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# ON THE SPACES OF EULER ALMOST NULL AND EULER ALMOST CONVERGENT SEQUENCES*

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Abstract. Let $E^r$ denotes the Euler means of order $r$. The Euler sequence spaces $e_0^r$, $e_c^r$ and $e_p^r$, $e_\infty^r$ consisting of all sequences whose $E^r$-transforms are in the spaces $c_0$, $c$ and $\ell_p$, $\ell_\infty$ are introduced by Altay and Başar [2], Altay et al. [3], and Mursaleen et al. [22]. Recently, Polat and Başar have studied the Euler spaces of difference sequences of order $m$, in [24].

The concept *almost convergence* of a bounded sequence introduced by Lorentz [19]. Quite recently, Başar and Kirişci have worked the domain of the generalized difference matrix $B(r,s)$ in the sequence spaces $f_0$ and $f$ of almost null and almost convergent sequences, in [8]. In this paper, following Başar and Kirişci [8], we essentially deal with the domains $(f_0)_{E^r}$ and $f_{E^r}$ of the Euler means of order $r$ in the spaces $f_0$ and $f$. Therefore, we add two new spaces to the Euler sequence spaces.

## 1. Introduction

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains $\phi$, the set of all finitely non-zero sequences, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write $\ell_\infty$, $c$ and $c_0$ for the classical spaces of all bounded, convergent and null sequences, respectively. Also by $bs$, $cs$, $\ell_1$ and $\ell_p$, we denote the space of all bounded, convergent, absolutely and $p$-absolutely convergent series, respectively.

Let $\lambda$ and $\mu$ be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of complex numbers $a_{nk}$, where $k, n \in \mathbb{N}$. Then, we say that $A$ defines a matrix

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mapping from $\lambda$ into $\mu$, and we denote it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$. The sequence $Ax = \{(Ax)_n\}$, the $A$-transform of $x$, is in $\mu$; where

$$(Ax)_n = \sum_k a_{nk} x_k \quad \text{for each} \quad n \in \mathbb{N}. \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. By $(\lambda : \mu)$, we denote the class of all matrices $A$ such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and each $x \in \lambda$ and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence $x$ is said to be $A$-summable to $l$ if $Ax$ converges to $l$ which is called the $A$-limit of $x$. If there is some notion of limit or sum in $\lambda$ and $\mu$, then we write $(\lambda, \mu; p)$ to denote the subclass of $(\lambda : \mu)$, which preserves the limit or sum. Further, $A \in (\lambda : c)$ is said to be strongly-multiplicative $s$, if $\lim Ax = s(f - \lim\ x_k)$ for each $x = (x_k) \in \lambda$, where $\lambda \in \{f, f(E)\}$. By $(\lambda : \mu)_s$, we denote the class of all such matrices. It is now trivial in the case $s = 1$ that the class $(\lambda, \mu)_s$ coincides with the class $(\lambda, \mu; p)$ and thus it is immediate that $(\lambda, \mu; p) \subset (\lambda, \mu)_s \subset (\lambda, \mu)$.

The matrix domain $\lambda_A$ of an infinite matrix $A$ in a sequence space $\lambda$ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\} \tag{1.2}$$

which is a sequence space. If $A = (a_{nk})$ is triangle, i.e., $a_{nn} \neq 0$ and $a_{nk} = 0$ for all $k > n$, then one can easily observe that the sequence spaces $\lambda_A$ and $\lambda$ are linearly isomorphic, i.e., $\lambda_A \cong \lambda$.

The main purpose of present paper is to introduce the spaces $f_0(E)$ and $f(E)$ of Euler almost null and Euler almost convergent sequences, and to determine the $\beta$- and $\gamma$- duals of these spaces. Furthermore, some classes of matrix mappings on the space of Euler almost convergent sequences are characterized.

We shall write throughout for brevity that

$$\begin{aligned}
\overline{a}_{nk} &= \sum_{j=k}^{\infty} \binom{j}{k} (r-1)^{j-k} r^{-j} a_{nj}, \\
a(n,k) &= \sum_{j=0}^{n} a_{jk}, \\
a(n,k,m) &= \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j,k}, \\
\Delta a_{nk} &= a_{nk} - a_{n,k+1}, \\
\Delta a(n,k,m) &= a(n,k,m) - a(n,k+1,m)
\end{aligned}$$

for all $k, m, n \in \mathbb{N}$.

## 2. Euler Sequence Spaces

Firstly, we give the definitions of some sequence spaces in the existing literature.

The Euler sequence spaces $e_0^r$ and $e_c^r$ were defined by Altay and Başar [2] and the spaces $e_p^r$ and $e_\infty^r$ were defined by Altay et al. [3], as follows:

$$\begin{aligned}
e_0^r &= \left\{x=(x_k)\in\omega : \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k x_k = 0\right\},\\
e_c^r &= \left\{x=(x_k)\in\omega : \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k x_k \quad \text{exists}\right\},\\
e_p^r &= \left\{x=(x_k)\in\omega : \sum_{n}\left|\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k x_k\right|^p < \infty\right\}, \quad (1\le p<\infty),\\
e_\infty^r &= \left\{x=(x_k)\in\omega : \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k x_k\right| < \infty\right\},
\end{aligned}$$

where $E^r=(e_{nk}^r)$ denotes the Euler means of order $r$ defined by

$$e_{nk}^r = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k & , \quad (0\le k\le n),\\ 0 & , \quad (k>n)\end{cases}$$

for all $k,n\in\mathbb{N}$. It is known that the method $E^r$ is regular for $0<r<1$ and $E^r$ is invertible such that $(E^r)^{-1}=E^{1/r}$ with $r\neq 0$. We assume unless stated otherwise that $0<r<1$.

Altay and Başar [2] gave the inclusion relations between the sequence spaces $e_0^r$ and $e_c^r$ with the classical sequence spaces, determined the Schauder basis for these spaces. They also calculated the alpha-, beta-, gamma- and continuous duals of the Euler sequence spaces, and characterized some matrix mappings on $e_0^r$ and $e_c^r$.

Altay et al. [3] calculated the dual spaces of the sequence spaces $e_p^r$ and $e_\infty^r$, and constructed the Schauder basis of the sequence space $e_p^r$. In [22], Mursaleen et al. characterized the classes $(e_p^r:\ell_\infty)$, $(e_1^r:\ell_p)$ and $(e_p^r:f)$ of infinite matrices for $1<p\le\infty$ and gave the characterizations of some other matrix mappings from the space $e_p^r$ to the Euler, Riesz, difference, etc., sequence spaces, also Mursaleen et al. [22] emphasized on some geometric properties such as Banach–Saks property, weak Banach–Saks property, fixed point property, Banach-Saks type $p$ of the space $e_p^r$.

Kara et al. [15] introduced the Euler sequence spaces $e^r(p)$ of nonabsolute type and proved that the spaces $e^r(p)$ and $\ell(p)$ are linearly isomorphic. Also the alpha- beta- and gamma-duals of the Euler sequence spaces $e^r(p)$ of nonabsolute type are computed in [15]. Kara et al. [15] defined a modular on the generalized Euler sequence spaces $e^r(p)$ and considered it equipped with the Luxemburg norm. Therefore, they gave some relationships between the modular and Luxemburg norm on the space $e^r(p)$ has property $(H)$ but is not rotund $(R)$.

Let $m$ be a positive integer. We define the operators $\Delta^{(m)}, \sum^{(m)} : \omega \to \omega$ by

$$\left(\Delta^{(1)}x\right)_k = x_k - x_{k-1}, \quad \left(\sum^{(1)} x\right)_k = \sum_{j=0}^{k} x_j \quad \text{for all} \quad k \in \mathbb{N},$$

$$\Delta^{(m)}x = \Delta^{(1)} \circ (\Delta^{(m-1)})x, \quad \sum^{(m)} x = \left(\sum^{(1)} \circ \sum^{(m-1)}\right) x \quad \text{for all} \quad m \geq 2.$$

The following equalities hold for $m \geq 1$ and $k = 0, 1, 2, \ldots$

$$\left(\Delta^{(m)}x\right)_k = \sum_{j=0}^{m} (-1)^j \binom{m}{j} x_{k-j},$$

$$\left(\sum^{(m)} x\right)_k = \sum_{j=0}^{m} \binom{m+k-j-1}{k-j} x_j,$$

$$\Delta^{(m)} \circ \sum^{(m)} = \sum^{(m)} \circ \Delta^{(m)} = I,$$

where $I$ is the identity on $\omega$. We write $\Delta$ and $\Sigma$ for the matrices with $\Delta_{nk} = \left(\Delta^{(1)}(e^{(k)}\right)_n$ and $\sum_{nk} = \left(\sum(e^{(k)})\right)_n$ for all $n, k \in \mathbb{N}$. So the operators $\Delta^{(1)}$ and $\sum^{(1)}$ are given by the matrices $\Delta$ and $\sum$. Similarly, the operators $\Delta^{(m)}$ and $\sum^{(m)}$ are given by the composition of $\Delta$ and $\sum$ with themselves $m$ times.

Altay and Polat [4] defined the Euler sequence spaces with difference operator $\Delta$ as follows:

$$\begin{aligned}
e_0^r(\Delta) &= \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k \Delta x_k = 0 \right\}, \\
e_c^r(\Delta) &= \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k \Delta x_k \quad \text{exists} \right\}, \\
e_\infty^r(\Delta) &= \left\{ x = (x_k) \in \omega : \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k \Delta x_k \right| < \infty \right\},
\end{aligned}$$

where $\Delta x_k = x_k - x_{k-1}$. Following Altay and Polat [4], Polat and Başar [24] gave the new sequence spaces $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ consisting of all sequences $x = (x_k)$ such that their $\Delta^{(m)}-$transforms are in Euler the spaces $e_0^r$, $e_c^r$ and $e_\infty^r$, respectively, that is,

$$e_0^r\left(\Delta^{(m)}\right) = \left\{ x = (x_k) \in \omega : \Delta^{(m)}x \in e_0^r \right\},$$

$$e_c^r\left(\Delta^{(m)}\right) = \left\{ x = (x_k) \in \omega : \Delta^{(m)}x \in e_c^r \right\},$$

$$e_{\infty}^{r}\left(\Delta^{(m)}\right)=\left\{x=(x_k)\in\omega:\Delta^{(m)}x\in e_{\infty}^{r}\right\}.$$

The sequence spaces $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ are reduced in the case $m=1$ to the spaces $e_0^r(\Delta)$, $e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ of Altay and Polat[4].

Başarır and Kayıkçı [10] defined the matrix $B^{(m)}=(b_{nk}^{(m)})$ by

$$b_{nk}^{(m)}=\begin{cases}\binom{m}{n-k}r^{m-n+k}s^{n-k} & , \quad (\max\{0,n-m\}\leq k\leq n),\\ 0 & , \quad (0\leq k<\max\{0,n-m\} \text{ or } k>n)\end{cases}$$

for all $k,n\in\mathbb{N}$ which is reduced to the $m$th order difference matrix $\Delta^{(m)}$ in case $r=1$, $s=-1$, where $\Delta^{(m)}=\Delta(\Delta^{(m-1)})$ and $m\in\mathbb{N}$. Kara and Başarır [16] introduced the $B^m-$Euler difference sequence spaces $e_0^r(B^{(m)})$, $e_c^r(B^{(m)})$ and $e_\infty^r(B^{(m)})$ as the set of all sequences whose $B^m-$transforms are in the Euler spaces $e_0^r$, $e_c^r$ and $e_\infty^r$, respectively, that is,

$$e_0^r\left(B^{(m)}\right)=\{x=(x_k)\in\omega:B^m x\in e_0^r\},$$

$$e_c^r\left(B^{(m)}\right)=\{x=(x_k)\in\omega:B^m x\in e_c^r\},$$

$$e_\infty^r\left(B^{(m)}\right)=\{x=(x_k)\in\omega:B^m x\in e_\infty^r\}.$$

Karakaya and Polat [17] defined the new paranormed Euler sequence spaces with difference operator $\Delta$ as follows:

$$\begin{aligned} e_0^r(\Delta,p) &= \left\{x=(x_k)\in\omega:\lim_{n\to\infty}\left|\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k\Delta x_k\right|^{p_n}=0\right\},\\ e_c^r(\Delta,p) &= \left\{x=(x_k)\in\omega:\exists l\in\mathbb{C}\ni\lim_{n\to\infty}\left|\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k(\Delta x_k-l)\right|^{p_n}=0\right\},\\ e_\infty^r(\Delta,p) &= \left\{x=(x_k)\in\omega:\sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k\Delta x_k\right|^{p_n}<\infty\right\}.\end{aligned}$$

The new sequence spaces $e_0^r(\Delta,p)$, $e_c^r(\Delta,p)$ and $e_\infty^r(\Delta,p)$ are reduced to some sequence spaces corresponding to special cases of $(p_k)$. For instance, in the case $p_k=1$ for all $k\in\mathbb{N}$, the sequence spaces $e_0^r(\Delta,p)$, $e_c^r(\Delta,p)$ and $e_\infty^r(\Delta,p)$ are reduced to the sequence spaces $e_0^r(\Delta)$, $e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ defined by Altay and Polat [4].

Demiriz and Çakan [11] introduced the sequence spaces $e_0^r(u,p)$ and $e_c^r(u,p)$ of nonabsolute type, as the sets of all sequences such that their $E^{r,u}-$transforms are in the spaces $c_0(p)$ and $c(p)$, respectively, that is,

$$\begin{aligned} e_0^r(u,p) &= \left\{x=(x_k)\in\omega:\lim_{n\to\infty}\left|\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k u_k x_k\right|^{p_n}=0\right\},\\ e_c^r(u,p) &= \left\{x=(x_k)\in\omega:\exists l\in\mathbb{C}\ni\lim_{n\to\infty}\left|\sum_{k=0}^{n}\binom{n}{k}(1-r)^{n-k}r^k(u_k x_k-l)\right|^{p_n}=0\right\},\end{aligned}$$

where $u = (u_k)$ is the sequence of non-zero reals. In the case $(u_k) = (p_k) = e = (1, 1, 1, \ldots)$, the sequence spaces $e_0^r(u, p)$ and $e_c^r(u, p)$ are, respectively, reduced to the sequence spaces $e_0^r$ and $e_c^r$ introduced by Altay and Başar [2].

Djolović and Malkowsky [12] added a new supplementary aspect to research of Polat and Başar [24] by characterizing classes of compact operators on those spaces. In [12], the spaces are treated as the matrix domains of a triangle in the classical sequence spaces $c_0, c$ and $\ell_\infty$. The main tool for their characterizations is the Hausdorff measure of noncompactness.

## 3. Spaces of Euler Almost Null and Euler Almost Convergent Sequences

In this section, we study some properties of the spaces of the almost null and almost convergent Euler sequences.

The shift operator $P$ is defined on $\omega$ by $(Px)_n = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit $L$ is defined on $\ell_\infty$ as a non-negative linear functional, such that $L(Px) = L(x)$ and $L(e) = 1$. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit $l$ if all Banach limits of $x$ is $l$ [19], and is denoted by $f - \lim\ x_k = l$. Let $P^i$ be the composition of $P$ with itself $i$ times and write for a sequence $x = (x_k)$

$$t_{mn}(x) := \frac{1}{m+1} \sum_{i=0}^{m} (P^i x)_n \quad \text{for all} \quad m, n \in \mathbb{N}. \tag{3.1}$$

Lorentz [19] proved that $f - \lim\ x_k = l$ if and only if $\lim_{m\to\infty} t_{mn}(x) = l$ uniformly in $n$. It is well-known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. By $f$ and $fs$, we denote the space of all almost convergent sequences and series, respectively, i.e.,

$$\begin{aligned} f &= \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{j=0}^{m} \frac{x_{n+j}}{m+1} = l \ \text{ uniformly in } \ n \right\}, \\ fs &= \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{j=0}^{n+k} \frac{x_j}{m+1} = l \ \text{ uniformly in } \ n \right\}. \end{aligned}$$

It is proved in [8] that $f$ is a Banach space with the norm

$$\|x\|_f := \sup_{m,n\in\mathbb{N}} |t_{mn}(x)|,$$

where $t_{mn}(x)$ is defined as in (3.1).

Başar and Kirişci [8] have defined the sequence spaces $\widehat{f_0}$ and $\widehat{f}$ derived by the domain of generalized difference matrix $B(r, s)$ in the sequence spaces $f_0$ and $f$, that is

$$\begin{aligned} \widehat{f_0} &= \{x = (x_k) \in \omega : B(r, s)x \in f_0\}, \\ \widehat{f} &= \{x = (x_k) \in \omega : B(r, s)x \in f\}, \end{aligned}$$

where the generalized difference matrix $B(r,s)=\{b_{nk}(r,s)\}$ is defined by

$$b_{nk}(r,s)=\begin{cases} r & , \quad (k=n), \\ s & , \quad (k=n-1), \\ 0 & , \quad (0\le k<n-1 \text{ or } k>n) \end{cases}$$

for all $k,n\in\mathbb{N}$.

We introduce the sequence spaces $f_0(E)$ and $f(E)$ as the sets of all sequences whose $E^r$-transforms are in the spaces $f_0$ and $f$, that is

$$f_0(E) = \left\{x=(x_k)\in\omega : \lim_{m\to\infty}\sum_{j=0}^{m}\sum_{k=0}^{n+j}\frac{\binom{n+j}{k}(1-r)^{n+j-k}r^k x_k}{m+1}=0 \ \text{ uniformly in } \ n\right\},$$

$$f(E) = \left\{x=(x_k)\in\omega : \exists l\in\mathbb{C}\ni \lim_{m\to\infty}\sum_{j=0}^{m}\sum_{k=0}^{n+j}\frac{\binom{n+j}{k}(1-r)^{n+j-k}r^k x_k}{m+1}=l \ \text{ unif. in } n\right\}.$$

With the notation of (1.2), we can redefine the spaces $f_0(E)$ and $f(E)$ as follows:

$$f(E)=(f)_{E^r} \quad \textit{and} \quad f_0(E)=(f_0)_{E^r}$$

It is trivial that $f_0(E)\subset f(E)$.

Define the sequence $y=\{y_k(r)\}$ by the $E^r-$transform of a sequence $x=(x_k)$, i.e.,

$$y_k(r)=\sum_{j=0}^{k}\binom{k}{j}(1-r)^{k-j}r^j x_j \quad \text{for all} \quad k\in\mathbb{N}.$$

It is trivial that $\|\cdot\|_{f(E)}$ is a norm on the spaces $f_0(E)$ and $f(E)$, where $\|x\|_{f(E)}=\sup_{m,n\in\mathbb{N}}|t_{mn}(y)|$.

Now we give some inclusion relations between the sequence spaces $f_0(E)$, $f(E)$, $c$ and $\ell_\infty$.

**Theorem 3.1.** *The inclusion $f(E)\subset\ell_\infty$ is strict.*

*Proof.* It is clear that $f(E)\subset\ell_\infty$. Now, we should show that this inclusion is strict. Define the sequence $x=E^{1/r}y$ with the sequence $y$ in the set $\ell_\infty\setminus f$ given by Miller and Orhan [21] as $y=\{0,\dots,0,1,\dots,1,0,\dots,0,1,\dots,1,\dots\}$, where the blocks of 0's are increasing by factors of 100 and blocks of 1's are increasing by factors of 10. Then, the sequence $x$ is not in $f(E)$ but in the space $\ell_\infty$, as desired. □

**Theorem 3.2.** *The inclusion $c\subset f(E)$ strictly holds.*

*Proof.* It is clear that $c\subset f(E)$. Now we show that this inclusion is strict.

Now we consider the sequence $x=(x_k)$ defined by $x_k(r)=(-r)^{-k}$ for all $k\in\mathbb{N}$. The sequence is not convergent but is in the space $f(E)$. □

**Theorem 3.3.** *The spaces $f_0(E)$ and $f(E)$ are linearly isomorphic to the spaces $f_0$ and $f$, respectively, i.e., $f_0(E)\cong f_0$ and $f(E)\cong f$.*

*Proof.* To prove this theorem, we should show the existence of a linear bijection between the spaces $f(E)$ and $f$. Consider the transformation $T$ from $f(E)$ to $f$ by $y = Tx = E^r x$. The linearity of $T$ is clear. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence $T$ is injective.

Let us take any $y \in f$ and define the sequence $x = \{x_k(r)\}$ by

$$x_k(r) = \sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j \quad \text{for all} \quad k \in \mathbb{N}.$$

Then, one can see that

$$(E^r x)_n = \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k \left[ \sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j \right] = y_n \quad \text{for all} \quad n \in \mathbb{N}$$

which shows that $E^r x \in f$, i.e., $x \in f(E)$. Consequently, we see from here that $T$ is surjective. Hence $T$ is a linear bijection which therefore says us that the spaces $f(E)$ and $f$ are linearly isomorphic, as was desired.

Since one can show in the similar way that $f_0(E) \cong f_0$, we omit the detail. □

Başar and Kirişci [8] proved that sequence space $f$ is a $BK-$space with the norm $\|\cdot\|_\infty$ and non-separable closed subspace of $(\ell_\infty, \|\cdot\|_\infty)$. So, the sequence space $f$ has no Schauder basis. Jarrah and Malkowsky [1] showed that the matrix domain $\lambda_A$ of a normed sequence space $\lambda$ has a basis if and only if $\lambda$ has a basis whenever $A = (a_{nk})$ is triangle. Then;

The sequence spaces $f_0(E)$ and $f(E)$ have no Schauder basis.

## 4. Duals of the Spaces of Euler Almost Null and Euler Almost Convergent Sequences

The set $S(\lambda, \mu)$ defined by

$$S(\lambda, \mu) = \{z = (z_k) \in \omega : xz = (x_k z_k) \in \mu \quad \text{for all} \quad x = (x_k) \in \lambda\} \tag{4.1}$$

is called the *multiplier space* of the sequence spaces $\lambda$ and $\mu$. One can eaisly observe for a sequence space $\upsilon$ with $\lambda \supset \upsilon \supset \mu$ that the inclusions

$$S(\lambda, \mu) \subset S(\upsilon, \mu) \quad \text{and} \quad S(\lambda, \mu) \subset S(\lambda, \upsilon)$$

hold. With the notation of (4.1), the alpha-, beta- and gamma-duals of a sequence space $\lambda$, which are respectively denoted by $\lambda^\alpha$, $\lambda^\beta$ and $\lambda^\gamma$ are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \quad \lambda^\beta = S(\lambda, cs) \quad \text{and} \quad \lambda^\gamma = S(\lambda, bs).$$

The alpha-, beta- and gamma-duals of a sequence space are also referred as *Köthe-Toeplitz dual*, *generalized Köthe-Toeplitz dual* and *Garling dual* of a sequence space, respectively.

We give the beta- and gamma-duals of the sequence spaces $f_0(E)$ and $f(E)$. For this, we need the following lemma:

**Lemma 4.1.** *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

(i) $A \in (f : \ell_\infty)$ if and only if

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty.$$

(ii) (cf. [25]). $A \in (f : c)$ if and only if (4.2) holds and

$$\lim_{n\to\infty} a_{nk} = \alpha_k \quad \text{for each fixed} \quad k \in \mathbb{N}, \tag{4.2}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha, \tag{4.3}$$

$$\lim_{n\to\infty} \sum_k |\Delta(a_{nk} - \alpha_k)| = 0. \tag{4.4}$$

(iii) (cf. [13]). $A \in (f : f)$ if and only if (4.2) holds and

$$f - \lim_{n\to\infty} a_{nk} = \alpha_k \quad \text{for each fixed} \quad k \in \mathbb{N}, \tag{4.5}$$

$$f - \lim_{n\to\infty} \sum_k a_{nk} = \alpha, \tag{4.6}$$

$$\lim_{m\to\infty} \sum_k |\Delta[a(n,k,m) - \alpha_k]| = 0 \quad \text{uniformly in} \quad n. \tag{4.7}$$

(iv) (cf. [13]). $A \in (\ell_\infty : f)$ if and only if (4.2), (4.6) and (4.8) hold.

**Theorem 4.2.** *Define the sets $d_1^r$, $d_2^r$, $d_3^r$, $d_4^r$, $d_5^r$ defined as follows:*

$$d_1^r = \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \right| < \infty \right\},$$

$$d_2^r = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \quad \textit{exists} \right\},$$

$$d_3^r = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \left[ \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} \right] a_k \quad \textit{exists} \right\},$$

$$d_4^r = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \right| = 0 \right\},$$

$$d_5^r = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \left| \sum_{j=n+1}^{\infty} (\Delta\overline{a}_{jk} - \alpha_k) \right| = 0 \right\}.$$

*Then, the $\beta-$dual of the sequence space $f(E)$ is $\bigcap_{n=1}^{5} d_n^r$.*

*Proof.* Let $a = (a_k) \in \omega$ and consider the equality

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[ \sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j \right] a_k \tag{4.8}$$
$$= \sum_{k=0}^{n} \left[ \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \right] y_k = (T^r y)_n,$$

where $T^r = (t^r_{nk})$ is defined by

$$t^r_{nk} = \begin{cases} \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j & , \quad (0 \leq k \leq n), \\ 0 & , \quad (k > n), \end{cases} \tag{4.9}$$

for all $k, n \in \mathbb{N}$. Thus, we deduce from Part (ii) of Lemma 4.1 with (4.9) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in f(E)$ if and only if $T^r y = \{(T^r y)_n\} \in c$ whenever $y = (y_k) \in f$, where $T^r = (t^r_{nk})$ is defined by (4.10). Therefore, we derive from (4.2), (4.3), (4.4) and (4.5) that

$$\sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \right| < \infty,$$

$$\lim_{n \to \infty} \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j = \alpha_k \quad \text{for each fixed} \quad k \in \mathbb{N},$$

$$\lim_{n \to \infty} \sum_k \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j = \alpha,$$

$$\lim_{n \to \infty} \sum_k \left| \Delta \left[ \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j - \alpha_k \right] \right| = 0$$

which shows that $\{f(E)\}^{\beta} = \bigcap_{n=1}^{5} d^r_n$. □

**Theorem 4.3.** *The $\gamma-$dual of the sequence spaces $f_0(E)$ and $f(E)$ is the set $d^r_1$.*

*Proof.* This is similar to the proof of Theorem 4.2 with Part (i) of Lemma 4.1 instead of Part (ii) of Lemma 4.1. So, we omit the detail. □

## 5. Matrix transformations Related to the Sequence Space $f(E)$

In the present section, we characterize the matrix transformations from $f(E)$ into any given sequence space $\mu$.

Since $f(E) \cong f$, it is trivial that the equivalence "$x \in f(E)$ if and only if $y \in f$" holds.

**Theorem 5.1.** *Suppose that the entries of the infinite matrices* $A = (a_{nk})$ *and* $D = (d_{nk})$ *are connected with the relation*

$$d_{nk} = \overline{a}_{nk}$$

*for all* $k, n \in \mathbb{N}$ *and* $\mu$ *be any given sequence space. Then* $A \in (f(E) : \mu)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in f(E)^{\beta}$ *for all* $n \in \mathbb{N}$ *and* $D \in (f : \mu)$.

*Proof.* Let $\mu$ be any given sequence space. Suppose that (5.1) holds between $A = (a_{nk})$ and $D = (d_{nk})$, and take into account that the spaces $f(E)$ and $f$ are linearly isomorphic.

Let $A \in (f(E) : \mu)$ and take any $y = (y_k) \in f$. Then $DE^r$ exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in \bigcap_{i=1}^{5} d_i^r$ which yields that $\{d_{nk}\}_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, $Dy$ exists and thus

$$\sum_k d_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$. We have that $Dy = Ax$ which leads us to the consequence $D \in (f : \mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(E)\}^{\beta}$ for each $n \in \mathbb{N}$ and $D \in (f : \mu)$ hold, and take any $x = (x_k) \in f(E)$. Then, $Ax$ exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \left[ \sum_{j=k}^{m} \binom{j}{k} (r-1)^{j-k} r^{-j} a_{nj} \right] y_k$$

for all $n \in \mathbb{N}$, as $m \to \infty$ that $Dy = Ax$ and this shows that $A \in (f(E) : \mu)$. This completes the proof. □

By changing the roles of the spaces $f(E)$ with $\mu$ in Theorem 5.1, we have:

**Theorem 5.2.** *Suppose that the elements of the infinite matrices* $A = (a_{nk})$ *and* $B = (b_{nk})$ *are connected with the relation*

$$b_{nk} := \sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j a_{jk} \quad \text{for all} \quad k, n \in \mathbb{N}.$$

*Let* $\mu$ *be any given sequence space. Then,* $A = (a_{nk}) \in (\mu : f(E))$ *if and only if* $B \in (\mu : f)$.

*Proof.* Let $z = (z_k) \in \mu$ and consider the following equality

$$\sum_{k=0}^{m} b_{nk} z_k = \sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j \left( \sum_{k=0}^{m} a_{jk} z_k \right) \quad \text{for all} \quad m, n \in \mathbb{N},$$

which yields as $m \to \infty$ that $(Bz)_n = \{E^r(Az)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in f(E)$ whenever $z \in \mu$ if and only if $Bz \in f$ whenever $z \in \mu$. This completes the proof. □

Of course, Theorems 5.1 and 5.2 have several consequences depending on the choice of the sequence space $\mu$. Whence by Theorem 5.1 and Theorem 5.2, the necessary and sufficient conditions for $(f(E):\mu)$ and $(\mu:f(E))$ may be derived by replacing the entries of $C$ and $A$ by those of the entries of $D=CE^{1/r}$ and $B=E^{r}A$, respectively; where the necessary and sufficient conditions on the matrices $D$ and $B$ are read from the concerning results in the existing literature.

Now, we list the following conditions on an infinite matrix $A=(a_{nk})$ transforming the sequences from/in the sequence space $f$:

$$\sup_{n\in\mathbb{N}}\sum_{k}|\Delta a_{nk}|<\infty, \tag{5.1}$$

$$\lim_{k\to\infty} a_{nk}=0 \qquad \text{for each fixed } n\in\mathbb{N}\ , \tag{5.2}$$

$$\lim_{n\to\infty}\sum_{k}|\Delta^2 a_{nk}|=\alpha, \tag{5.3}$$

$$\lim_{m\to\infty}\sum_{k}|a(n,k,m)-\alpha_k|=0 \quad \text{uniformly in } n\ , \tag{5.4}$$

$$\lim_{q\to\infty}\sum_{k}\frac{1}{q+1}\Big|\sum_{i=0}^{q}\Delta[(a(n+i,k)-\alpha_k)]\Big|=0 \quad \text{uniformly in } n\ , \tag{5.5}$$

$$\sup_{n\in\mathbb{N}}\sum_{k}|\Delta a(n,k)|<\infty, \tag{5.6}$$

$$f-\lim a(n,k)=\alpha_k \quad \text{exists for each fixed } k\in\mathbb{N}\ , \tag{5.7}$$

$$\lim_{q\to\infty}\sum_{k}\frac{1}{q+1}\Big|\sum_{i=0}^{q}\Delta^2[a(n+i,k)-\alpha_k]\Big|=0 \quad \text{uniformly in } n\ , \tag{5.8}$$

$$\sup_{n\in\mathbb{N}}\sum_{k}|a(n,k)|<\infty, \tag{5.9}$$

$$\sum_{k}a_{nk}=\alpha_k \quad \text{for each fixed } k\in\mathbb{N}\ , \tag{5.10}$$

$$\sum_{n}\sum_{k}a_{nk}=\alpha, \tag{5.11}$$

$$\lim_{n\to\infty}\sum_{k}|\Delta[a(n,k)-\alpha_k]|=0. \tag{5.12}$$

**Lemma 5.3.** *Let $A=(a_{nk})$ be an infinite matrix. Then,*

(i) $A=(a_{nk})\in(\ell_\infty:f)$ if and only if (4.2), (4.6) and (5.5) hold, [13].
(ii) $A=(a_{nk})\in(f:f)$ if and only if (4.2), (4.6),(4.7) and (4.8) hold, [13].
(iii) $A=(a_{nk})\in(fs:\ell_\infty)$ if and only if (5.2) and (5.3) hold.
(iv) $A=(a_{nk})\in(fs:c)$ if and only if (4.3) and (5.2)-(5.4) hold, [23].
(v) $A=(a_{nk})\in(c:f)$ if and only if (4.2), (4.6) and (4.7) hold, [18].

(vi) $A = (a_{nk}) \in (bs : f)$ if and only if (4.6), (5.2), (5.3) and (5.9) hold, [9].
(vii) $A = (a_{nk}) \in (fs : f)$ if and only if (4.6), (4.8), (5.3) and (5.6) hold, [5].
(viii) $A = (a_{nk}) \in (cs : f)$ if and only if (4.6) and (5.2) hold, [7].
(ix) $A = (a_{nk}) \in (bs : fs)$ if and only if (5.3) and (5.6)-(5.8) hold, [9].
(x) $A = (a_{nk}) \in (fs : fs)$ if and only if (5.6)-(5.9) hold, [5].
(xi) $A = (a_{nk}) \in (cs : fs)$ if and only if (5.7) and (5.8) hold, [7].
(xii) $A = (a_{nk}) \in (f : cs)$ if and only if (5.10)-(5.13) hold, [5].

Now, we can give the following results:

**Corollary 1.** *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i) $A \in (f(E) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(E)\}^\beta$ for all $n \in \mathbb{N}$ and (4.2) holds with $\overline{a}_{nk}$ instead of $a_{nk}$.
(ii) $A \in (f(E) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(E)\}^\beta$ for all $n \in \mathbb{N}$ and (4.2)-(4.5) hold with $\overline{a}_{nk}$ instead of $a_{nk}$.
(iii) $A \in (f(E) : c_0)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(E)\}^\beta$ for all $n \in \mathbb{N}$ and (4.2) holds, (4.3) and (4.5) hold with $\alpha_k = 0$, and (4.4) holds and $\alpha = 0$ as $\overline{a}_{nk}$ instead of $a_{nk}$.
(iv) $A \in (f(E) : f)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(E)\}^\beta$ for all $n \in \mathbb{N}$ and (4.2), (4.6)-(4.8) hold with $\overline{a}_{nk}$ instead of $a_{nk}$.
(v) $A \in (f(E) : bs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(E)\}^\beta$ for all $n \in \mathbb{N}$ and (5.10) holds.
(vi) $A \in (f(E) : cs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(E)\}^\beta$ for all $n \in \mathbb{N}$ and (5.10)-(5.13) hold with $\overline{a}_{nk}$ instead of $a_{nk}$.

**Corollary 2.** *Let $A = (a_{nk})$ be an infinite matrix and $b_{nk}$ be defined by (5.2). Then, following statements hold:*

(i) $A = (a_{nk}) \in (\ell_\infty : f(E))$ if and only if (4.2), (4.6) and (5.5) hold with $b_{nk}$ instead of $a_{nk}$.
(ii) $A = (a_{nk}) \in (f : f(E))$ if and only if (4.2), (4.6), (4.7) and (4.8) hold with $b_{nk}$ instead of $a_{nk}$.
(iii) $A = (a_{nk}) \in (c : f(E))$ if and only if (4.2), (4.6) and (4.7) hold with $b_{nk}$ instead of $a_{nk}$.
(iv) $A = (a_{nk}) \in (bs : f(E))$ if and only if (5.2), (5.3), (4.6) and (5.6) hold with $b_{nk}$ instead of $a_{nk}$.
(v) $A = (a_{nk}) \in (fs : f(E))$ if and only if (5.3), (4.6), (4.8) and (5.6) hold with $b_{nk}$ instead of $a_{nk}$.
(vi) $A = (a_{nk}) \in (cs : f(E))$ if and only if (5.2) and (4.6) hold with $b_{nk}$ instead of $a_{nk}$.
(vii) $A = (a_{nk}) \in (bs : fs(E))$ if and only if (5.3), (5.6)-(5.8) hold with $b_{nk}$ instead of $a_{nk}$, where $fs(E)$ denotes the domain of the matrix $E^r$ in the sequence space $fs$.
(viii) $A = (a_{nk}) \in (fs : fs(E))$ if and only if (5.6)-(5.9) hold with $b_{nk}$ instead of $a_{nk}$.

(ix) $A = (a_{nk}) \in (cs : fs(E))$ if and only if (5.7) and (5.8) hold with $b_{nk}$ instead of $a_{nk}$.

Now, we can give some consequences, below:

**Corollary 3.** *$A \in (f(E) : c)_s$ if and only if (4.2) holds, (4.3) and (4.5) hold with $\alpha_k = 0$ for each $k \in \mathbb{N}$ and (4.4) also holds with $\alpha = s$ with $\overline{a}_{nk}$ instead of $a_{nk}$.*

**Corollary 4.** *$A \in (f(E) : f)_s$ if and only if (4.2), (4.6) and (4.8) hold with $\alpha_k = 0$ and (4.7) also holds with $\alpha = s$ with $\overline{a}_{nk}$ instead of $a_{nk}$.*

Now, we may mention about Steinhaus type theorems which were formulated by Maddox [20], as follows: Consider the class $(\lambda : \mu)_1$ of 1-multiplicative matrices and $\upsilon$ be a sequence space such that $\upsilon \supset \lambda$. Then a result of the form $(\lambda : \mu)_1 \bigcap (\upsilon : \mu) = \emptyset$, where $\emptyset$ denotes the empty set, is called a theorem of the Steinhaus type. Now, we can give the next Steinhaus type theorem concerning with the strongly-multiplicative and coercive matrix classes:

**Theorem 5.4.** *The classes $(f(E) : c)_s$ and $(\ell_\infty : c)$ are disjoint.*

*Proof.* Suppose that the converse of this is true, that is $(f(E) : c)_s \bigcap (\ell_\infty : c) \neq \emptyset$. Then there exists at least one infinite matrix $A$ satisfying the conditions of Corollary 5.6 and Schur's theorem. Then, we can easily see that $\lim_{n\to\infty} \overline{a}_{nk} = 0$ which contradicts the condition $\lim_{n\to\infty} \sum_k \overline{a}_{nk} = s$ of Corollary 5.4. This completes the proof. □

**Theorem 5.5.** *The classes $(f(E) : f)_s$ and $(\ell_\infty : f)$ are disjoint.*

*Proof.* This is similar to the proof of Theorem 5.8. So, we omit the detail. □

## 6. Conclusion

The construction of new sequence spaces with the Euler mean were studied by Altay and Başar [2], Altay et al. [3] and Mursaleen et al. [22]. After Altay and Polat [4], Polat and Başar [24] studied the Euler difference sequence spaces of order $m$. Also, Karakaya and Polat [17] extended the Euler sequence spaces $e_0^r(\Delta)$, $e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ defined by Altay and Polat [4] to the paranormed case. Kara et al. [15] studied some topological and geometrical properties of the generalized Euler spaces. Further Başarır and Kayıkçı[10] defined Euler $B^{(m)}$-difference sequence spaces. Demiriz and Çakan [11] introduced the sequence spaces $e_0^r(u,p)$ and $e_c^r(u,p)$ of nonabsolute type, as the sets of all sequences such that $E^{r,u}$-transforms of them are in the spaces $c_0(p)$ and $c(p)$. Djolović and Malkowsky [12] added a new supplementary aspect to research of Polat and Başar [24] by characterizing classes of compact operators on those spaces.

The concept of almost convergence has been employed many mathematicians since 1948. Başar and Kirişci [8] established new almost convergent sequence spaces with the generalized difference matrix $B(r,s)$ and Sönmez [26] studied the concept of almost convergence with the triple band matrix $B(r,s,t)$. Başar and Kirişci [8] proved that the space $f$ is a $BK-$space with the sup-norm, and is a non-seperable closed subspace of $(\ell_\infty, \|\cdot\|_\infty)$. Since the space $f$ is non-seperable, this space and the spaces isomorphic to the space $f$ have no Schauder basis.

In this paper, we combine the almost convergence with the Euler means. Since the domain of generalized difference matrix $B(r,s)$ in the space $f$ is studied by Başar and Kirişci [8], the present paper is its natural continuation.

Finally, we should note from now on that the investigation of the domain of some particular limitation matrices, namely the composition of Euler means with the $m^{th}$ order difference matrix or generalized weighted mean, the matrix $\Lambda$, etc., in the space $f$ will lead us to new results. Also it can study various matrix transformations, such as sequence-to-sequence, sequence-to-series, series-to-sequences and series-to-series, between the new almost Euler sequence spaces and other spaces.

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