

ON LORENTZIAN TRANS-SASAKIAN MANIFOLDS

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ABSTRACT. The object of the present paper is to study the Trans-Sasakian structure on a manifold with Lorentzian metric. Several interesting results are obtained on the manifold. Also conformally flat Lorentzian Trans-Sasakian manifolds have been studied. Next, in three- dimensional Lorentzian Trans-Sasakian manifolds, explicit formulae for Ricci operator, Ricci tensor and curvature tensor are obtained. Also it is proved that a three-dimensional Lorentzian Trans-Sasakian manifold of type (α, β) is locally ϕ - symmetric if and only if the scalar curvature r is constant provided α and β are constants. Finally, we give some examples of three-dimensional Lorentzian Trans-Sasakian manifold.

1. INTRODUCTION

Let M be an odd dimensional manifold with Riemannian metric g . It is well known that an almost contact metric structure (ϕ, ξ, η) (with respect to g) can be defined on M by a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1- form η . If M has a Sasakian structure (Kenmotsu structure), then M is called a Sasakian manifold (Kenmotsu manifold). Sasakian manifolds and Kenmotsu manifolds have been studied by several authors.

In the classification of Gray and Hervella [8] of almost Hermitian manifolds there appears a class, W_4 , of Hermitian manifolds which are closely related to locally conformally Kaehler manifolds. An almost contact metric structure (ϕ, ξ, η, g) on M is Trans-Sasakian [17] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4 , where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt}),$$

for all vector fields X on M , f is a smooth function on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [2]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \quad (1.1)$$

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for smooth functions α and β on M . Hence we say that the Trans-Sasakian structure is of type (α, β) . In particular, it is normal and it generalizes both α -Sasakian and β -Kenmotsu structures. From the formula (1.1) one easily obtains

$$\nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi). \quad (1.2)$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \quad (1.3)$$

In 1981, Janssens and Vanhecke introduced the notion of α -Sasakian and β -Kenmotsu manifolds where α and β are non zero real numbers. It is known that [6] Trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic ([1], [2]), β -Kenmotsu ([6]) and α -Sasakian ([6]) respectively. The local structure of Trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by Marrero [10]. He proved that a Trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α -Sasakian or β -Kenmotsu manifold. Trans-Sasakian manifolds have been studied by several authors ([3], [4], [5], [11], [18]).

Let (x, y, z) be cartesian co-ordinates in \mathbb{R}^3 , then (ϕ, ξ, η, g) given by

$$\xi = \frac{\partial}{\partial z}, \eta = dz - ydx,$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, \quad g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix}$$

is a Trans-Sasakian structure of type $(\frac{-1}{2e^z}, \frac{1}{2})$ in \mathbb{R}^3 [2]. In general, in a three-dimensional K -contact manifold with structure tensors (ϕ, ξ, η, g) for a non-constant function f , if we define $\tilde{g} = fg + (1 - f)\eta \otimes \eta$; then $(\phi, \xi, \eta, \tilde{g})$ is a Trans-Sasakian structure of type $(\frac{1}{f}, \frac{1}{2}\xi(\ln f))$ [10].

Let M be a differentiable manifold. When M has a Lorentzian metric g , that is, a symmetric non degenerate $(0, 2)$ tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case give interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1- dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric.

Therefore, it is very natural and interesting idea to define both a Trans-Sasakian structure and a Lorentzian metric on an odd dimensional manifold.

The paper is organized as follows. In Section 1, we give a brief account of Lorentzian Trans-Sasakian manifolds. After preliminaries, some basic results are given. In Section 4, we study conformally flat Lorentzian Trans-Sasakian manifolds.

In the next section, explicit formulae for Ricci operator, Ricci tensor and curvature tensor are obtained for three-dimensional Trans-Sasakian manifolds. Also it is proved that a three-dimensional Lorentzian Trans-Sasakian manifold of type (α, β) is locally ϕ - symmetric if and only if the scalar curvature r is constant provided α and β are constants. Finally we construct some examples of three-dimensional Lorentzian Trans-Sasakian manifolds.

2. LORENTZIAN TRANS-SASAKIAN MANIFOLDS

A differentiable manifold M of dimension $(2n + 1)$ is called a Lorentzian Trans-Sasakian manifold if it admits a $(1, 1)$ tensor field ϕ , a contravariant vector field ξ , a covariant vector field η and the Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \quad (2.1)$$

$$\phi^2 = I + \eta \otimes \xi, \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.3)$$

$$g(X, \xi) = \eta(X), \phi\xi = 0, \eta(\phi X) = 0, \quad (2.4)$$

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \quad (2.5)$$

for all $X, Y \in T(M)$.

Also a Lorentzian Trans-Sasakian manifold M satisfies

$$\nabla_X \xi = -\alpha(\phi X) - \beta(X + \eta(X)\xi), \quad (2.6)$$

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \quad (2.7)$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g .

If $\alpha = 0$ and $\beta \in \mathbb{R}$, the set of real numbers, then the manifold reduces to a Lorentzian β -Kenmotsu manifold studied by Funda Yaliniz, Yildiz, and Turan [20]. If $\beta = 0$ and $\alpha \in \mathbb{R}$, then the manifold reduces to a Lorentzian α - Sasakian manifold studied by Yildiz, Turan and Murathan [21]. If $\alpha = 0$ and $\beta = 1$, then the manifold reduces to a Lorentzian Kenmotsu manifold introduced by Mihai, Oiaga and Rosca [15]. Furthermore, if $\beta = 0$ and $\alpha = 1$, then the manifold reduces to a Lorentzian Sasakian manifold studied by Ikawa and Erdogan [15]. Also Lorentzian para contact manifolds were introduced by Matsumoto [12] and further studied by the authors ([13],[14],[16]). Trans Lorentzian para Sasakian manifolds have been used by Gill and Dube [7].

3. SOME BASIC RESULTS

In this section, we prove some Lemmas which are needed in the rest of the sections.

Lemma 3.1. *In a Lorentzian Trans-Sasakian manifold, we have*

$$\begin{aligned} R(X, Y)\xi &= (\alpha^2 + \beta^2)(\eta(Y)X - \eta(X)Y) \\ &\quad + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) + (Y\alpha)\phi X \\ &\quad - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \quad (3.1)$$

where R is the curvature tensor.

Proof. We have

$$\begin{aligned} \nabla_X \nabla_Y \xi &= \nabla_X(-\alpha(\phi Y) - \beta(Y + \eta(Y)\xi)) \\ &= -(X\alpha)\phi Y - \alpha\nabla_X(\phi Y) - (X\beta)\phi^2 Y \\ &\quad - \beta\nabla_X Y - \beta(X\eta(Y))\xi + \alpha\beta\eta(Y)\phi X \\ &\quad + \beta^2\eta(Y)X + \beta^2\eta(X)\eta(Y)\xi, \end{aligned}$$

where (2.2) and (2.6) have been used. Hence, in view of the above equation and (2.6), we get

$$\begin{aligned} R(X, Y)\xi &= \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X, Y]}\xi \\ &= -(X\alpha)\phi Y + (Y\alpha)\phi X - \alpha((\nabla_X \phi Y) - (\nabla_Y \phi X)) \\ &\quad - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X - \beta((\nabla_X \eta)Y - (\nabla_Y \eta)X)\xi \\ &\quad + \alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) + \beta^2(\eta(Y)X - \eta(X)Y), \end{aligned}$$

which in view of (2.5) and (2.7) gives (3.1). \square

Lemma 3.2. *For a Lorentzian Trans-Sasakian manifold, we have*

$$\eta(R(X, Y)Z) = (\alpha^2 + \beta^2)(g(X, Z)\eta(Y) - g(Y, Z)\eta(X)). \quad (3.2)$$

Proof. We have from (3.1),

$$\begin{aligned} g(R(X, Y)\xi, Z) &= (\alpha^2 + \beta^2)(\eta(Y)g(X, Z) - \eta(X)g(Y, Z)) \\ &\quad + 2\alpha\beta(\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)) + (Y\alpha)g(\phi X, Z) \\ &\quad - (X\alpha)g(\phi Y, Z) + (Y\beta)g(\phi^2 X, Z) - (X\beta)g(\phi^2 Y, Z), \end{aligned}$$

Now interchanging ξ and Z in the above equation, we get

$$\begin{aligned} -g(R(X, Y)Z, \xi) &= (\alpha^2 + \beta^2)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)) \\ &\quad + 2\alpha\beta(\eta(Y)g(\phi X, \xi) - \eta(X)g(\phi Y, \xi)) + (Y\alpha)g(\phi X, \xi) \\ &\quad - (X\alpha)g(\phi Y, \xi) + (Y\beta)g(\phi^2 X, \xi) - (X\beta)g(\phi^2 Y, \xi). \end{aligned}$$

After simplification, we find,

$$g(R(X, Y)Z, \xi) = (\alpha^2 + \beta^2)(g(X, Z)\eta(Y) - g(Y, Z)\eta(X)),$$

which gives (3.2). \square

Lemma 3.3. *For a Lorentzian Trans-Sasakian manifold, we have*

$$R(\xi, Y)\xi = (\alpha^2 + \beta^2 - \xi\beta)\phi^2 Y + (2\alpha\beta - \xi\alpha)\phi Y. \quad (3.3)$$

Proof. Replacing X by ξ in (3.1), we get (3.3). \square

Lemma 3.4. *In a $(2n + 1)$ - dimensional Lorentzian Trans-Sasakian manifold, we have*

$$\begin{aligned} S(X, \xi) &= (2n(\alpha^2 + \beta^2) - \xi\beta)\eta(X) + (2n - 1)(X\beta) \\ &\quad - (\phi X)\alpha + \psi(2\alpha\beta\eta(X) + X\alpha), \end{aligned} \quad (3.4)$$

$$\begin{aligned} Q\xi &= (2n(\alpha^2 + \beta^2) - \xi\beta)\xi + (2n - 1)\text{grad}\beta \\ &\quad - \phi(\text{grad}\alpha) + \psi(2\alpha\beta\xi + \text{grad}\alpha), \end{aligned} \quad (3.5)$$

where S is the Ricci curvature and Q is the Ricci operator given by

$$S(X, Y) = g(QX, Y) \quad \text{and} \quad \psi = \sum_{i=1}^{2n+1} \epsilon_i g(\phi e_i, e_i).$$

Proof. Let M be an $(2n + 1)$ - dimensional Lorentzian Trans-Sasakian manifold. Then the Ricci tensor S of the manifold M is defined by

$$S(X, Y) = \sum_{i=1}^{2n+1} \epsilon_i g(R(e_i, X)Y, e_i),$$

where $\epsilon_i = g(e_i, e_i)$, $\epsilon_i = \pm 1$. From (3.1), we have

$$\begin{aligned} S(X, \xi) &= (\alpha^2 + \beta^2)[\eta(X) \sum_{i=1}^{2n+1} g(e_i, e_i)g(e_i, e_i) - \sum_{i=1}^{2n+1} \eta(e_i)g(e_i, e_i)g(X, e_i)] \\ &\quad + 2\alpha\beta[\eta(X) \sum_{i=1}^{2n+1} g(e_i, e_i)g(\phi e_i, e_i) - \sum_{i=1}^{2n+1} \eta(e_i)g(e_i, e_i)g(\phi X, e_i)] \\ &\quad - \sum_{i=1}^{2n+1} (e_i\alpha)g(e_i, e_i)g(\phi X, e_i) + \sum_{i=1}^{2n+1} (X\alpha)g(e_i, e_i)g(\phi e_i, e_i) \\ &\quad - \sum_{i=1}^{2n+1} (e_i\beta)g(e_i, e_i)g(\phi^2 X, e_i) + \sum_{i=1}^{2n+1} (X\beta)g(e_i, e_i)g(\phi^2 e_i, e_i) \\ &= (2n(\alpha^2 + \beta^2) - \xi\beta)\eta(X) + (2n - 1)(X\beta) \\ &\quad - (\phi X)\alpha + \psi(2\alpha\beta\eta(X) + X\alpha) \end{aligned}$$

and hence from (3.4), we get (3.5). \square

Remark 3.5. If in a $(2n + 1)$ - dimensional Lorentzian Trans-Sasakian manifold of type (α, β) we consider $\phi(\text{grad}\alpha) = (2n - 1)\text{grad}\beta$, then

$$\begin{aligned}\xi\beta &= g(\xi, \text{grad}\beta) = \frac{1}{2n-1}g(\xi, \phi(\text{grad}\alpha)) \\ &= \frac{1}{2n-1}\eta(\phi(\text{grad}\alpha)) = 0\end{aligned}$$

and

$$\begin{aligned}X\beta &= g(X, \text{grad}\beta) = \frac{1}{2n-1}g(X, \phi(\text{grad}\alpha)) \\ &= \frac{1}{2n-1}g(\phi X, (\text{grad}\alpha)) = \frac{1}{2n-1}(\phi X)\alpha\end{aligned}$$

and hence (3.4) and (3.5) are reduced to

$$S(X, \xi) = 2n(\alpha^2 + \beta^2)\eta(X) + \psi(2\alpha\beta\eta(X) + X\alpha) \quad (3.6)$$

and

$$Q\xi = (2n(\alpha^2 + \beta^2) - \xi\beta)\xi + \psi(2\alpha\beta\xi + \text{grad}\alpha), \quad (3.7)$$

respectively.

4. CONFORMALLY FLAT LORENTZIAN TRANS-SASAKIAN MANIFOLDS

In this section we consider conformally flat Lorentzian Trans-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ ($n > 1$). The conformal curvature tensor C is given by

$$\begin{aligned}C(X, Y)Z &= R(X, Y)Z - \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY] + \frac{r}{(2n)(2n-1)}[g(Y, Z)X - g(X, Z)Y], \quad (4.1)\end{aligned}$$

where r is the scalar curvature of M .

For conformally flat manifold, we have $C(X, Y)Z = 0$ for $n > 1$ and hence from (4.1) we have

$$\begin{aligned}\tilde{R}(X, Y, Z, W) &= \frac{1}{2n-1}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \\ &\quad + g(Y, Z)S(X, W) - g(X, Z)S(Y, W)] \\ &\quad - \frac{r}{(2n)(2n-1)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)], \quad (4.2)\end{aligned}$$

where $g(R(X, Y)Z, U) = \tilde{R}(X, Y, Z, U)$. Setting $W = \xi$ in (4.2) we get

$$\begin{aligned}
\eta(R(X, Y)Z) &= \frac{1}{2n-1}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y) \\
&\quad + g(Y, Z)S(X, \xi) - g(X, Z)S(Y, \xi)] \\
&\quad - \frac{r}{(2n)(2n-1)}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \quad (4.3)
\end{aligned}$$

Replacing Y by ξ in (4.3) and using (3.2) and (3.6), we get

$$\begin{aligned}
S(X, Z) &= [(\alpha^2 + \beta^2)(1 - 4n) + \frac{r}{2n} + (\xi\alpha - 2\alpha\beta)\psi]g(X, Z) \\
&\quad + [(\alpha^2 + \beta^2)(1 - 6n) + \frac{r}{2n} - 4\alpha\beta\psi]\eta(X)\eta(Z) \\
&\quad - [\eta(Z)(X\alpha) + \eta(X)(Z\alpha)]\psi. \quad (4.4)
\end{aligned}$$

This leads to the following:

Theorem 4.1. *A conformally flat Lorentzian Trans Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ ($n > 1$) is an η -Einstein manifold provided $\psi = \text{trace}\phi = 0$ and $\phi(\text{grad}\alpha) = (2n-1)\text{grad}\beta$.*

Corollary 1. *A conformally flat Lorentzian β -Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ ($n > 1$) is an η -Einstein manifold.*

5. THREE- DIMENSIONAL LORENTZIAN TRANS- SASAKIAN MANIFOLDS

Since the conformal curvature tensor vanishes in a three-dimensional Riemannian manifold, therefore we get

$$\begin{aligned}
R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \\
&\quad - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y], \quad (5.1)
\end{aligned}$$

where Q is the Ricci operator, that is, $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of the manifold.

From Lemma 2.4, in a three- dimensional Lorentzian Trans-Sasakian manifold we have

$$\begin{aligned}
S(X, \xi) &= (2(\alpha^2 + \beta^2) - \xi\beta)\eta(X) + (X\beta) \\
&\quad - (\phi X)\alpha + \psi(2\alpha\beta\eta(X) + X\alpha), \quad (5.2)
\end{aligned}$$

$$\begin{aligned}
Q\xi &= (2(\alpha^2 + \beta^2) - \xi\beta)\xi + \text{grad}\beta \\
&\quad - \phi(\text{grad}\alpha) + \psi(2\alpha\beta\xi + \text{grad}\alpha). \quad (5.3)
\end{aligned}$$

Now, in the following theorem, we obtain an expression for Ricci operator in a three-dimensional Lorentzian Trans-Sasakian manifold.

Theorem 5.1. *In a three- dimensional Lorentzian Trans Sasakian manifold, the Ricci operator is given by*

$$\begin{aligned}
QX &= \left(\frac{r}{2} + \xi\beta - (\alpha^2 + \beta^2) + \psi(\xi\alpha - 2\alpha\beta)\right)X \\
&+ \left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi\right)\eta(X)\xi \\
&- \eta(X)(\text{grad}\beta - \phi(\text{grad}\alpha) + \psi(\text{grad}\alpha)) - (X\beta - (\phi X)\alpha + \psi(X\alpha))\xi \\
&+ (2\alpha\beta - \xi\alpha)\phi X. \tag{5.4}
\end{aligned}$$

Proof. For a three- dimensional Lorentzian Trans Sasakian manifold, from (5.1) and (5.2), we have

$$\begin{aligned}
R(X, Y)\xi &= \eta(Y)QX - \eta(X)QY \\
&- \left(\frac{r}{2} + \xi\beta - 2(\alpha^2 + \beta^2) - 2\alpha\beta\psi\right)[X\eta(Y) - Y\eta(X)] \\
&+ (Y\beta - (\phi Y)\alpha + (Y\alpha)\psi)X - (X\beta - (\phi X)\alpha + (X\alpha)\psi)Y. \tag{5.5}
\end{aligned}$$

In view of (3.1) and (5.5), we obtain

$$\begin{aligned}
2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) &+ (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y \\
&= \eta(Y)QX - \eta(X)QY - \left(\frac{r}{2} + \xi\beta - (\alpha^2 + \beta^2) - 2\alpha\beta\psi\right) \\
&[\eta(Y)X - \eta(X)Y] + (Y\beta - (\phi Y)\alpha + (Y\alpha)\psi)X \\
&- (X\beta - (\phi X)\alpha + (X\alpha)\psi)Y.
\end{aligned}$$

Putting $Y = \xi$ in the above equation, we get (5.4) . □

Corollary 2. *In a three- dimensional Lorentzian Trans Sasakian manifold, Ricci tensor and curvature tensor are given respectively by*

$$\begin{aligned}
S(X, Y) &= \left(\frac{r}{2} + \xi\beta - (\alpha^2 + \beta^2) + \psi(\xi\alpha - 2\alpha\beta)\right)g(X, Y) \\
&+ \left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi\right)\eta(X)\eta(Y) \\
&+ \eta(X)[-Y\beta + (\phi Y)\alpha - \psi(Y\alpha)] - \eta(Y)(X\beta - (\phi X)\alpha + \psi(X\alpha)) \\
&+ (2\alpha\beta - \xi\alpha)g(\phi X, Y). \tag{5.6}
\end{aligned}$$

and

$$\begin{aligned}
R(X, Y)Z &= \left(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 + \beta^2) + 2\psi(\xi\alpha - 2\alpha\beta)\right)[g(Y, Z)X - g(X, Z)Y] \\
&+ g(Y, Z)\left[\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi\right)\eta(X)\xi \right. \\
&+ \eta(X)(\phi(\text{grad}\alpha) - \psi(\text{grad}\alpha) - \text{grad}\beta) - (X\beta - (\phi X)\alpha + \psi(X\alpha))\xi] \\
&+ g(X, Z)\left[\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi\right)\eta(Y)\xi \right. \\
&+ \eta(Y)(\phi(\text{grad}\alpha) - \psi(\text{grad}\alpha) - \text{grad}\beta) - (Y\beta - (\phi Y)\alpha + \psi(Y\alpha))\xi] \\
&+ \left[\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi\right)\eta(Y)\eta(Z) \right. \\
&+ \eta(Y)(-Z\beta + (\phi Z)\alpha - \psi(Z\alpha)) - \eta(Z)(Y\beta - (\phi Y)\alpha + \psi(Y\alpha))]X \\
&- \left[\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi\right)\eta(X)\eta(Z) \right. \\
&+ \eta(X)(-Z\beta + (\phi Z)\alpha - \psi(Z\alpha)) - \eta(Z)(X\beta - (\phi X)\alpha + \psi(X\alpha))]Y \\
&+ (2\alpha\beta - \xi\alpha)[g(\phi Y, Z)X - g(\phi X, Z)Y]. \tag{5.7}
\end{aligned}$$

Equation (5.6) follows from (5.4). Using (5.4) and (5.6) in (5.1), the curvature tensor in a three- dimensional Lorentzian Trans-Sasakian manifold is given by (5.7).

6. LOCALLY ϕ - SYMMETRIC THREE-DIMENSIONAL LORENTZIAN TRANS-SASAKIAN MANIFOLDS WITH *trace* $\phi = \psi = 0$

The notion of locally ϕ -symmetry was first introduced by T.Takahashi [19] on a Sasakian manifold. In this paper we study locally ϕ - symmetric three-dimensional Lorentzian Trans-Sasakian manifolds.

Definition 6.1. A three-dimensional Lorentzian Trans-Sasakian manifold is said to be locally ϕ - symmetric if

$$\phi^2(\nabla_W R)(X, Y)Z = 0, \tag{6.1}$$

where W, X, Y, Z are horizontal vector fields, that is W, X, Y, Z are orthogonal to ξ .

Let M be a three- dimensional Lorentzian Trans-Sasakian manifold with $trace\phi = \psi = 0$. Then its curvature tensor is given by

$$\begin{aligned}
R(X, Y)Z &= \left(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 + \beta^2)\right)[g(Y, Z)X - g(X, Z)Y] \\
&+ g(Y, Z)\left[\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2)\right)\eta(X)\xi \right. \\
&+ \eta(X)(\phi(grad\alpha) - grad\beta) - (X\beta - (\phi X)\alpha)\xi] \\
&+ g(X, Z)\left[\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2)\right)\eta(Y)\xi \right. \\
&+ \eta(Y)(\phi(grad\alpha) - grad\beta) - (Y\beta - (\phi Y)\alpha)\xi] \\
&+ \left[\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2)\right)\eta(Y)\eta(Z) \right. \\
&+ \eta(Y)(-Z\beta + (\phi Z)\alpha) - \eta(Z)(Y\beta - (\phi Y)\alpha)]X \\
&- \left[\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2)\right)\eta(X)\eta(Z) \right. \\
&+ \eta(X)(-Z\beta + (\phi Z)\alpha) - \eta(Z)(X\beta - (\phi X)\alpha)]Y \\
&+ (2\alpha\beta - \xi\alpha)[g(\phi Y, Z)X - g(\phi X, Z)Y]. \tag{6.2}
\end{aligned}$$

Differentiating (6.2) we get

$$\begin{aligned}
(\nabla_W R)(X, Y)Z &= \left[\frac{dr(W)}{2} + 2(\nabla_W(\xi\beta)) - 4(d\alpha(W) + d\beta(W))\right] \\
&[g(Y, Z)X - g(X, Z)Y] + g(Y, Z)\left[\left(\frac{dr(W)}{2} + (\nabla_W(\xi\beta)) \right. \right. \\
&- 6(d\alpha(W) + d\beta(W)))\eta(X)\xi + \left(\frac{r}{2} + \xi\beta - 3(\alpha^2 \right. \\
&+ \beta^2))((\nabla_W\eta)(X)\xi + \eta(X)(\nabla_W\xi)) \\
&+ (\nabla_W\eta)(X)(\phi(grad\alpha) - grad\beta) + \eta(X)(\nabla_W(\phi(grad\alpha) - grad\beta)) \\
&+ (\nabla_W(X\beta - (\phi X)\alpha))\xi + (X\beta - (\phi X)\alpha)\nabla_W\xi] \\
&- g(X, Z)\left[\left(\frac{dr(W)}{2} + (\nabla_W(\xi\beta)) - 6(d\alpha(W) + d\beta(W))\right)\eta(Y)\xi \right. \\
&+ \left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2))((\nabla_W\eta)(Y)\xi + \eta(Y)(\nabla_W\xi)) \right. \\
&+ (\nabla_W\eta)(Y)(\phi(grad\alpha) - grad\beta) + \eta(Y)(\nabla_W(\phi(grad\alpha) - grad\beta)) \\
&+ (\nabla_W(Y\beta - (\phi Y)\alpha))\xi + (Y\beta - (\phi Y)\alpha)\nabla_W\xi] \\
&- Y[(\nabla_W(Y\beta - (\phi Y)\alpha))\eta(Z) + (Y\beta - (\phi Y)\alpha)(\nabla_W\eta)Z \\
&+ \nabla_W(Z\beta - (\phi Z)\alpha)\eta(Y) + (Z\beta - (\phi Z)\alpha)(\nabla_W\eta)Y \\
&- \left(\frac{dr(W)}{2} + (\nabla_W(\xi\beta)) - 6(d\alpha(W) + d\beta(W))\right)\eta(Y)\eta(Z)
\end{aligned}$$

$$\begin{aligned}
& -\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2)\right) \\
& ((\nabla_W \eta)Y\eta(Z) + \eta(Y)(\nabla_W \eta)Z) \\
& + X[(\nabla_W(X\beta - (\phi X)\alpha))\eta(Z) + (X\beta - (\phi X)\alpha)(\nabla_W \eta)Z \\
& + \nabla_W(Z\beta - (\phi Z)\alpha)\eta(X) + (Z\beta - (\phi Z)\alpha)(\nabla_W \eta)X \\
& - \left(\frac{dr(W)}{2} + (\nabla_W(\xi\beta)) - 6(d\alpha(W) + d\beta(W))\right)\eta(X)\eta(Z) \\
& - \left(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2)\right) \\
& ((\nabla_W \eta)X\eta(Z) + \eta(X)(\nabla_W \eta)Z) \\
& + (2(\nabla_W(\alpha\beta)) - (\nabla_W(\xi\alpha)))[g(\phi Y, Z)X - g(\phi X, Z)Y]. \tag{6.3}
\end{aligned}$$

Suppose that α and β are constants and X, Y, Z, W are orthogonal to ξ . Then using $\phi\xi = 0$ and (6.1), we get

$$\phi^2(\nabla_W R)(X, Y)Z = \left(\frac{dr(W)}{2}\right)[g(Y, Z)X - g(X, Z)Y]. \tag{6.4}$$

Thus we can state the following:

Theorem 6.2. *A three-dimensional Lorentzian Trans-Sasakian manifold of type (α, β) is locally ϕ -symmetric if and only if the scalar curvature r is constant provided α and β are constants.*

7. EXAMPLES

Example 7.1: We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard co-ordinate of \mathbb{R}^3 .

The vector fields

$$e_1 = z\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M .

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g , we have

$$\eta(e_3) = -1,$$

$$\phi^2 Z = Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an Lorentzian structure on M .

Let ∇ be the Levi-Civita connection with respect to metric g and R be the curvature tensor of g . Then we have

$$[e_1, e_2] = ye_2 - z^2e_3, \quad [e_1, e_3] = -\frac{1}{z}e_1 \quad \text{and} \quad [e_2, e_3] = -\frac{1}{z}e_2.$$

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric g , we can easily calculate

$$\begin{aligned} \nabla_{e_1}e_3 &= -\frac{1}{z}e_1 + \frac{1}{z^2}e_2, & \nabla_{e_1}e_2 &= -\frac{1}{2}z^2e_3, \\ \nabla_{e_1}e_1 &= -\frac{1}{z}e_3, & \nabla_{e_2}e_3 &= -\frac{1}{z}e_2 + \frac{1}{2}z^2e_1, \\ \nabla_{e_2}e_2 &= ye_1 - \frac{1}{z}e_3, & \nabla_{e_2}e_1 &= \frac{1}{2}z^2e_3 - ye_2, \\ \nabla_{e_3}e_3 &= 0, & \nabla_{e_3}e_2 &= \frac{1}{2}z^2e_1, & \nabla_{e_3}e_1 &= -\frac{1}{2}z^2e_2. \end{aligned}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an Lorentzian Trans-Sasakian structure on M . Consequently $M^3(\phi, \xi, \eta, g)$ is an Lorentzian Trans-Sasakian manifold with $\alpha = \frac{1}{2}z^2 \neq 0$ and $\beta = \frac{1}{z} \neq 0$.

Example 7.2: We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard co-ordinate of \mathbb{R}^3 .

The vector fields

$$e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = z \frac{\partial}{\partial z}$$

are linearly independent at each point of M .

Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1, e_3) &= g(e_1, e_2) = g(e_2, e_3) = 0, \\ g(e_1, e_1) &= g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1 \end{aligned}$$

that is, the form of the metric becomes

$$g = \frac{dx^2 + dy^2 - dz^2}{z^2}.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g , we have

$$\begin{aligned} \eta(e_3) &= -1, \\ \phi^2 Z &= Z + \eta(Z)e_3, \end{aligned}$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an Lorentzian structure on M .

Let ∇ be the Levi-Civita connection with respect to metric g . Then we have

$$\begin{aligned} [e_1, e_3] &= e_1 e_3 - e_3 e_1 \\ &= z \frac{\partial}{\partial x} \left(z \frac{\partial}{\partial z} \right) - z \frac{\partial}{\partial z} \left(z \frac{\partial}{\partial x} \right) \\ &= z^2 \frac{\partial^2}{\partial x \partial z} - z^2 \frac{\partial^2}{\partial z \partial x} - z \frac{\partial}{\partial x} \\ &= -e_1. \end{aligned}$$

Similarly

$$[e_1, e_2] = 0 \quad \text{and} \quad [e_2, e_3] = -e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned} \quad (7.1)$$

which known as Koszul's formula.

Using (7.1) we have

$$\begin{aligned} 2g(\nabla_{e_1} e_3, e_1) &= -2g(e_1, e_1) \\ &= 2g(-e_1, e_1). \end{aligned} \quad (7.2)$$

Again by (7.1)

$$2g(\nabla_{e_1} e_3, e_2) = 0 = 2g(-e_1, e_2) \quad (7.3)$$

and

$$2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(-e_1, e_3). \quad (7.4)$$

From (7.2), (7.3) and (7.4) we obtain

$$2g(\nabla_{e_1} e_3, X) = 2g(-e_1, X),$$

for all $X \in \chi(M)$.

Thus

$$\nabla_{e_1} e_3 = -e_1.$$

Therefore, (7.1) further yields

$$\begin{aligned} \nabla_{e_1} e_3 &= -e_1, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_1 &= -e_3, \\ \nabla_{e_2} e_3 &= -e_2, & \nabla_{e_2} e_2 &= -e_3, & \nabla_{e_2} e_1 &= 0, \\ \nabla_{e_3} e_3 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_1 &= 0. \end{aligned} \quad (7.5)$$

(7.5) tells us that the manifold satisfies (1.3) for $\alpha = 0$, $\beta = 1$ and $\xi = e_3$. Hence the manifold is a Lorentzian Trans-Sasakian manifold of type $(0, 1)$. It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (7.6)$$

With the help of the above results and using (7.6) it can be easily verified that

$$\begin{aligned} R(e_1, e_2)e_3 &= 0, & R(e_2, e_3)e_3 &= -e_2, & R(e_1, e_3)e_3 &= -e_1, \\ R(e_1, e_2)e_2 &= -e_1, & R(e_2, e_3)e_2 &= -e_3, & R(e_1, e_3)e_2 &= 0, \\ R(e_1, e_2)e_1 &= e_2, & R(e_2, e_3)e_1 &= 0, & R(e_1, e_3)e_1 &= -e_3. \end{aligned}$$

From the expression of the curvature tensor it follows that the manifold is of constant curvature -1 . Hence the manifold is locally ϕ -symmetric. Also from the above expressions of the curvature tensor, we obtain

$$\begin{aligned} S(e_1, e_1) &= g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) \\ &= -2. \end{aligned}$$

Similarly, we have

$$S(e_2, e_2) = -2, S(e_3, e_3) = 2.$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = -6.$$

Thus the scalar curvature r is constant. Hence Theorem 6.1 is verified.

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