathbb{R})$. If $\tau_F - \lim X_n = X_0$ then $M - \lim X_n = X_0$.*

Proof. Assume that $\tau_F - \lim X_n = X_0$. By Definition 3.3, for every $\varepsilon_1 \in \mathcal{F}^+(\mathbb{R})$ there exists an $n_0 = n_0(\varepsilon_1) \in \mathbb{N}$ such that $d_G(X_n, X_0) \prec \varepsilon_1$ for all $n > n_0$. Then we have

$$\chi_{\left[\left|\underline{X_n}^1 - \underline{X_0^1}\right|, \sup\limits_{\mu\in[\alpha,1]} \max\left\{\left|\underline{X_n}^\mu - \underline{X_0}^\mu\right|, \left|\overline{X_n}^\mu - \overline{X_0}^\mu\right|\right\}\right]} \prec \varepsilon_1$$

for every $\alpha \in [0,1]$ and $n > n_0$. Thus we get

$$\sup_{\mu\in[\alpha,1]} \max\left\{\left|\underline{X_n}^\mu - \underline{X_0}^\mu\right|, \left|\overline{X_n}^\mu - \overline{X_0}^\mu\right|\right\} < \overline{\varepsilon_1}^\alpha.$$

Since $\sup\limits_{\alpha\in[0,1]} \max\left\{\left|\underline{X_n}^\alpha - \underline{X_0}^\alpha\right|, \left|\overline{X_n}^\alpha - \overline{X_0}^\alpha\right|\right\} = d_M(X_n, X_0)$, we have $d_M(X_n, X_0) < \varepsilon$ for all $n > n_0$ and for every $\varepsilon > 0$. Consequently, we have $M - \lim X_n = X_0$. □

The converse of the theorem above does not hold in general as can be seen in the following example.

Example 3.6. Define the SFN $\{X_n\}$ for every $x \in \mathbb{R}$ by

$$X_n(x) := \begin{cases} 0 & , \ x \in (-\infty, 3 - \frac{1}{n}] \cup [5 - \frac{1}{n}, \infty) \\ x - \left(3 - \frac{1}{n}\right) & , \ x \in \left(3 - \frac{1}{n}, 4 - \frac{1}{n}\right) \\ \left(5 - \frac{1}{n}\right) - x & , \ \text{otherwise} \end{cases},$$

and let

$$X_0(x) := \begin{cases} 0 & , \ x \in (-\infty, 3] \cup [5, \infty) \\ x - 3 & , \ x \in (3, 4) \\ 5 - x & , \ \text{otherwise} \end{cases}.$$

Then $M - \lim X_n = X_0$. Now we show that $\tau_F - \lim X_n \neq X_0$. Let $P \in \mathcal{F}^+(\mathbb{R})$ be defined as

$$P(x) := \begin{cases} 0 & , \ x \in (-\infty, 0] \cup [2, \infty) \\ x & , \ x \in (0, 1] \\ 2 - x & , \ \text{otherwise} \end{cases}.$$

We have

$$\begin{aligned} d_G(X_n, X_0) &= \sup_{\alpha\in[0,1]} \alpha\chi_{\left[\left|\underline{X_n}^1 - \underline{X_0^1}\right|, \sup\limits_{\mu\in[\alpha,1]} \max\left\{\left|\underline{X_n}^\mu - \underline{X_0}^\mu\right|, \left|\overline{X_n}^\mu - \overline{X_0}^\mu\right|\right\}\right]} \\ &= \sup_{\alpha\in[0,1]} \alpha\chi_{\left[\left|\left(4 - \frac{1}{n}\right) - 4\right|, \frac{1}{n}\right]} \\ &= \sup_{\alpha\in[0,1]} \alpha\chi_{\left[\frac{1}{n}, \frac{1}{n}\right]} = \left(\frac{1}{n}\right)_1. \end{aligned}$$

In this case $P \nsim \left(\frac{1}{n}\right)_1$, i.e., $\left(\frac{1}{n}\right)_1 \nprec P$ for every $n \in \mathbb{N}$. Consequently, $\tau_F - \lim X_n \neq X_0$.

Remark 3.7. We should note that if we define

$$\mathcal{B}_G := \{K(X,\varepsilon_1) : X \in \mathcal{F}(\mathbb{R}), \varepsilon > 0\} \subset \mathcal{P}(\mathcal{F}(\mathbb{R})),$$

where $K(X,\varepsilon_1) := \{Z \in \mathcal{F}(\mathbb{R}) : d_G(X,Z) \prec \varepsilon_1,\ \varepsilon > 0\}$. It is easy to show that the set $\mathcal{B}_G$ form basis for a topology τ_G on $\mathcal{F}(\mathbb{R})$. Note that the topology τ_F is finer than τ_G so that the convergences with respect to these topologies are not equivalent. In Definition 3.3, if we introduce a new type of convergence by using more positive fuzzy numbers, instead of just the positive characteristic functions used in Guangquan's [10, 11] definition of convergence. We note that this convergence should not be perceived as a generalization of ordinary convergence. If we replace the set of positive fuzzy numbers with the set of characteristic functions of positive real numbers, we obtain the $G-$convergence (namely, τ_G-convergence) defined by Guangquan [10].

Definition 3.8 ($G-$Convergence). [10] *Let $X = \{X_n\} \subset \mathcal{F}(\mathbb{R})$ and $X_0 \in \mathcal{F}(\mathbb{R})$. $\{X_n\}$ is said to be $G-$* convergent *to X_0, which is denoted by*

$$G - \lim X_n = X_0 \text{ or } \{X_n\} \xrightarrow{G} X_0 \ (n \to \infty),$$

provided that for any $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$d_G(X_n, X_0) \prec \varepsilon_1 \qquad \text{as } n > n_0.$$

In this case, $G-$convergence is equivalent to $M-$convergence as can be seen by the following lemma. The first version of this lemma was obtained by Wen-yi Zeng [25].

Lemma 3.9. *Let $X = \{X_n\} \subset \mathcal{F}(\mathbb{R})$ and $X_0 \in \mathcal{F}(\mathbb{R})$. Then $G - \lim X_n = X_0$ if, and only if, $M - \lim X_n = X_0$.*

Proof. Necessity. Let $G - \lim X_n = X_0$. By Definition 3.8, for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d_G(X_n, X_0) \prec \varepsilon_1$ for all $n > n_0$. We have

$$\chi_{\left[\left|\underline{X_n}^1 - \underline{X_0}^1\right|,\ \sup\limits_{\mu\in[\alpha,1]} \max\left\{\left|\underline{X_n}^\mu - \underline{X_0}^\mu\right|, \left|\overline{X_n}^\mu - \overline{X_0}^\mu\right|\right\}\right]} \prec \chi_{[\varepsilon,\varepsilon]} = \varepsilon_1$$

for every $\alpha \in [0,1]$ and $n > n_0$. Therefore we have

$$\sup_{\mu\in[\alpha,1]} \max\left\{\left|\underline{X_n}^\mu - \underline{X_0}^\mu\right|, \left|\overline{X_n}^\mu - \overline{X_0}^\mu\right|\right\} < \varepsilon$$

for every $\alpha \in [0,1]$, i.e.,

$$\sup_{\mu\in[0,1]} \max\left\{\left|\underline{X_n}^\mu - \underline{X_0}^\mu\right|, \left|\overline{X_n}^\mu - \overline{X_0}^\mu\right|\right\} = d_M(X_n, X_0) < \varepsilon$$

for every $n > n_0$. Consequently, $M - \lim X_n = X_0$.

Sufficiency. Let $M - \lim X_n = X_0$. Then for each $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d_M(X_n, X_0) < \varepsilon$ for every $n > n_0$. We have

$$\sup_{\mu\in[\alpha,1]} \max\left\{\left|\underline{X_n}^\mu - \underline{X_0}^\mu\right|, \left|\overline{X_n}^\mu - \overline{X_0}^\mu\right|\right\} < \varepsilon$$

for every $\alpha \in [0,1]$ and $n > n_0$. Therefore

$$\chi_{\left[\left|\underline{X_n}^1 - \underline{X_0}^1\right|, \sup_{\mu\in[\alpha,1]} \max\left\{\left|\underline{X_n}^\mu - \underline{X_0}^\mu\right|, \left|\overline{X_n}^\mu - \overline{X_0}^\mu\right|\right\}\right]} \prec \chi_{[\varepsilon,\varepsilon]} = \varepsilon_1$$

for every $\alpha \in [0,1]$ and $n > n_0$. Hence $d_G(X_n, X_0) \prec \varepsilon_1$ for every $n > n_0$. So, $G - \lim X_n = X_0$. □

Now we present sufficient conditions for an $M-$convergent SFN to be τ_F-convergent.

Theorem 3.10. *Let $X = \{X_n\} \subset \mathcal{F}(\mathbb{R})$ and $X_0 \in \mathcal{F}(\mathbb{R})$. If $M-\lim X_n = X_0$ and there exists an $\widetilde{n} \in \mathbb{N}$ such that $\underline{X_n}^1 = \underline{X_0^1}$ for every $n > \widetilde{n}$, then $\tau_F-\lim X_n = X_0$.*

Proof. Assume that $M - \lim X_n = X_0$. Then for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d_M(X_n, X_0) < \varepsilon$ for all $n > n_0$. Define $N = N(\varepsilon) := \max\{n_0, \widetilde{n}\}$. Now we show that $d_G(X_n, X_0) \prec P$ for all $P \in \mathcal{F}^+(\mathbb{R})$ and $n > N$. To the contrary, suppose that there exists a $P \in \mathcal{F}^+(\mathbb{R})$ such that $d_G(X_n, X_0) \not\prec P$ for infinitely many $n \in \mathbb{N}$. In this case, we have either $d_G(X_n, X_0) \succeq P$ or $d_G(X_n, X_0) \nsim P$. First assume that there exists a $P \in \mathcal{F}^+(\mathbb{R})$ such that $d_G(X_n, X_0) \succeq P$ for infinitely many n. Then we have $\underline{d_G(X_n, X_0)}^\alpha \geq \underline{P}^\alpha$ and $\overline{d_G(X_n, X_0)}^\alpha \geq \overline{P}^\alpha$ for every $\alpha \in [0,1]$. Since $P \in \mathcal{F}^+(\mathbb{R})$, we have $\overline{P}^\alpha > 0$ for all $\alpha \in [0,1]$. Define $\varepsilon := \overline{P}^0$. Hence, by definitions of d_G and d_M, we have $\overline{d_G(X_n, X_0)}^0 \geq \varepsilon$, i.e., $d_M(X_n, X_0) \geq \varepsilon$ for infinitely many n. This contradicts to $M - \lim X_n = X_0$. Now we assume that $d_G(X_n, X_0) \nsim P$ for infinitely many n. Then the following two cases are possible: There exists an $\alpha_0 \in [0,1]$ such that

(i) $\underline{d_G(X_n, X_0)}^{\alpha_0} > \underline{P}^{\alpha_0}$, or
(ii) $\overline{d_G(X_n, X_0)}^{\alpha_0} > \overline{P}^{\alpha_0}$.

Since we have $\underline{X_n}^1 = \underline{X_0^1}$, we can write $\underline{d_G(X_n, X_0)}^{\alpha_0} = 0$, by the definition of d_G, except finitely many n. Hence the case (i) is not valid because $P \in \mathcal{F}^+(\mathbb{R})$. In the case (ii), we have

$$\sup_{\mu\in[\alpha_0,1]} \max\left\{\left|\underline{X_n}^\mu - \underline{X_0}^\mu\right|, \left|\overline{X_n}^\mu - \overline{X_0}^\mu\right|\right\} > \overline{P}^{\alpha_0}$$

for infinitely many $n \in \mathbb{N}$. Define $\varepsilon := \overline{P}^{\alpha_0}$. By definition of d_M, we have $d_M(X_n, X_0) > \varepsilon$ for infinitely many $n \in \mathbb{N}$. However, this contradicts the fact that $M - \lim X_n = X_0$. □

Throughout the rest of paper, we will present fuzzy analogues of some results in classical mathematical analysis, in the context of τ_F- convergence.

Theorem 3.11. *If $\tau_F - \lim X_n = X_0$ and $\tau_F - \lim X_n = Y_0$, then $X_0 = Y_0$.*

Proof. Assume that $\tau_F - \lim X_n = X_0$ and $\tau_F - \lim X_n = Y_0$. Then for each $P \in \mathcal{F}^+(\mathbb{R})$ there exists an $n_1 = n_1(P) \in \mathbb{N}$ such that $d_G(X_n, X_0) \prec P$ for all

$n > n_1$. Similarly, there exists an $n_2 = n_2(P) \in \mathbb{N}$ such that $d_G(X_n, Y_0) \prec P$ for all $n > n_2$. Define $N := \max\{n_1, n_2\}$. Then we have

$$d_G(X_0, Y_0) \preceq d_G(X_n, X_0) + d_G(X_n, Y_0) \prec P + P = 2P$$

for every $P \in \mathcal{F}^+(\mathbb{R})$ and $n > N$. Hence we have $X_0 = Y_0$. □

Now we introduce the concept of τ_F-limit point of an SFN, and compare it with the concept of $M-$limit point.

Definition 3.12 (τ_F-limit point). *A fuzzy number λ is a τ_F-*limit point *of the SFN $X = \{X_n\}$ provided that there is a subsequence of X that τ_F- converges to λ. We denote the set of all τ_F- limit points of $X = \{X_n\}$* by L_X^F.

Corollary 1. *$L_X^F \subset L_X^M$ for every $X = \{X_n\} \subset \mathcal{F}(\mathbb{R})$.*

Proof. If $\lambda \in L_X^F$ then there is a subsequence $\{X_{n_k}\}$ such that $\tau_F - \lim_{k\to\infty} X_{n_k} = \lambda$. By Theorem 3.5, we have $M - \lim_{k\to\infty} X_{n_k} = \lambda$, so $\lambda \in L_X^M$. □

Remark 3.13. In Example 3.6, $L_X^F = \emptyset$, but $L_X^M = \{X_0\}$, i.e., the inclusion relation given in Corollary 1 is strict.

4. Conclusion

In order to introduce a more general convergence in Guangquan's fuzzy metric space, we have defined a new neigbourhood of a fuzzy number using positive fuzzy numbers, and thus we have obtained τ_F-convergence of a sequence of fuzzy numbers with respect to the topology generated by such neigbourhoods. This new type of convergence is a natural extension of Guangquan's definition of convergence in a fuzzy metric space. Even though ours is a simple idea, it puts forward a new concept of convergence which is equivalent neither to the convergence with respect to the supremum metric nor to the convergence in the sense of Guangquan.

Furthermore, we point out that the definitions and results presented here significantly differ from those in classical analysis. For instance, in Example 3.6, we have shown that a monotone increasing and bounded sequence of fuzzy numbers is not necessarily τ_F-convergent. In detail, although $\{X_n\}$ is an SFN where $\alpha-$cuts of its terms are close to the $\alpha-$cuts of X_0, the sequence $\{X_n\}$ may not be τ_F-convergent to X_0, unless $\underline{X_n^1} = \underline{X_0^1}$ except for a finite number of terms. Thus, the theory of τ_F-convergence requires extra conditions.

Finally, it should be noted that one may obtain some results that are just parallel to those in classical analysis by modifying the metric d_G in a more general context.

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