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CALCULATION METHOD FOR QUADRATIC PROGRAMMING PROBLEM IN HILBERT SPACES, PARTIALLY ORDERED BY CONE WITH EMPTY INTERIOR

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Abstract. In the article, a numerical method for convex programming problem (with linear inequality) in Hilbert spaces is given. Firstly, by Khun-Tucker conditions problem is reduced to minimize a convex funtional under nonnegative variables. Then, last problem is solved by coordinate descent method.

1. Introduction

Various quadratic programming problems in finite dimensional spaces have been solved by using of simplex method. Unfortunately there is no an analog of simplex method in infinite dimensional spaces and naturally the solution of quadratic programming problems in infinite dimensional spaces requires different approaches. During of the last years many papers are devoted to the solution of quadratic programming problems in infinite dimensional spaces [2]-[5].

In this paper we present a method of solution for quadratic programming problems in Hilbert spaces partially ordered by a cone with empty interior. By using Kuhn-Tucker condition the problem is reduced to simplier problem solvable by known methods.

2. Formulation of the problem.

Let C_{ij} and A_{ij} be linear bounded operator from Hilbert space H into itself. The space H is partially ordered by convex closed cone K.

Consider the following quadratic programming problem:

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$$\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i, C_{ij}x_j) + \sum_{j=1}^{n}(p_i, x_j) \to min$$

$$\sum_{j=1}^{n} A_{ij}x_j \le b_i,\ i = 1, \ldots, m$$

where $b_i \in H, p_i \in H$.

The problem can be reformulated as

$$(x, Cx) + (p, x) \to: min \tag{2.1}$$

$$Ax \le b \tag{2.2}$$

where $C = [c_{ij}]$ is a $n \times n$ matrix and $A = [a_{ij}]$ is a $m \times n$ matrix.

Suppose that the matrix C is self-conjugated and positively determined: there is a positive number γ such that for any $x \in H^n$

$$(x, Cx) \ge \gamma ||x||^2$$

Definition. The restriction (2.2) is said to be strongly simultaneous, if there is a real number $\epsilon_0 > 0$ such that for all $\bar{b}$ from the set $M = \{\bar{b} \in H^m : ||\bar{b} - b|| \le \epsilon_0\}$ the condition $Ax \le \bar{b}$ is consistent.

3. **The description of the method.**

Suppose that the restriction (2.2) is strongly simultaneous and for each $i, 1 \le i \le m$ there is j_i such that the operator A_{ij_i} satisfies the following condition:

there is $\gamma_{ij_i} > 0$ such that for any $x \in H$

$$||A_{ij_i}|| \ge \gamma_{ij_i} ||x||$$

Since the strongly simultaneity condition is satisfied we can derive Kuhn-Tucker conditions [1] for the problem (2.1), (2.2):

$$2Cx + p + A^*w = 0 \tag{3.1}$$

$$Ax + y = 0 \tag{3.2}$$

$$(y, u) = 0 \tag{3.3}$$

$$y \ge 0, u \ge 0 \tag{3.4}$$

From (3.1) we get

$$x = -\frac{1}{2}C^{-1}(A^*u + p) \tag{3.5}$$

By inserting x from(3.5) into (3.2) we get the following Kuhn-Tucker conditions:

$$2Gu + h - y = 0 \tag{3.6}$$

$$(y, u) = 0 \tag{3.7}$$

$$y \geq 0, u \geq 0 \tag{3.8}$$

where $G = \frac{1}{4}AC^{-1}A^*, h = \frac{1}{2}AC^{-1}p + b$.

Clearly G is a nonnegative operator: for any $z \in H^m$ we have $(z, Gz) \geq 0$. Diagonal element of matrix G are positively determined operators, but the operator G need not to be positively determined operator. Indeed,

$$g_{ii} = (A_{i_1}, \ldots, A_{i_n})C^{-1}\begin{pmatrix} A_{i_1} \\ \vdots \\ A_{i_n} \end{pmatrix}$$

Let us denote $\begin{pmatrix} A_{i_1} \\ \vdots \\ A_{i_n} \end{pmatrix}$ by A_i^*. Then

$$(u_i, g_{ii}u_i) = (A_i^*u_i, C^{-1}A_i^*u_i) \geq \gamma_1||A_i^*u_i||^2$$

$$= \gamma_1(\sum_{j=1}^{n}||A_i^*u_i||)^2 \geq \gamma_1||A_{ij_i}^*u_i||^2 \geq \gamma\gamma_{ij_i}||u_i||^2$$

Conditions (3.6) − (3.8) are Kuhn-Tucker conditions for the problem

$$\phi(u) = (u, Gu) + (h, u) \to min \tag{3.9}$$

$$u \geq 0 \tag{3.10}$$

For the solution of (3.9),(3.10) problem we propose the following method of non-coordinate descent. We choose an initial $u^{(0)} \geq 0$. Components of $n - th$ iteration are determined by minimization of $\phi(u)$ with respect to u_i at $u_i \geq 0$.The other components take their values at the last iteration. Clearly,

$$\phi(u^{k+1}) \leq \phi(u^k), k = 1, 2, \dots \tag{3.11}$$

Theorem 3.1. *The sequence $\{u^k\}$ converges to the solution of the problem* (3.9),(3.10).

Proof. Let $u^{k+1} = Pu^{(k)}$. It can be easily shown that the operator P is continuous. Due to (3.11)

$$\phi(Pu) \leq \phi(u)$$

Let us show that $\phi(Pu) = \phi(u)$ if and only if $u = \tilde{u}$, where $\tilde{u}$ is a solution of (3.9),(3.10). The necessity of this condition is obvious. Let us prove the sufficiency. From $\phi(P\tilde{u}) = \phi(\tilde{u})$ it follows that for all $i = 1, \dots, m$

$$min_{u_i \geq 0} : \phi(\tilde{u_1}, \dots, \tilde{u_{i-1}}, u_i, \dots, \tilde{u_m}) = \phi(\tilde{u_1}, \dots, \tilde{u_m})$$

Now

$$\frac{\partial \phi(\tilde{u_1}, \dots, \tilde{u_m})}{\partial u_i} \geq 0, : i = 1, \dots, m \tag{3.12}$$

$$(\frac{\partial \phi(\tilde{u_1}, \dots, \tilde{u_m})}{\partial u_i}, u_i) = 0, : i = 1, \dots, m \tag{3.13}$$

The conditions(3.12), (3.13) are Kuhn-Tucker conditions of the problem (3.9),(3.10). Thus, $\bar{u}$ is a solution of the problem (3.9),(3.10).

Proof. The sequence $\{u^{(k)}\}$ is a subsequence of the sequence $\{q^p\}$, where $q^0 = u^{(0)}$ and q^p is a one coordinate descent of $q^{(p-1)}$. Since $\phi(u)$ is a strongly convex functional of u_i, the following inequality is held: □

$$||q^p - q^{(p-1)}||^2 \leq \frac{2}{\rho}(\phi(q^{p-1}) - \phi(q^p)), : p = 1, \dots \tag{3.14}$$

where $0 < \rho < \gamma : min_{1 \leq i \leq m} : \{\gamma_{iy_i}\}$. The sequence $\phi(q^{(p)})$ converges as a monotonically decreasing and bounded from below sequence. From (3.14) it follows that the sequence $q^{(p)}$ also converges. The sequence $\{u^{(k)}\}$ as a subsequence of convergent sequence $\{q^{(p)}\}$ also converges. Let u^* be a limit of the sequence $\{u^{(k)}\}$. Let us show that u^* is a solution of the problem(3.9),(3.10).

The continuity of $\phi(u)$ implies that $\phi(u^{(k)}) \to \phi(u^{(*)})$. Now $\phi(u^{(k+1)}) = \phi(Pu^{(k)}) \to \phi(Pu^{(*)})$. Since the limit is unique we get $\phi(u^{(*)}) = \phi(Pu^{(*)})$.

The proof of the theorem is completed.

□

As it was mentioned above, in order to find components of each iteration one have to minimize $\phi(u)$ with respect to u_i. For this purpose one of the known methods can be applied (for example the method of gradient projection).

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