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LOCAL AND EXTREMAL SOLUTIONS OF SOME FRACTIONAL INTEGRODIFFERENTIAL EQUATION WITH IMPULSES*

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Abstract. The subject of this work is to prove existence, uniqueness, and continuous dependence upon the data of solution to integrodifferential hyperbolic equation with integral conditions. The proofs are based on a priori estimates and Laplace transform method. Finally, the solution by using a numerical technique for inverting the Laplace transforms is obtained.

Introduction

The concept of fractional analysis like differentiation and integration can be considered as a generalization of ordinary ones with integer order. However, it remains a lot to be done before assuming that this generalization is really established. Fractional differential equations have been extensively applied in many fields, for example, in probability, viscoelasticity and electrical circuits. Different theoretical studies about the subject were done by many famous mathematicians over the years like Liouville, Riemann, Fourier, Abel, Leibniz. For more details, we refer to the books [6, 9, 10].

On the other side, the interest in studying impulsive differential equations is related to their utility for modeling phenomena subject to considerable short-term changes during their evolution. The fact that the duration of the perturbations is negligible in comparison with the duration of the phenomena requires us to consider them in the form of impulses. The theory of impulsive differential equations has been well developed during these twenty last years; to know more see [2, 7]

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It is therefore interesting that the impulsive effects may be a part of studies of fractional differential problems. This topic has awoken the curiosity of many researchers in recent years. Recently, some authors [1, 3, 5, 11] discussed existence results of solutions of impulsive fractional differential equations under different conditions, boundary ones, non local ones etc. The results are obtained by using fixed point principles. We point out that in the papers [1, 3, 5, 11] the authors used an incorrect formula of solutions; for this reason in [4] the authors introduced the right formula for solutions of some given impulsive Cauchy problem with Caputo fractional derivative.

In this paper, we study the existence of local and extremal solutions for some integrodifferential fractional equation involving Caputo's derivative subject to impulses in fixed moments by using fixed-point theory and fractional analysis under suitable assumptions. This, taking into account the discontinuous nature of impulsive differential problems compared with non impulsive differential ones precisely for the fractional order. A non impulsive fractional problem was treated in [8].

The paper is divided into three sections. In Section 2 we recall some basic notions which will be used in the remainder of the paper. In Section 3 we establish existence results, first, of local solution based on Schauder fixed point theorem then of extremal solutions by using impulsive fractional inequalities.

1. Preliminaries

1.1. **Fractional calculus.** We will introduce notations and definitions that are used in this paper and can be found in [6]. Let a $(-\infty < a < \infty)$ a constant on the real axis $\mathbb{R}$, the Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined by

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > a.$$

Among the great amount of definitions dealing with fractional derivatives of order $\alpha \geq 0$ we recall the Riemann-Liouville one which is defined by

$$D_{a^+}^{\alpha} f(t) = D^n I_{a^+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds, \tag{1.1}$$

provided that the right-hand-side exists; where $n = [\alpha] + 1$, $\Gamma(\alpha)$ is the classical Gamma function. Remark that $D_{a^+}^{\alpha} K \neq 0$, for any constant K.

The following Caputo's definition is also widely used due to its practical formulation in real world problems:

$$^{c}D_{a^+}^{\alpha} f(t) = I_{a^+}^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds. \tag{1.2}$$

It's clear that $^{c}D_{a^+}^{\alpha} K = 0$, for any constant K. Thus, the following properties hold

$$^{c}D_{a^+}^{n} f(t) = f^{(n)}(t), \quad {}^{c}D_{a^+}^{0} f(t) = I_{a^+}^{0} f(t) = f(t), \quad {}^{c}D_{a^+}^{\alpha} I_{a^+}^{\alpha} f(t) = f(t).$$

The function $f(t) = c_0 + c_1 (t-a) + \cdots . + c_{n-1} (t-a)^{n-1}$ is a solution of the equation ${}^cD^{\alpha}_{a^+} f(t) = 0$ with c_0, $c_1, \ldots, c_{n-1}$ arbitrary real constants, then for $f \in C^n [a,b]$ we have

$$\begin{aligned} I^{\alpha}_{a^+}\ {}^cD^{\alpha}_{a^+} f(t) &= f(t) + c_1 (t-a) + \cdots . + c_{n-1} (t-a)^{n-1}; \\ \text{for } n-1 &< \alpha < n. \end{aligned} \tag{1.3}$$

In particular, when $0 < \alpha < 1$, we have

$$\begin{aligned} {}^cD^{\alpha}_{a^+} f(t) &= \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \\ I^{\alpha}_{a^+}\, {}^cD^{\alpha}_{a^+} f(t) &= f(t) + c_0, \quad c_0 \in \mathbb{R}. \end{aligned}$$

1.2. **Impulsive effects.** The most real case of the instants of impulsive effects is as follows: A finite or infinite number of fixed moments noted t_k given by an increasing sequence without accumulation points, i.e., $t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = +\infty$.

Let us denote the right and left limits of $x(t)$ at $t = t_k$ respectively by

$$x\left(t_k^+\right) = \lim_{h\to 0^+} x(t_k + h)\,;\ x\left(t_k^-\right) = \lim_{h\to 0^-} x(t_k + h)\,.$$

Definition 1.1. The impulsive effects said impulsive condition is measured by the difference between the limits of the state function $x(t)$ on the right and left of the moment of impulses t_k and is noted

$$\Delta x(t_k) = x\left(t_k^+\right) - x\left(t_k^-\right),\ k = 1, 2, 3, \ldots .$$

Remark 1.2. Submitting a system to such conditions deprives the state function of its continuity but improves significantly other properties especially the numerical results.

To study an impulsive initial value problem on the interval $[t_0, t_0 + T]$, where the number of impulses is m; we proceed as follows: $[t_0, t_0 + T]$ is subdivided into $m + 1$ intervals and we act as if we had a classical Cauchy problem on each interval $(t_k, t_{k+1}]$, $k = 0, \ldots, m$, where $t_{m+1} = t_0 + T$. To ensure the existence of a solution we must assume the continuity of $x(t)$ on $(t_k, t_{k+1}]$, $k = 0, \ldots, m$ and its right limit exists at t_k for $k = 0, \ldots, m$. Hence, the solutions should belong to the space of piece continuous functions denoted by $\mathcal{PC}$ and defined by $\mathcal{PC}([t_0, t_0 + T], \mathbb{R}) = \{x : [t_0, t_0 + T] \to \mathbb{R} : x(t)$ is continuous for $t \neq t_k$, left continuous at $t = t_k$ and $x\left(t_k^+\right)$ exists for $k = 1, \ldots, m\}$ which is a Banach space once endowed with the norm

$$\|x\|_{\mathcal{PC}} = \max\left\{ \sup_{t\in(t_k, t_{k+1}]} |x(t)|, k = 0, 1, \ldots, m \right\}.$$

2. Main results

2.1. **Impulsive fractional integrodifferential initial value problem.** We are concerned by the following scalar integrodifferential equation of fractional order $0<\alpha<1$,

$$ {}^{c}D_{t_0^+}^{\alpha}x(t)+G(t,x(t))=\int_{t_0}^{t}K(t,s,x(s))\,ds;\quad t\neq t_k;\quad k=1,\ldots,m; \tag{2.1}$$

with the initial condition

$$x(t_0)=x_0;\ t_0\geq 0, \tag{2.2}$$

and the impulsive conditions

$$\Delta x(t_k)=J_k\left(x\left(t_k^-\right)\right);\quad k=1,\ldots,m. \tag{2.3}$$

We set the following assumptions:

($\mathbf{A}_1$) The instants of impulsive effects $t_k, k=1,\ldots,m$ are such that $t_0<t_1<\cdots<t_k<t_{k+1}<\cdots<t_m<t_0+T$.

($\mathbf{A}_2$) $G(t,x)\in C([t_0,t_0+T]\times\mathbb{R},\mathbb{R})$; $K(t,s,x)\in C([t_0,t_0+T]\times[t_0,t_0+T]\times\mathbb{R},\mathbb{R})$ and $J_k\in C(\mathbb{R},\mathbb{R})$, $k=1,\ldots,m$.

($\mathbf{A}_3$) The integrals $\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{t_0}^{s}K(s,\sigma,x(\sigma))d\sigma ds$ and $\int_{t_0}^{t}(t-s)^{\alpha-1}G(s\,,x(s))ds$ are pointwise defined on $(t_0,t_0+T]$.

Definition 2.1. A function $x\in\mathcal{PC}([t_0,t_0+T],R)$ with its α-derivatives existing on $[t_0,t_0+T]\setminus\{t_k\}_{k=1,\ldots,m}$ for $0<\alpha<1$, is said to be a solution of the problem (2.1)-(2.3) if x satisfies the fractional integrodifferential equation (2.1) on $[t_0,t_0+T]\setminus\{t_k\}_{k=1,\ldots,m}$, the impulsive conditions (2.2) for $t=t_k$; $k=1,\ldots,m$; and the initial condition (2.3) for $t=t_0$.

2.2. **Impulsive fractional integral equation.** We begin with the following lemma which allows us to discuss the properties of the impulsive fractional integral equation (2.4) rather than the impulsive fractional integrodifferential problem (2.1)-(2.3).

Lemma 2.2. *A function $x\in\mathcal{PC}([t_0,t_0+T],\mathbb{R})$ is a solution of the problem (2.1)-(2.3) if and only if x satisfies the integral equation of the form*

$$\begin{aligned} x(t) &= x_0+\sum_{t_0<t_k<t}J_k(x(t_k))+\frac{1}{\Gamma(\alpha)} \\ &\quad\times\int_{t_0}^{t}(t-s)^{\alpha-1}\left\{\int_{t_0}^{s}K(s,\sigma,x(\sigma))\,d\sigma-G(s,x(s))\right\}ds. \end{aligned} \tag{2.4}$$

Proof. Let $x(t)$ be a solution of the problem (2.1)-(2.3). Using appropriate properties of fractional calculus for $0<\alpha\leq 1$, after applying $I_{t_0^+}^{\alpha}$ to (2.1) on $[t_0,t_1]$, we obtain

$$\begin{aligned} I_{t_0^+}^{\alpha c} D_{t_0^+}^{\alpha} x(t) &= x(t) + c_0 \\ &= I_{t_0^+}^{\alpha} \left[\int_{t_0}^{t} K(t, s, x(s))\, ds - G(t, x(t)) \right]. \end{aligned} \tag{2.5}$$

From (2.2)we get $c_0 = -\, x_0$, then

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left\{ \int_{t_0}^{s} K(s, \sigma, x(\sigma))\, d\sigma - G(s, x(s)) \right\} ds.$$

Doing the same thing on $(t_1, t_2]$ we obtain from (2.5)

$$c_0 = -x\left(t_1^+\right) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} \left\{ \int_{t_0}^{s} K(s, \sigma, x(\sigma))\, d\sigma - G(s, x(s)) \right\} ds,$$

where

$$\begin{aligned} x\left(t_1^+\right) &= x\left(t_1^-\right) + J_1\left(x\left(t_1^-\right)\right) \\ &= x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} \left\{ \int_{t_0}^{s} K(s, \sigma, x(\sigma))\, d\sigma - G(s, x(s)) \right\} ds \\ &\quad + J_1(x(t_1)), \end{aligned}$$

with $x\left(t_k^-\right) = x(t_k), k = 1, \ldots, m$. So, on $(t_1, t_2]$ we get

$$\begin{aligned} x(t) &= x_0 + J_1(x(t_1)) + \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_{t_0}^{t} (t-s)^{\alpha-1} \left\{ \int_{t_0}^{s} K(s, \sigma, x(\sigma))\, d\sigma - G(s, x(s)) \right\} ds. \end{aligned}$$

Then, we obtain by induction for $t \in (t_m, t_0 + T]$ the form of integral equation satisfied by $x(t)$

$$x(t) = x_0 + \sum_{k=1}^{m} J_k(x(t_k)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left\{ \int_{t_0}^{s} K(s, \sigma, x(\sigma))\, d\sigma - G(t, x(t)) \right\} ds.$$

This shows the first implication. For the other implication, we apply ${}^{c}D_{t_0^+}^{\alpha}$ to (2.4) to get (2.1). Conditions (2.2) and (2.3) are obtained easily from (2.4) respectively for $t = t_0$ and $t = t_k, k = 1, \ldots, m$. □

2.3. **Local existence.** From the fixed-point theory, we recall the following theorem which will be used in the sequel.

Schauder's fixed-point theorem :

If E is a closed, bounded, convex subset of a Banach space and the mapping $A : E \to E$ is completely continuous, then A has a fixed point in E.

Theorem 2.3. *We assume that*

($\mathbf{A}_4$) *for $t_0 < s \le t \le t_0 + T$ and $x \in \mathbb{R}$ we have*

(i) $|K(t,s,x)| \leq h(t,s)\varphi(|x|)$; *where* $h(t,s) \in C([t_0,t_0+T]\times[t_0,t_0+T],\mathbb{R}_+)$ *and* $\varphi \in C(\mathbb{R}_+,\mathbb{R}_+)$ *is nondecreasing;*

(ii) $|G(t,x)| \leq a(t)g(|x|)$, *where* $a \in C([t_0,t_0+T],\mathbb{R}_+)$ *and* $g \in C(\mathbb{R}_+,\mathbb{R}_+)$ *is nondecreasing.*

(iii) $|J_k(x)| \leq \varphi_k(|x|)$, *where* $\varphi_k \in C(\mathbb{R}_+,\mathbb{R}_+)$ *is nondecreasing,* $k = 1,\ldots,m$.

Then there exists at least one solution $x(t)$ *of the problem (2.1)-(2.3) in* $\mathcal{PC}([t_0,t_0+\beta],\mathbb{R})$ *for some positive number* β.

Proof. We introduce the following notation

$$\Omega = \{x \in \mathcal{PC}([t_0,t_0+\beta],\mathbb{R}) \text{ such that } \|x-x_0\|_{PC} \leq b\},$$

for some β such that $0<\beta\leq T$ and

$$0 < \sum_{t_0<t_k<\beta} a_k + \frac{\beta^\alpha}{\alpha\Gamma(\alpha)}\left(M_1\frac{\beta}{\alpha+1}+M_2\right) \leq b. \tag{2.6}$$

From the continuity of the functions given in $(\mathbf{A}_4)$ on their domains we can find positive constants M_1, M_2 and a_k, $k=1,\ldots,m$ such that for $x\in\Omega$ we have

$$\begin{aligned}
|K(t,s,x(s))| &\leq \sup_{t_0<s\leq t\leq t_0+\beta} h(t,s)\varphi(\|x\|_{\mathcal{PC}}) := M_1,\\
|G(t,x(t))| &\leq \sup_{t\in[t_0,t_0+\beta]} a(t)g(\|x\|_{\mathcal{PC}}) := M_2,\\
|\ J_k(x(t_k))| &\leq \varphi_k(\|x\|_{\mathcal{PC}}) := a_k, \quad k=1,\ldots,m.
\end{aligned}$$

For applying Schauder's theorem we need to check that Ω is a non empty closed, bounded and convex subset of the Banach $\mathcal{PC}([t_0,t_0+\beta],\mathbb{R})$ which is an easy task. Let us define the operator A on Ω by

$$\begin{aligned}
Ax(t) &= x_0 + \sum_{t_0<t_k<t} J_k(x(t_k)) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} \\
&\quad\times\left\{\int_{t_0}^{s} K(s,\sigma,x(\sigma))\,d\sigma - G(s,x(s))\right\}ds.
\end{aligned} \tag{2.7}$$

It is clear that for each $x\in\Omega$ we have from (2.7)

$$\begin{aligned}
|Ax(t)-x_0| &\leq \sum_{t_0<t_k<t}|J_k(x(t_k))| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}[M_1(s-t_0)+M_2]\,ds\\
&\leq \sum_{t_0<t_k<\beta} a_k + \frac{1}{\Gamma(\alpha)}\left[M_1\int_{t_0}^{t}(t-s)^{\alpha-1}(s-t_0)\,ds + \frac{(t-t_0)^\alpha}{\alpha}M_2\right]\\
&\leq \sum_{t_0<t_k<\beta} a_k + \frac{1}{\alpha\Gamma(\alpha)}\left[M_1\frac{(t-t_0)^{\alpha+1}}{\alpha+1} + \beta^\alpha M_2\right],
\end{aligned}$$

then,

$$\|Ax - x_0\|_{\mathcal{PC}} \leq b. \tag{2.8}$$

Hence A maps Ω into itself. To show that A is completely continuous we will show it is continuous and $A\Omega$ is relatively compact in $\mathcal{PC}\left(\left[t_0, t_0+\beta\right], \mathbb{R}\right)$. Let $(y_n)_{n\geq 0}$ be a sequence such that $y_n \to y$ in Ω when $n \to \infty$. Then, for each $t \in [t_0, t_0+\beta]$

$$\begin{aligned}
&\left|Ay_n\left(t\right) - Ay\left(t\right)\right| \leq \sum_{t_0 < t_k < t} \left|J_k\left(y_n\left(t_k\right)\right) - J_k\left(y\left(t_k\right)\right)\right| \\
&+\frac{1}{\Gamma\left(\alpha\right)}\left|\int_{t_0}^{t}\left(t-s\right)^{\alpha-1}\int_{t_0}^{s}\left|K\left(s,\sigma,y_n\left(\sigma\right)\right) - K\left(s,\sigma,y\left(\sigma\right)\right)\right|d\sigma ds \\
&+\frac{1}{\Gamma\left(\alpha\right)}\int_{t_0}^{t}\left(t-s\right)^{\alpha-1}\left|G\left(s,y_n\left(s\right)\right)d\sigma - G\left(s,y\left(s\right)\right)\right|ds.
\end{aligned}$$

Since the functions K, G, J_k with $k = 1, \ldots, m$ are continuous and by the dominated convergence theorem we have $\|Ay_n - Ay\|_{\mathcal{PC}} \to 0$, when $n \to \infty$. Thus, A is continuous. In view of Arzela-Ascoli theorem it suffices to show that $A\Omega$ is uniformly bounded and equicontinuous in $\mathcal{PC}\left(\left[t_0, t_0+\beta\right], \mathbb{R}\right)$ indeed to show that $A\Omega$ is relatively compact. From (2.8) we get the following

$$\|Ax\|_{\mathcal{PC}} \leq |x_0| + b.$$

Thus, the functions of $A\Omega$ are uniformly bounded in $\mathcal{PC}\left(\left[t_0, t_0+\beta\right], \mathbb{R}\right)$. To prove that the functions of $A\Omega$ are equicontinuous, we consider τ_1, $\tau_2 \in [t_0, t_0+\beta]$ such that $\tau_1 < \tau_2$, it follows that

$$\left|Ax\left(\tau_2\right) - Ax\left(\tau_1\right)\right| \leq \sum_{\tau_1 \leq t_k < \tau_2} \left|J_k\left(x(t_k)\right)\right|$$

$$\begin{aligned}
&+\frac{1}{\Gamma\left(\alpha\right)}\left|\int_{t_0}^{\tau_2}\left(\tau_2-s\right)^{\alpha-1}\left\{\int_{t_0}^{s}K\left(s,\sigma,x\left(\sigma\right)\right)d\sigma - G\left(s,x\left(s\right)\right)\right\}ds\right. \\
&\left.-\int_{t_0}^{\tau_1}\left(\tau_1-s\right)^{\alpha-1}\left\{\int_{t_0}^{s}K\left(s,\sigma,x\left(\sigma\right)\right)d\sigma - G\left(s,x\left(s\right)\right)\right\}ds\right|,
\end{aligned}$$

so

$$\begin{aligned}
&\left|Ax\left(\tau_2\right) - Ax\left(\tau_1\right)\right| \leq \sum_{\tau_1 \leq t_k < \tau_2} a_k + \frac{M_2}{\Gamma\left(\alpha\right)} \\
&\times\left[\int_{t_0}^{\tau_1}\left[\left(\tau_2-s\right)^{\alpha-1} - \left(\tau_1-s\right)^{\alpha-1}\right]ds + \int_{\tau_1}^{\tau_2}\left(\tau_2-s\right)^{\alpha-1}ds\right] \\
&+\frac{M_1}{\Gamma\left(\alpha\right)}\left[\int_{t_0}^{\tau_1}\left[\left(\tau_2-s\right)^{\alpha-1} - \left(\tau_1-s\right)^{\alpha-1}\right]\left(s-t_0\right)ds + \int_{\tau_1}^{\tau_2}\left(\tau_2-s\right)^{\alpha-1}\left(s-t_0\right)ds\right].
\end{aligned}$$

Therefore

$$\begin{aligned}\left\|Ax(\tau_2)-Ax(\tau_1)\right\|_{\mathcal{PC}} &\leq \sum_{\tau_1\leq t_k<\tau_2} a_k+\frac{M_2}{\alpha\Gamma(\alpha)}\left[(\tau_2-t_0)^{\alpha}-(\tau_1-t_0)^{\alpha}\right]\\ &+\frac{M_1}{\alpha(\alpha+1)\Gamma(\alpha)}\left[(\tau_2-t_0)^{\alpha+1}-(\tau_1-t_0)^{\alpha+1}\right]\\ &+\frac{M_1}{\alpha\Gamma(\alpha)}(\tau_1-t_0)\left[(\tau_2-t_0)^{\alpha}-(\tau_1-t_0)^{\alpha}\right];\end{aligned}$$

from which we get $\left\|Ax(\tau_2)-Ax(\tau_1)\right\|_{\mathcal{PC}}\to 0$, when $\tau_2\to\tau_1$, that is, $\{Ax(t)\}$ is an equicontinuous family on $[t_0,t_0+\beta]$. Hence $\overline{A\Omega}$ is compact and so A is completely continuous. Finally, we conclude by virtue of Schauder's theorem that A has at least one fixed-point in Ω which is a solution of the problem (2.1)-(2.3). The proof is complete. □

2.4. **Extremal solutions.** We shall prove the existence of extremal solution of the problem (2.1)-(2.3) through the following steps by using comparison principles and the notion of convergence.

First we give results regarding the impulsive fractional inequalities in the following lemmas.

Lemma 2.4. *Further (A_1)-(A_4), assume that for $t_0<s\leq t\leq t_0+T$; $x\in\mathbb{R}$,*

($\boldsymbol{A}_5$) $K(t,s,x)$ is nondecreasing with respect to x and $G(t,x)$ is nonincreasing with respect to x.

Let $x,y\in\mathcal{PC}([t_0,t_0+T],\mathbb{R})$ satisfying respectively the following inequalities

$$\begin{aligned}x(t) &< x(t_0)+\sum_{t_0<t_k<t}J_k(x(t_k))\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left\{\int_{t_0}^{s}K(s,\sigma,x(\sigma))\,d\sigma-G(s,x(s))\right\}ds\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}y(t) &\geq y(t_0)+\sum_{t_0<t_k<t}J_k(y(t_k))\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left\{\int_{t_0}^{s}K(s,\sigma,y(\sigma))\,d\sigma-G(s,y(s))\right\}ds.\end{aligned}\tag{2.10}$$

If $x(t_0)<y(t_0)$, $\Delta(x(t_k))<\Delta(y(t_k))$, $k=1,\ldots,m$, then

$$x(t)<y(t)\ \text{for every}\ \ t\in[t_0,t_0+T].\tag{2.11}$$

Proof. Suppose that the inequality (2.11) does not hold. Then, there exists some $\tau_1\in(t_0,t_0+T]$ such that

$$x(\tau_1)=y(\tau_1)\ .\tag{2.12}$$

Using the fact that $x(t_0) < y(t_0)$, $J_k(x(t_k)) < J_k(y(t_k))$, $k = 1, \ldots, m$, and the inequalities (2.9), (2.10) we get

$$\begin{aligned} y(\tau_1) &\geq y(t_0) + \sum_{t_0<t_k<\tau_1} J_k(y(t_k)) + \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_{t_0}^{\tau_1} (\tau_1 - s)^{\alpha-1} \left\{ \int_{t_0}^{s} K(s,\sigma,y(\sigma))\, d\sigma - G(s,y(s)) \right\} ds \\ &> x(t_0) + \sum_{t_0<t_k<\tau_1} J_k(x(t_k)) + \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_{t_0}^{\tau_1} (\tau_1 - s)^{\alpha-1} \left\{ \int_{t_0}^{s} K(s,\sigma,x(\sigma))\, d\sigma - G(s,x(s)) \right\} ds \\ &> x(\tau_1). \end{aligned}$$

This contradicts (2.14), which completes the proof. □

Remark 2.5. One of the two inequalities (2.9) or (2.10) strictly holds.

Remark 2.6. The condition on the jumps of the state functions $\Delta(x(t_k)) < \Delta(y(t_k))$, $k = 1, \ldots, m$, can be replaced by the nondecrease of $J_k(x)$, $k = 1, \ldots, m$ for $x \in \mathbb{R}$.

Lemma 2.7. *Let (A_1)-(A_4) hold and $x, y \in \mathcal{PC}([t_0, t_0+T], \mathbb{R})$ be solutions of (2.1)-(2.3) satisfying the following*

$${}^{c}D^{\alpha}_{t_0^+} x(t) <{}^{c} D^{\alpha}_{t_0^+} y(t), \ t \neq t_k, k = 1, \ldots, m, \quad t \in (t_0, t_0+T]. \tag{2.13}$$

If $x(t_0) < y(t_0)$; $\Delta(x(t_k)) < \Delta(y(t_k))$, $k = 1, \ldots, m$; then inequality (2.11) holds.

Proof. Suppose that the inequality (2.11) is not true. Then, there exists some $\tau_1 \in (t_0, t_0+T]$ such that

$$x(\tau_1) = y(\tau_1). \tag{2.14}$$

Using the fact that $x(t_0) < y(t_0)$, $J_k(x(t_k)) < J_k(y(t_k))$, $k = 1, \ldots, m$, and the inequalities (2.9), (2.10) we get

$$\begin{aligned} y(\tau_1) &\geq y(t_0) + \sum_{t_0<t_k<\tau_1} J_k(y(t_k)) + \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_{t_0}^{\tau_1} (\tau_1 - s)^{\alpha-1} \left\{ \int_{t_0}^{s} K(s,\sigma,y(\sigma))\, d\sigma - G(s,y(s)) \right\} ds \\ &> x(t_0) + \sum_{t_0<t_k<\tau_1} J_k(x(t_k)) + \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_{t_0}^{\tau_1} (\tau_1 - s)^{\alpha-1} \left\{ \int_{t_0}^{s} K(s,\sigma,x(\sigma))\, d\sigma - G(s,x(s)) \right\} ds \\ &> x(\tau_1). \end{aligned}$$

This contradicts (2.14) which completes the proof. □

Now, we can give the main result of this subsection dealing with existence of minimal and maximal solutions called extremal solutions.

Theorem 2.8. *Assume that* (A_1)*-*(A_5) *are satisfied.*
Then the problem (2.1)-(2.3) has extremal solutions on $[t_0, t_0+\beta]$ *for* $0<\beta\le T$.

Proof. First, prove the existence of a maximal solution. Consider the impulsive fractional initial value problem

$$\begin{cases} {}^{c}D^{\alpha}_{t_0^+}x(t)+G(t,x(t))=\int_{t_0}^{t}K(t,s,x(s))\,ds+\epsilon; \\ \quad t\neq t_k;\quad k=1,\ldots,m;\quad 0<\varepsilon\le \dfrac{b}{2(m+1)}, \\ x(t_0)=x_0+\varepsilon;\ t_0\ge 0\ ; \\ \Delta x(t_k)=J_k\left(x\left(t_k^-\right)\right)+\epsilon;\quad t_k\in[t_0,t_0+T],\ k=1,\ldots,m. \end{cases} \tag{2.15}$$

Define the closed bounded set in Ω

$$\Omega_\epsilon=\left\{x\in\mathcal{PC}([t_0,t_0+\beta],\mathbb{R})\ \text{ such that }\ \|x-(x_0+\epsilon)\|_{PC}\le b/2\right\},$$

for some $0<\beta\le T$ chosen such that

$$0<\sum_{t_0<t_k<\beta}a_k+\frac{mb}{2(m+1)}+\frac{\beta^\alpha}{\alpha\Gamma(\alpha)}\left[M_1\frac{\beta}{\alpha+1}+M_2+\frac{b}{2(m+1)}\right]\le\frac{b}{2}.$$

It is clear that all assumptions of Theorem 2.3 are satisfied, then problem (2.15) has a solution $x(t,\varepsilon)\in\mathcal{PC}([t_0,t_0+\beta],\mathbb{R})$. From Lemma 2.2 the solution of (2.15) satisfies the integral equation of the form

$$\begin{aligned} x(t,\epsilon) &= x(t_0,\varepsilon)+\sum_{t_0<t_k<t}J_k(x(t_k,\epsilon))+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} \\ &\quad\times\left\{\int_{t_0}^{s}K(s,\sigma,x(\sigma,\epsilon))\,d\sigma+\epsilon-G(s,x(s,\epsilon))\right\}ds, \end{aligned} \tag{2.16}$$

with $x(t_0,\varepsilon)=x_0+\epsilon$ and $J_k(x(t_k,\epsilon))=J_k(x(t_k))+\epsilon, k=1,\ldots,m$.

Let $\{x(t,\varepsilon)\}$ be a family of functions in $\mathcal{PC}([t_0,t_0+\beta],\ \mathbb{R})$ for $\epsilon>0$ satisfying (2.16). So, under the assumptions $(A_1)-(A_3)$ we have

$$\begin{aligned} |x(t,\epsilon)| &\le |x_0|+\epsilon+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\{[M_1(s-t_0)+\epsilon+M_2]\}\,ds+\sum_{t_0<t_k<\beta}(a_k+\epsilon) \\ &\le |x_0|+(m+1)\epsilon+\sum_{t_0<t_k<\beta}a_k+\frac{1}{\alpha\Gamma(\alpha)}\left[M_1\frac{\beta^{\alpha+1}}{\alpha+1}+\beta^\alpha M_2+\beta^\alpha\epsilon\right] \\ &\le |x_0|+\sum_{t_0<t_k<\beta}a_k+\frac{b}{2}+\frac{\beta^\alpha}{\Gamma(\alpha+1)}\left[M_1\frac{\beta}{\alpha+1}+M_2+\frac{b}{2(m+1)}\right] \\ &\le |x_0|+b. \end{aligned}$$

Then, $\{x(t,\varepsilon)\}$ is a uniformly bounded family in $\mathcal{PC}([t_0,t_0+\beta],\mathbb{R})$. It is also equicontinuous on $[t_0,t_0+\beta]$. Indeed, for $\tau_1,\,\tau_2\in[t_0,t_0+\beta]$ such that $\tau_1<\tau_2$ we have

$$|x(\tau_2,\epsilon)-x(\tau_1,\epsilon)|\leq\sum_{\tau_1\leq t_k<\tau_2}\left|J_k\left(x(t_k^-)\right)+\epsilon\right|$$

$$+\frac{1}{\Gamma(\alpha)}\left|\int_{t_0}^{\tau_2}(\tau_2-s)^{\alpha-1}\left\{\int_{t_0}^{s}K(s,\sigma,x(\sigma,\epsilon))\,d\sigma+\epsilon-G(s,x(s,\epsilon))\right\}ds\right.$$
$$\left.-\int_{t_0}^{\tau_1}(\tau_1-s)^{\alpha-1}\left\{\int_{t_0}^{s}K(s,\sigma,x(\sigma,\epsilon))\,d\sigma+\epsilon-G(s,x(s,\epsilon))\right\}ds\right|.$$

Then, by the same arguments used in the proof of Theorem 2.3, we obtain

$$\|x(\tau_2,\epsilon)-x(\tau_1,\epsilon)\|_{\mathcal{PC}}\to0\text{ when }\tau_1\to\tau_2.$$

On the other hand, we point out that for each ε_1, ε_2 such that $0<\varepsilon_1<\varepsilon_2\leq\varepsilon$ we have

$$x(t_0,\varepsilon_1)=x_0+\varepsilon_1<x_0+\varepsilon_2=x(t_0,\varepsilon_2),$$

and

$$\Delta(x(t_k,\varepsilon_1))=\ \Delta x(t_k)+\epsilon_1<\Delta x(t_k)+\epsilon_2=\Delta x(t_k,\epsilon_2),\ k=1,\ldots,m.$$

Let $F(t,x(t,\epsilon))=\int\limits_{t_0}^{t}K(t,s,x(s,\epsilon))\,ds-G(t,x(t,\epsilon))$, $F_\epsilon(t,x)=F(t,x)+\epsilon$ and $J_k\left(x\left(t_k^-\right)\right)+\varepsilon_1=J_k\left(x\left(t_k^-,\varepsilon_1\right)\right)=J_k(x(t_k,\varepsilon_1))$; we get

$$\begin{aligned}x(t,\varepsilon_1)\quad&<\quad x(t_0,\varepsilon_1)+\sum_{t_0<t_k<t}J_k(x(t_k,\varepsilon_1))+\frac{1}{\Gamma(\alpha)}\\&\qquad\times\int_{t_0}^{t}(t-s)^{\alpha-1}\left\{\int_{t_0}^{s}K(s,\sigma,x(\sigma,\epsilon_1))\,d\sigma+\varepsilon_2-G(s,x(s,\epsilon_1))\right\}ds\\&<\quad x(t_0,\varepsilon_1)+\sum_{t_0<t_k<t}J_k(x(t_k,\varepsilon_1))+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}F_{\varepsilon_2}(s,x(s,\varepsilon_1))\,ds\end{aligned}$$

and

$$x(t,\varepsilon_2)\geq x(t_0,\varepsilon_2)+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}F_{\varepsilon_2}(s,x(s,\varepsilon_2))\,ds+\sum_{t_0<t_k<t}J_k(x(t_k,\varepsilon_2)).$$

We infer from Lemma 2.4 that $x(t,\varepsilon_1)<x(t,\varepsilon_2)$, for $t\in[t_0,t_0+\beta]$.

We conclude by Arzela-Ascoli lemma there is a decreasing sequence $\{\varepsilon_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $x(t,\varepsilon_n)$ satisfies the form

$$\begin{aligned} x(t,\varepsilon_n) &= x_0 + \varepsilon_n + \sum_{t_0<t_k<t} (J_k(x(t_k)) + \varepsilon_n) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \\ &\quad \times \left\{ \int_{t_0}^{s} K(s,\sigma,x(\sigma,\varepsilon_n))\, d\sigma + \varepsilon_n - G(s,x(s,\varepsilon_n)) \right\} ds. \end{aligned}$$

Since K and G are uniformly continuous, we get the following integral equation by letting $n \to \infty$

$$\begin{aligned} x(t) &= x_0 + \sum_{t_0<t_k<t} J_k(x(t_k)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \\ &\quad \times \left\{ \int_{t_0}^{s} K(s,\sigma,x(\sigma))\, d\sigma - G(s,x(s)) \right\} ds. \end{aligned}$$

Then $\lim_{n\to\infty} x(t,\varepsilon_n) = x(t)$ uniformly on $[t_0, t_0+\beta]$ with $x(t_0) = x_0$. Therefore, $x(t)$ is a solution of (2.1)-(2.3) on $[t_0, t_0+\beta]$. Now, we have to prove that $x(t)$ is the maximal solution of (2.1)-(2.3)on $[t_0, t_0+\beta]$. Let $y(t)$ be any solution of (2.1)-(2.3) on $[t_0, t_0+\beta]$. It is clear that

$$\begin{aligned} y(t_0) &= x_0 < x_0 + \varepsilon = x(t_0,\varepsilon); \\ \text{and } \Delta\, y(t_k) &= \Delta x(t_k) < \Delta x(t_k) + \epsilon = \Delta x(t_k,\varepsilon),\ k = 1,\ldots,m. \end{aligned}$$

The fact that $y(t)$ satisfies (2.1) implies that

$${}^cD_{t_0}^{\alpha} y(t) <{}^c D_{t_0}^{\alpha} x(t,\varepsilon),\ t \neq t_k, k = 1,\ldots,m, \quad t \in [t_0, t_0+\beta];\ 0 < \varepsilon \leq \frac{b}{2(m+1)}.$$

Then we have by Lemma 2.7 , $y(t) < x(t,\varepsilon)$, $t \in [t_0, t_0+\beta]$. Since the maximal solution is unique, then $\lim_{\varepsilon\to 0} x(t,\varepsilon) = x(t)$ uniformly on $[t_0, t_0+\beta]$.

Likewise, we can prove by the same arguments the existence of a unique minimal solution; this completes the proof. □

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