Commun.Fac.Sci.Univ.Ank.Series A1
Volume 63, Number 1, Pages 11–24 (2014)
ISSN 1303–5991

INVARIANT HYPERSURFACES WITH SEMI-SYMMETRIC METRIC CONNECTION OF $F_a(K,1)-$STRUCTURE MANIFOLD

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Abstract. The aim of this paper is to define induced structure on the tangent bundle of invariant hypersurface with semi-symmetric metric connection of a $F_a(K,1)-$structure manifold and to obtain relations with respect to this induced structure.

1. Introduction

A nonzero tensor field $\hat{F}$ of the type $(1,1)$ and class C^∞ on an $n-$dimensional differentiable manifold M is supposed to satisfy

$$\hat{F}^K - a^2\hat{F} = 0 \tag{1.1}$$

where a is a complex number not equal to zero and $K > 2$ is a positive integer [9].

Let the operators $\hat{\ell}$ and $\hat{t}$ on M be defined as [9]:

$$\hat{\ell} = \frac{\hat{F}^{K-1}}{a^2} \text{ and } \hat{t} = \hat{I} - \frac{\hat{F}^{K-1}}{a^2} \tag{1.2}$$

where $\hat{I}$ denotes the identity operator on M. From (1.2), we have

$$\hat{\ell} + \hat{t} = \hat{I}, \quad \hat{\ell}\hat{t} = \hat{t}\hat{\ell} = 0, \quad \hat{\ell}^2 = \hat{\ell}, \quad \hat{t}^2 = \hat{t}. \tag{1.3}$$

The equation (1.3) shows that there exist two complementary distributions $\hat{L}$ and $\hat{T}$ in M corresponding to the projection operators $\hat{\ell}$ and $\hat{t}$, respectively. When the rank of $\hat{F}$ is constant and equal to r on M, then $\hat{L}$ is $r-$dimensional and $\hat{T}$ is $(n-r)-$dimensional. Such a structure is called $F_a(K,1)-$structure of rank r and the manifold M with this structure is called a $F_a(K,1)-$structure manifold [9].

We have the following results [9]

$$\hat{F}\hat{\ell} = \hat{\ell}\hat{F} = \hat{F}, \quad \hat{F}\hat{t} = \hat{t}\hat{F} = 0 \ , \tag{1.4}$$

Received by the editors Agu. 02, 2013; Accepted: March 31, 2014.
2000 *Mathematics Subject Classification.* 53C15, 53B05(53B20), 53C40.
Key words and phrases. $F_a(K,1)-$structure, semi-symmetric metric connection, invariant hypersurface.

$$\hat{F}^2\hat{\ell} = \hat{\ell}\hat{F}^2 = \hat{F}^2, \quad \hat{F}^2\hat{t} = \hat{t}\hat{F}^2 = 0 \ , \tag{1.5}$$

$$\hat{F}^{K-2}\hat{\ell} = \hat{\ell}\hat{F}^{K-2} = \hat{F}^{K-2}, \quad \hat{F}^{K-2}\hat{t} = \hat{t}\hat{F}^{K-2} = 0 \ , \tag{1.6}$$

$$\hat{F}^{K-1}\hat{\ell} = a^2\hat{\ell}, \quad \hat{F}^{K-1}\hat{t} = 0 \ . \tag{1.7}$$

Then, $\hat{F}^{K-1}$ acts on $\hat{L}$ as a $GF-$structure and on $\hat{T}$ as a null operator. Additionly, if the rank of $\hat{F}$ is maximal then $F_a(K,1)-$structure is a $GF-$structure.

The Nijenhius tensor of $\hat{F}$ is a tensor field of the type $(1,2)$ given by [3]

$$\hat{N}(\hat{X},\hat{Y}) = [\hat{F}\hat{X},\hat{F}\hat{Y}] - \hat{F}[\hat{F}\hat{X},\hat{Y}] - \hat{F}[\hat{X},\hat{F}\hat{Y}] + \hat{F}^2[\hat{X},\hat{Y}] \tag{1.8}$$

for any $\hat{X},\hat{Y} \in \chi(M)$. Then, the integrability conditions of $\hat{F}$ in terms of $\hat{N}$ follow from [9]:

Theorem 1.1. *A necessary and sufficient condition for the distribution $\hat{T}$ to be integrable is that*

$$\hat{\ell}\hat{N}(\hat{t}\hat{X},\hat{t}\hat{Y}) = 0$$

for any $\hat{X},\hat{Y} \in \Im_0^1(M)$.

Theorem 1.2. *In order that the distribution $\hat{L}$ be integrable, it is necessary and sufficient condition that the equation*

$$\hat{t}\hat{N}(\hat{\ell}\hat{X},\hat{\ell}\hat{Y}) = 0$$

is satisfied for any $\hat{X},\hat{Y} \in \Im_0^1(M)$.

Theorem 1.3. *A necessary and sufficient condition for $\hat{F}$ to be partially integrable is that the equation*

$$\hat{N}(\hat{\ell}\hat{X},\hat{\ell}\hat{Y}) = 0$$

is satisfied for any $\hat{X},\hat{Y} \in \Im_0^1(M)$.

Theorem 1.4. *In order that $\hat{F}$ be integrable, it is necessary and sufficient condition that the equation*

$$\hat{N}(\hat{X},\hat{Y}) = 0$$

for any $\hat{X},\hat{Y} \in \Im_0^1(M)$.

2. Invariant Hypersurfaces and The Induced Structure

S is a $(m-1)-$dimensional imbedded submanifold of M and its imbedding is denoted by $\imath : S \longrightarrow M$ [3, 7]. The differential mapping $d\imath$ is a mapping from TS into TM, which is called the tangent map of $\imath$, where TS and TM are the tangent bundles of S and M, respectively. The tangent map $d\imath$ is denoted by B and $B : TS \rightarrow T\imath(S)$ is an isomorphism. For $X,Y \in \chi(S)$, the following holds:

$$B[X,Y] = [BX,BY] \tag{2.1}$$

Definition 2.1. If the tangent space $T_p(\imath(S))$ of $\imath(S)$ is invariant by the linear mapping $\hat{F}_p$ at each $p \in S$, then S is called an invariant hypersurface of M, that is, $\hat{F}(\chi(\imath(S))) \subset \chi(\imath(S))$ [2].

In this paper, we shall assume that M is a $F_a(K,1)-$structure manifold and S is an invariant hypersurface of M. Since S is an invariant hypersurface, we have

$$\hat{F}(BX) = B\acute{X} \tag{2.2}$$

for $X \in \chi(S)$, where $\acute{X}$ is a vector field in S. Thus, we define a tensor field of type $(1,1)$ in S such that

$$F : \chi(S) \to \chi(S), FX = \acute{X}$$

From (2.2), we obtain

$$\hat{F}(BX) = B(FX). \tag{2.3}$$

Definition 2.2. The tensor field F defined by the equation (2.3) is called induced structure from $\hat{F}$ to S [2].

By using the induction method, the equation (2.3) can be generalized as follows:

$$\hat{F}^{K-1}(BX) = B(F^{K-1}X) \tag{2.4}$$

Theorem 2.3. *Let $\hat{N}$ and N be the Nijenhius tensors of $\hat{F}$ and F, respectively. Then, we have*

$$\hat{N}(BX, BY) = BN(X,Y) \tag{2.5}$$

for $X, Y \in \chi(S)$ [2].

We can easily see that there are two cases for any invariant hypersurface S of M. Now, we consider these cases.

Case 1. *The distribution $\hat{T}$ is never tangential to S.*

Then, there is no vector field of the type $\hat{t}(BX)$, where $X \in \chi(S)$. That is, vector fields of the type BX belong to the distribution $\hat{L}$ or $\hat{t}(BX) = 0$. In contrast to we assume that with $\hat{t}(BX) \neq 0$. Then, using the equations (1.2) and (2.4), we obtain

$$\hat{t}(BX) = B\left(I - \frac{1}{a^2}F^{K-1}\right)X \tag{2.6}$$

where I is the identity operator on S. Contrary to the hipothesis, this equation show that $\hat{t}(BX) \in T(\imath(S))$. This is a contradiction. Thus, $\hat{t}(BX) = 0$.

Theorem 2.4. *Let the distribution $\hat{T}$ be never tangential to S. Then, F is a induced $GF-$structure in S.*

Proof. From the equation (2.4), we get

$$\begin{aligned} B(F^{K-1}X) &= \hat{F}^{K-1}(BX) \\ &= a^2\hat{\ell}(BX) \\ &= a^2(\hat{I}-\hat{t})(BX) \\ &= a^2(BX) \\ &= B(a^2X) \end{aligned}$$

for $X \in \chi(S)$. Since B is an isomorphism, $F^{K-1} = a^2I$. Therefore, F is an induced $GF-$structure in S. □

Let $\hat{g}$ be a Riemann metric on M and $\hat{\nabla}$ be also the Riemann connection on the Riemann manifold $(M, \hat{g})$. Then, the semi-symmetric metric connection $\bar{\nabla}$ on $(M, \hat{g})$ is given by

$$\bar{\nabla}_{\hat{X}}\hat{Y} = \hat{\nabla}_{\hat{X}}\hat{Y} + \hat{w}(\hat{Y})\hat{X} - \hat{g}(\hat{X},\hat{Y})\hat{P}$$

for arbitrary vector fields $\hat{X}$ and $\hat{Y}$ in $(M, \hat{g})$, where $\hat{w}$ is a $1-$form in $(M, \hat{g})$ and $\hat{P}$ is a vector field defined by $\hat{g}(\hat{P}, \hat{X}) = \hat{w}(\hat{X})$ for any vector field $\hat{X}$ in $(M, \hat{g})$ [3].

Now, we define a tensor field $\bar{S}$ of the type $(1, 2)$ in $(M, \hat{g})$ as follows:

$$\bar{S}(\hat{X},\hat{Y}) = \hat{N}(\hat{X},\hat{Y}) - \bar{\nabla}_{\hat{X}}(\hat{t}\hat{Y}) - \bar{\nabla}_{\hat{Y}}(\hat{t}\hat{X}) - \hat{t}[\hat{X},\hat{Y}] \tag{2.7}$$

for $\hat{X}, \hat{Y} \in \chi(M)$.

Theorem 2.5. *Let the distribution $\hat{T}$ be never tangential to S. Then, we have*

$$\bar{S}(BX, BY) = BN(X, Y) \tag{2.8}$$

for $X, Y \in \chi(S)$.

Proof. If the distribution $\hat{T}$ is never tangential to S, then $\hat{t}(BX) = 0$. The proof is completed from the equations (2.5) and (2.7). □

Definition 2.6. The $F_a(K, 1)-$structure is said to be *normal* with respect to $\bar{\nabla}$ in M if $\bar{S} = 0$.

Theorem 2.7. *Let the distribution $\hat{T}$ be never tangential to S. If $\hat{F}$ is normal with respect to $\bar{\nabla}$ in M, then F is integrable in S.*

Proof. If the distribution $\hat{T}$ be never tangential to S, then $\hat{t}(BX) = 0$. Let $\hat{F}$ be normal with respect to $\bar{\nabla}$ in M. Therefore, from Definition 3 and the equation (2.8), we obtain $BN(X, Y) = 0$, for $X, Y \in \chi(S)$. Since B is a isomorphism, $N(X, Y) = 0$, for $X, Y \in \chi(S)$. Then, F is integrable in S. □

Case 2. *The distribution $\hat{T}$ is always tangential to S.*

By considering (2.6), we can define a tensor field of the type $(1,1)$ on S by

$$t = I - \frac{1}{a^2}F^{K-1}. \tag{2.9}$$

Therefore, we have

$$\hat{t}(BX) = B(tX) \tag{2.10}$$

for $X \in \chi(S)$.

Theorem 2.8. *Let ℓ be a tensor field of the type $(1,1)$ on S, which is defined by $\ell = \frac{1}{a^2}F^{K-1}$. Then,*

$$\hat{\ell}(BX) = B(\ell X) \tag{2.11}$$

for $X \in \chi(S)$.

Proof. Using the equation (2.4), we obtain

$$\begin{aligned} \hat{\ell}(BX) &= \frac{1}{a^2}\hat{F}^{K-1}(BX) \\ &= \frac{1}{a^2}B(F^{K-1}X) \\ &= B\left(\frac{1}{a^2}F^{K-1}X\right) \\ &= B(\ell X) \end{aligned}$$

for $X \in \chi(S)$. □

Theorem 2.9. *The tensor fields of the type $(1,1)$ t and ℓ defined by the equations (2.10) and (2.11) imply*

$$\begin{array}{cc} \ell + t = I, & \ell t = 0 \\ \ell^2 = \ell, & t^2 = t \end{array}. \tag{2.12}$$

Proof. For $X \in \chi(S)$, applying BX to both side of $\hat{\ell} + \hat{t} = \hat{I}$, we get $B(\ell X + tX) = X$. Since B is an isomorphism, $\ell X + tX = X$. Then, $\ell + t = I$.

The other equations can be shown similarly. □

The equation (2.12) show that, t and ℓ are complementary projection operators in S. Therefore,

$$B(F^K X) = B(a^2 F X)$$

for $X \in \chi(S)$. This implies that

$$F^K - a^2 F = 0. \tag{2.13}$$

Then, F acts as a $F_a(K,1)-$structure on S and is called *the induced $F_a(K,1)-$ structure* on S.

Theorem 2.10. *For the complementary projection operators t and ℓ, satisfying the equation (2.12) on S, there are the following relations:*

$$BN(\ell X, \ell Y) = \hat{N}(\hat{\ell}BX, \hat{\ell}BY),$$
$$BN(tX, tY) = \hat{N}(\hat{t}BX, \hat{t}BY),$$
$$BtN(X, Y) = \hat{t}\hat{N}(BX, BY).$$

Proof. The proof trivial from the equations (2.3), (2.10) and (2.11). □

For F satisfying the equation (2.13) on S, these exist complementary distributions T and L corresponding to the projection operators t and ℓ, respectively. Hence, the integrability conditions of F can be given by the following theorems.

Theorem 2.11. *The distribution $\hat{L}$ is integrable in M if and only if L is integrable in S.*

Proof. Let the distribution $\hat{L}$ be integrable in M. Then, we have $\hat{t}\hat{N}(BX, BY) = 0$ for $X, Y \in \chi(S)$. At this point, we get $BtN(X, Y) = 0$. Since B is an isomorphism, we obtain $tN(X, Y) = 0$. Therefore, L is integrable in S.

The other side can be shown similarly. □

Theorem 2.12. *The distribution $\hat{T}$ is integrable in M if and only if T is integrable in S.*

Proof. Let the distribution $\hat{T}$ be integrable in M. Then, we have $\hat{N}(\hat{t}BX, \hat{t}BY) = 0$ for $X, Y \in \chi(S)$.At this point, we get $BN(tX, tY) = 0$. Since B is an isomorphism, we obtain $N(tX, tY) = 0$. Therefore, T is integrable in S.

The other side can be shown similarly. □

Theorem 2.13. *$\hat{F}$ is partially integrable in M if and only if F is partially integrable in S.*

Proof. Let $\hat{F}$ be partially integrable in M. Then, we have $\hat{N}(\hat{\ell}BX, \hat{\ell}BY) = 0$ for $X, Y \in \chi(S)$. At this point, we get $BN(tX, tY) = 0$. Since B is an isomorphism, we obtain $N(\ell X, \ell Y) = 0$. Therefore, F is partially integrable in S.

The other side can be shown similarly. □

Theorem 2.14. *$\hat{F}$ is integrable in M if and only if F is integrable in S.*

Proof. Let $\hat{F}$ be integrable in M. Then, we have $\hat{N}(BX, BY) = 0$ for $X, Y \in \chi(S)$. At this point, we get $BN(X, Y) = 0$. Since B is an isomorphism, we obtain $N(X, Y) = 0$. Therefore, F is integrable in S.

The other side can be shown similarly. □

The hypersurface S is a Riemann manifold with the induced metric g defined by $g(X, Y) = \hat{g}(BX, BY)$, for $X, Y \in \chi(S)$. Then, $\mathring{\nabla}$ is the induced semi-symmetric metric connection on (S, g) from $\bar{\nabla}$, which satisfies the equation

$$\bar{\nabla}_{BX} BY = B\left(\mathring{\nabla}_X Y\right) + m(X, Y)N \tag{2.14}$$

for $X, Y \in \chi(S)$, where m is a tensor field type of $(0,2)$ in S. If m vanishes, then S is called totally geodesic with respect to $\mathring{\nabla}$ [4].

Now, we define a tensor field $\mathring{S}$ of type $(1,2)$ on S totally geodesic with respect to $\mathring{\nabla}$ by

$$\mathring{S}(X,Y) = N(X,Y) + \mathring{\nabla}_X(tY) - \mathring{\nabla}_Y(tX) - t[X,Y] \tag{2.15}$$

for $X, Y \in \chi(S)$.

Theorem 2.15. *Let S be totally geodesic with respect to $\mathring{\nabla}$. Then,*

$$\bar{S}(BX,BY) = B\mathring{S}(X,Y)$$

for $X, Y \in \chi(S)$.

Proof. Using the equations (2.5) and (2.10), we obtain

$$\begin{aligned} B\mathring{S}(X,Y) &= BN(X,Y) + B(\mathring{\nabla}_X tY) - B(\mathring{\nabla}_Y tX) - Bt[X,Y] \\ &= \hat{N}(BX,BY) + \bar{\nabla}_{BX}B(tY) - \bar{\nabla}_{BY}B(tX) - \hat{t}B[X,Y] \\ &= \hat{N}(BX,BY) + \bar{\nabla}_{BX}\hat{t}(BY) - \bar{\nabla}_{BY}\hat{t}(BX) - \hat{t}[BX,BY] \\ &= \bar{S}(BX,BY). \end{aligned}$$

□

Corollary 1. *If $\hat{F}$ is normal with respect to $\bar{\nabla}$ in M, then F is normal with respect to $\mathring{\nabla}$ in S.*

3. The Induced Structure on The Tangent Bundle of A Invariant Hypersurface

Let TM denote the tangent bundle of M with the projection $\pi_M : TM \to M$. According to [5], using the complete lift operation we have the following equalities:

$$\begin{aligned} \hat{\nabla}^C_{\hat{X}^C}\hat{Y}^C &= \left(\hat{\nabla}_{\hat{X}}\hat{Y}\right)^C, \\ [\hat{X}^C, \hat{Y}^C] &= [\hat{X}, \hat{Y}]^C, \\ \hat{F}^C(\hat{X}^C) &= \left(\hat{F}(\hat{X})\right)^C, \\ \hat{F}^C\hat{G}^C &= (\hat{F}\hat{G})^C, \\ \hat{F}^C + \hat{G}^C &= (\hat{F}+\hat{G})^C, \\ \hat{\nabla}^C_{X^C}Y^C &= \left(\hat{\nabla}_X Y\right)^C \\ \hat{N}^C_{\hat{F}^C} &= \left(\hat{N}_{\hat{F}}\right)^C, \\ \left(P(\hat{F})\right)^C &= P(\hat{F}^C) \end{aligned}$$

for $\hat{X}$, $\hat{Y} \in \chi(M)$; $\hat{F}$, $\hat{G} \in \Im^1_1(M)$, where $P(t)$ is a polinomial in one variable t.

Theorem 3.1. *$\hat{F}$ is an $F_a(K,1)-$structure in M if and only if the complete lift $\hat{F}^C$ of $\hat{F}$ is also an $F_a(K,1)-$structure in TM. Then, $\hat{F}$ is of rank r if and only if $\hat{F}^C$ is of rank $2r$* [10].

Theorem 3.2. *Let $\hat{F}$ be an $F_a(K,1)-$structure in M and S be a invariant hypersurface of M. Then,*

$$\left(\hat{F}(\bar{X})\right)^{\bar{C}} = \hat{F}^C(\bar{X})^{\bar{C}} \tag{3.1}$$

for $\bar{X} \in \chi(\imath(S))$. Here, $\bar{C}$ denotes the complete lift operation on $\pi_M^{-1}(\imath(S))$.

Proof. Since S is an invariant hypersurface, $\hat{F}(\bar{X})$ belongs to $\chi(\imath(S))$ for $\bar{X} \in \chi(\imath(S))$. According to [1], we obtain

$$\left(\hat{F}(\bar{X})\right)^{\bar{C}} = \#(\hat{F}(\hat{X}))^C = \#\hat{F}^C(\hat{X})^C = \hat{F}^C(\bar{X})^{\bar{C}}.$$

Then, $\hat{F}^C(\bar{X})^{\bar{C}}$ belongs to $\chi(T\imath(S))$. Here, $\#$ denotes the operation of restriction to $\pi_M^{-1}(\imath(S))$. □

Theorem 3.3. *Let $\hat{F}$ be a $F_a(K,1)-$structure in M. Then, S is a invariant hypersurface of M if and only if TS is a invariant submanifold of TM.*

Proof. Since S is an invariant hypersurface, $\hat{F}(\bar{X})$ belongs to $\chi(\imath(S))$ for $\bar{X} \in \chi(\imath(S))$. From the equation (3.1), $\hat{F}^C(\bar{X})^{\bar{C}}$ belongs to $\chi(T\imath(S))$. Also, $\bar{X}^{\bar{C}}$ is in $\chi(T\imath(S))$. Then, $\hat{F}^C$ is invariant on $\chi(T\imath(S))$. Therefore, TS is an invariant submanifold of TM.

The other side can be shown similarly. □

The tangent map of B is denoted by $\tilde{B}$, where $\tilde{B} : T(TS) \to T(T\imath(S))$ is an isomorphism.

Definition 3.4. The tensor field $\tilde{F}$ of type $(1,1)$ satisfies

$$\hat{F}^C(\tilde{B}X^C) = \tilde{B}(\tilde{F}X^C) \tag{3.2}$$

for $X \in \chi(S)$, is called induced structure from $\hat{F}^C$ to TS.

Similarly to (2.4), the equation (3.2) can be generalized as follows:

$$\left(\hat{F}^C\right)^{K-1}(\tilde{B}X^C) = \tilde{B}(\tilde{F}^K X^C). \tag{3.3}$$

Theorem 3.5. *For $X, Y \in \chi(S)$,*

$$\tilde{B}[X^C, Y^C] = [\tilde{B}X^C, \tilde{B}Y^C]. \tag{3.4}$$

Proof. Using the equation (3.10) in [1] and the equation (2.1), we get

$$\begin{aligned}
\tilde{B}[X^C, Y^C] &= \tilde{B}[X,Y]^C \\
&= (B[X,Y])^{\bar{C}} \\
&= [BX, BY]^{\bar{C}} \\
&= [(BX)^{\bar{C}}, (BY)^{\bar{C}}]) \\
&= [\tilde{B}X^C, \tilde{B}Y^C].
\end{aligned}$$

□

Theorem 3.6. *The induced structure $\tilde{F}$ on TS is the complete lift of the induced structure F on S.*

Proof. Using the equation (3.10) in [1] and the equation (3.2), we get

$$\begin{aligned}
\hat{F}^C(\tilde{B}X^C) &= \hat{F}^C(BX)^{\bar{C}} \\
&= (\hat{F}(BX))^{\bar{C}} \\
&= (B(FX))^{\bar{C}} \\
&= \tilde{B}(FX)^C \\
&= \tilde{B}(F^C X^C)
\end{aligned}$$

for $X \in \chi(S)$. From (3.2), we obtain $\tilde{B}(F^C X^C) = \tilde{B}(\tilde{F}X^C)$. Since $\tilde{B}$ is an isomorphism, $F^C = \tilde{F}$. □

Theorem 3.7. *Let $\tilde{N}$ and $\hat{N}^C$ be the Nijenhius tensors of $\tilde{F}$ and $\hat{F}^C$, respectively. Then,*

$$\hat{N}^C\left(\tilde{B}X^C, \tilde{B}Y^C\right) = \tilde{B}\tilde{N}\left(X^C, Y^C\right) \tag{3.5}$$

for $X, Y \in \chi(S)$.

Proof. Using the equation (3.3), we obtain

$$\begin{aligned}
\hat{N}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right) &= [\hat{F}^C(\tilde{B}X^C),\hat{F}^C(\tilde{B}Y^C)] - \hat{F}^C[\hat{F}^C(\tilde{B}X^C),\tilde{B}Y^C] \\
&\quad -\hat{F}^C[\tilde{B}X^C,\hat{F}^C(\tilde{B}Y^C)] + (\hat{F}^C)^2[\tilde{B}X^C,\tilde{B}Y^C] \\
&= [\tilde{B}(\tilde{F}X^C),\tilde{B}(\tilde{F}Y^C)] - \hat{F}^C[\tilde{B}(\tilde{F}X^C),\tilde{B}Y^C] \\
&\quad -\hat{F}^C[\tilde{B}X^C,\tilde{B}(\tilde{F}Y^C)] + (\hat{F}^C)^2[\tilde{B}X^C,\tilde{B}Y^C] \\
&= \tilde{B}[\tilde{F}X^C,\tilde{F}Y^C] - \hat{F}^C\tilde{B}[\tilde{F}X^C,Y^C] \\
&\quad -\hat{F}^C\tilde{B}[X^C,\tilde{F}Y^C] + (\hat{F}^C)^2\tilde{B}[X^C,Y^C] \\
&= \tilde{B}[\tilde{F}X^C,\tilde{F}Y^C] - \tilde{B}\tilde{F}[\tilde{F}X^C,Y^C] \\
&\quad -\tilde{B}\tilde{F}[X^C,\tilde{F}Y^C] + \tilde{B}\tilde{F}^2[X^C,Y^C] \\
&= \tilde{B}([\tilde{F}X^C,\tilde{F}Y^C] - \tilde{F}[\tilde{F}X^C,Y^C] \\
&\quad -\tilde{F}[X^C,\tilde{F}Y^C] + \tilde{F}^2[X^C,Y^C]) \\
&= \tilde{B}\tilde{N}(X^C,Y^C).
\end{aligned}$$

□

Theorem 3.8. *Let $\hat{N}^C$ be the Nijenhius tensors of $\hat{F}^C$. Then,*

$$\left(\hat{N}\left(\bar{X},\bar{Y}\right)\right)^{\bar{C}} = \hat{N}^C\left(\bar{X}^{\bar{C}},\bar{Y}^{\bar{C}}\right)$$

for $\bar{X},\bar{Y}\in\chi(\imath(S))$.

Proof. From the equation (2.5), $\hat{N}\left(\bar{X},\bar{Y}\right)$ belongs to $\chi(\imath(S))$. Therefore, we have

$$\begin{aligned}
\left(\hat{N}\left(\bar{X},\bar{Y}\right)\right)^{\bar{C}} &= \#\left(\hat{N}\left(\hat{X},\hat{Y}\right)\right)^C \\
&= \#\hat{N}^C(\hat{X}^C,\hat{Y}^C) \\
&= \hat{N}^C(\bar{X}^{\bar{C}},\bar{Y}^{\bar{C}}).
\end{aligned}$$

□

Corollary 2. *Let $\tilde{N}$ and N be the Nijenhius tensors of $\tilde{F}$ and F, respectively. Then, $\tilde{N}$ is the complete lift of N.*

Proof. From Theorem 23, $\hat{N}^C(\tilde{B}X^C,\tilde{B}Y^C) = (\hat{N}(BX,BY))^{\bar{C}}$ for $X,Y\in\chi(S)$. Then, we get

$$\begin{aligned}
\hat{N}^C(\tilde{B}X^C,\tilde{B}Y^C) &= (\hat{N}(BX,BY))^{\bar{C}} \\
&= (BN(X,Y))^{\bar{C}} \\
&= \tilde{B}(N(X,Y))^C \\
&= \tilde{B}N^C(X^C,Y^C).
\end{aligned}$$

Note that the equation (3.5), we obtain $\tilde{B}N^C(X^C,Y^C)=\tilde{B}\tilde{N}\left(X^C,Y^C\right)$. Since $\tilde{B}$ is a isomorphism, $\tilde{N}=N^C$. □

Theorem 3.9. *The distribution $\hat{T}$ never tangential to S if and only if the distribution $\hat{T}^C$ never tangential to TS.*

Proof. Let the distribution $\hat{T}$ be never tangential to S. Then, $\hat{t}(BX)=0$ for $X\in\chi(S)$. Since $(\hat{t}(BX))^{\bar{C}}=\hat{t}^C(\tilde{B}X^C)$, we obtain $\hat{t}^C(\tilde{B}X^C)=0$. Therefore, the distribution $\hat{T}^C$ never tangential to TS.

The other side can be shown similarly. □

Theorem 3.10. *Let the distribution $\hat{T}^C$ be never tangential to TS. Then, $\tilde{F}$ is induced $GF-$structure in TS.*

Proof. Similar to proof of the Theorem 6, we get the desired result. □

Theorem 3.11. *Let $\bar{\nabla}$ be a semi-symmetric metric connection with respect to $\hat{\nabla}$ Riemann connection in $(M,\hat{g})$. Then, $\bar{\nabla}^C$ is also a semi-symmetric metric connection with respect to $\hat{\nabla}^C$ Riemann connection in $(TM,\hat{g}^C)$* [11].

Noting that the equation (2.7) we obtain

$$\bar{S}^C\left(\hat{X}^C,\hat{Y}^C\right)=\hat{N}^C(\hat{X}^C,\hat{Y}^C)-\bar{\nabla}^C_{\hat{X}^C}\left(\hat{t}^C\hat{Y}^C\right)-\bar{\nabla}^C_{\hat{Y}^C}(\hat{t}^C\hat{X}^C)-\hat{t}^C\left[\hat{X}^C,\hat{Y}^C\right]$$

for $\hat{X},\hat{Y}\in\chi(M)$, on TM.

Theorem 3.12. *Let the distribution $\hat{T}^C$ be never tangential to TS. Then,*

$$\bar{S}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right)=\tilde{B}\tilde{N}(\tilde{X},\tilde{Y})$$

for $X,Y\in\chi(S)$.

Proof. Similar to proof of the Theorem 7, we get the desired result. □

Theorem 3.13. *$\hat{F}$ is normal with respect to $\bar{\nabla}$ in M if and only of $\hat{F}^C$ is normal with respect to $\bar{\nabla}^C$ in TM.*

Proof. The proof trivial from Definition 3. □

Theorem 3.14. *Let the distribution $\hat{T}^C$ be never tangential to TS. If $\hat{F}^C$ is normal with respect to $\bar{\nabla}^C$ in TM, then $\tilde{F}$ is integrable in TS.*

Proof. Similar to proof of the Theorem 8, we get the desired result. □

Theorem 3.15. *The distribution $\hat{T}^C$ is tangential to TS if and only of the distribution $\hat{T}$ is tangential to S.*

Proof. Let the distribution $\hat{T}$ be tangential to S. We have $\hat{t}(BX)\neq 0$ for $X\in\chi(S)$. Then, we obtain $\hat{t}^C(\tilde{B}X^C)\neq 0$. Therefore, the distribution $\hat{T}^C$ is tangential to TS.

The other side can be shown similarly. □

If the distribution $\hat{T}^C$ is tangential to TS, then ℓ^C and t^C are complementary projection operators in TS, for ℓ and t defined by the equations (2.10) and (2.11), respectively.

Let $\tilde{\ell}$ and $\tilde{t}$ be expressed by

$$\tilde{\ell} = \frac{1}{a^2}\tilde{F}^{K-1} \text{ and } \tilde{t} = I - \frac{1}{a^2}\tilde{F}^{K-1} \tag{3.6}$$

where $\tilde{\ell} = \ell^C$ and $\tilde{t} = t^C$.

Theorem 3.16. *The operators $\tilde{\ell}$ and $\tilde{t}$ satisfy*

$$\hat{\ell}^C\left(\tilde{B}X^C\right) = \tilde{B}\left(\tilde{\ell}X^C\right), \quad \hat{t}^C\left(\tilde{B}X^C\right) = \tilde{B}(\tilde{t}X^C)$$

for $X, Y \in \chi(S)$, on TS.

Proof. From (2.10), we have

$$\begin{aligned}\hat{t}^C\left(\tilde{B}X^C\right) &= (\hat{t}(BX))^{\bar{C}}\\ &= (B(tX))^{\bar{C}}\\ &= \tilde{B}(tX)^C\\ &= \tilde{B}(t^C X^C)\\ &= \tilde{B}(\tilde{t}X^C).\end{aligned}$$

The other equation can be shown similarly. □

Theorem 3.17. *Let the distribution $\hat{T}^C$ be tangential to TS. Then, $\tilde{F}$ is the induced $F_a(K,1)-$structure on TS.*

Proof. For $X \in \chi(S)$, we obtain

$$\begin{aligned}\tilde{B}(\tilde{F}^{K-1}X^C) &= \tilde{B}(a^2\tilde{\ell}X^C) = a^2\tilde{B}(\tilde{\ell}X^C) = a^2\hat{\ell}^C\left(\tilde{B}X^C\right)\\ &= a^2\hat{F}^C\left(\tilde{B}X^C\right) = a^2\tilde{B}(\tilde{F}X^C) = \tilde{B}(a^2\tilde{F}X^C).\end{aligned}$$

Since $\tilde{B}$ is an isomorphism, we get $\tilde{F}^{K-1} - a^2\tilde{F} = 0$. Then, $\tilde{F}$ is the induced $F_a(K,1)-$structure on TS. □

Theorem 3.18. *For the complementary projection operators $\tilde{\ell}$ and $\tilde{t}$, which imply the equation (3.6) on TS, there are the following relations:*

$$\begin{aligned}\tilde{B}\tilde{N}(\tilde{\ell}X^C, \tilde{\ell}Y^C) &= \hat{N}^C(\hat{\ell}^C\tilde{B}X^C, \hat{\ell}^C\tilde{B}Y^C),\\ \tilde{B}\tilde{N}(\tilde{t}X^C, \tilde{t}Y^C) &= \hat{N}^C(\hat{t}^C\tilde{B}X^C, \hat{t}^C\tilde{B}Y^C),\\ \tilde{B}\tilde{t}\tilde{N}(X^C, Y^C) &= \hat{t}^C\hat{N}^C(\tilde{B}X^C, \tilde{B}Y^C).\end{aligned}$$

Proof. Similar to proof of the Theorem 11, we get the desired result. □

Let $\tilde{T}$ and $\tilde{L}$ be the distributions corresponding to the projection operators $\tilde{t}$ and $\tilde{\ell}$, respectively. Then, $\tilde{T} = T^C$ and $\tilde{L} = L^C$. Therefore, similarly to Theorem 12, Theorem 13, Theorem 14 and Theorem15 the integrability conditions of $\tilde{F}$ are given in the following theorems.

Theorem 3.19. *The distributions $\tilde{T}$ and $\tilde{L}$ are integrable in TS if and only if the distributions $\hat{T}^C$ and $\hat{L}^C$ are integrable in TM.*

Theorem 3.20. *$\tilde{F}$ is partially integrable in TS if and only if $\hat{F}^C$ is partially integrable in TM.*

Theorem 3.21. *$\tilde{F}$ is integrable in TS if and only if $\hat{F}^C$ is integrable in TM.*

For the Riemann metric $\hat{g}$ in M, the complete lift $\hat{g}^C$ of $\hat{g}$ is the *pseudo-Riemann metric* in TM. Therefore, if we denote the induced metric from $\hat{g}^C$ on TS by $\tilde{g}$, then

$$\tilde{g}(X^C, Y^C) = \hat{g}^C(\tilde{B}X^C, \tilde{B}Y^C),$$

for arbitrary $X, Y \in \Im_0^1(S)$. Thus, the complete lift $\hat{\nabla}^C$ of the Riemann connection $\hat{\nabla}$ in $(M, \hat{g})$ is the Riemann connection in the *pseudo-Riemann manifold* $(TM, \hat{g}^C)$. Similarly, the complete lift ∇^C of the induced connection ∇ on (S, g) is also the Riemann connection in $(TS, \tilde{g})$ [1].

$\mathring{\nabla}^C$ is the induced semi-symmetric metric connection from $\bar{\nabla}^C$ to TS. Then, we have

$$\bar{\nabla}^C_{\tilde{B}X^C} \tilde{B}Y^C = \tilde{B}\left(\mathring{\nabla}^C_{X^C} Y^C\right) + m^V(X^C, Y^C)N^{\bar{C}} + m^C(X^C, Y^C)N^{\bar{V}}$$

for $X, Y \in \chi(S)$ [11].

Theorem 3.22. *TS is totally geodesic with respect to the semi-symmetric metric connection $\mathring{\nabla}^C$ if and only if S is totally geodesic with respect to the semi-symmetric metric connection $\mathring{\nabla}$* [11].

Let TS be totally geodesic with respect to $\mathring{\nabla}^C$. Then, we have

$$\bar{\nabla}^C_{\tilde{B}X^C} \tilde{B}Y^C = \tilde{B}\left(\mathring{\nabla}^C_{X^C} Y^C\right)$$

for $X, Y \in \chi(S)$ [11]. Therefore, we define a tensor field $\mathring{S}^C$ of type $(1,2)$ by

$$\mathring{S}^C(X^C, Y^C) = \tilde{N}(X^C, Y^C) + \mathring{\nabla}^C_{X^C}\left(\tilde{t}Y^C\right) - \mathring{\nabla}^C_{Y^C}\left(\tilde{t}X^C\right) - \tilde{t}[X^C, Y^C]$$

for $X, Y \in \chi(S)$ on TS.

Theorem 3.23. *Let TS be totally geodesic with respect to $\mathring{\nabla}^C$. Then,*

$$\bar{S}^C(\tilde{B}X^C, \tilde{B}Y^C) = \tilde{B}\mathring{S}^C(X^C, Y^C)$$

for $X, Y \in \chi(S)$.

Proof. Similar to proof of the Theorem 16, we get the desired result. □

Corollary 3. *If $\hat{F}^C$ is normal with respect to $\bar{\nabla}^C$ in TM, then $\tilde{F}$ is normal with respect to $\mathring{\nabla}^C$ in TS.*

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