Commun.Fac.Sci.Univ.Ank.Series A1
Volume 63, Number 1, Pages 55–71 (2014)
ISSN 1303–5991

QUASILINEARIZATION METHOD IN CAUSAL DIFFERENTIAL EQUATIONS WITH INITIAL TIME DIFFERENCE

COŞKUN YAKAR

Abstract. In this paper, the method of the quasilinearization technique in causal
differential equations is applied to obtain upper and lower sequences with initial time difference in terms of the solutions of the linear causal differential equations that start at different initial times. It is also shown that these sequences converge to the unique solution of the nonlinear equation in causal differential equations uniformly and superlinearly.

1. Introduction

The most important applications of the quasilinearization method in causal differential equations [5] has been to obtain a sequence of lower and upper bounds which are the solutions of linear causal differential equations that converge superlinearly. As a result, the method has been popular in applied areas. However, the convexity assumption that is demanded by the method of quasilinearization has been a stumbling block for further development of the theory. Recently, this method has been generalized, refined and extended in several directions so as to be applicable to a much larger class of nonlinear problems by not demanding convexity property. Moreover, other possibilities that have been explored make the method of generalized quasilinearization universally useful in applications [7]. In the investigation of initial value problems of causal differential equations [5], we have been partial to initial time all along in the sense that we only perturb the space variable and keep the initial time unchanged. However, it appears important to vary the initial time as well because it is impossible not to make errors in the starting time [4, 6, 7, 8, 9, 10, 11, 12, 13]. Recently, the investigations of initial value problems of causal differential equations where the initial time changes with each solution

Received by the editors Feb. 06, 2014, Accepted: May 21, 2014.

2010 *Mathematics Subject Classification.* (MOS) 34A12, 34A45, 34C11.
Key words and phrases. Causal differential equations, initial time difference, quasilinearization, superlinear convergence, inequalities.

in addition to the change of spatial variable have been initiated [1, 13] and some results on the comparison theorems, global existence, the method of variation of parameters, the method of lower and upper solutions and the method of monotone iterative techniques [3, 4, 7, 8, 11] have been obtained.

In this paper, the generalized quasilinearization technique in causal differential equations is used to obtain upper and lower sequences in terms of the solutions of linear causal differential equations that start at different initial times and bound the solutions of a given nonlinear causal differential equation [5]. It is also shown that these sequences converge to the unique solution of the nonlinear equation uniformly and superlinearly.

2. Preliminaries

In this section, we state some fundamental definitions and useful theorems for the future reference to prove the main result. First one is comparison result, the second one is existence result in terms of the upper and lower solutions with initial time difference.

An operator $N : E \to E, E = C[J, \mathbb{R}^n]$ is said to be causal operator if, for any x, $y \in E$ such that $x(s) = y(s)$, we have $N(x)(s) = N(y)(s)$ for $t_0 \leq s < t_0 + T$.

Let us consider the casual functional equation

$$x(t) = (Nx)(t), x(t_0) = x_0$$

where the causal operator $N : E \to E$ is continuous and $x(t_0) = x_0$ for $t_0 \geq 0$ denotes the initial value for any $x \in E$.

Let α_0, $\beta_0 \in C^1[J, \mathbb{R}]$ with $\alpha_0(t) \leq \beta_0(t)$ on $J = [t_0, t_0 + T]$, t_0, $T \in \mathbb{R}^+$ and

$$\Omega = \{u \in E : \alpha_0(t) \leq u \leq \beta_0(t),\ t \in J\}.$$

We consider the following initial value problem for casual differential equation

$$u'(t) = (Nu)(t),\ u(t_0) = u_0 \text{ for } t \geq t_0 \tag{2.1}$$

where $N : E \to E$ continuous causal operator, $E = C[J, \mathbb{R}]$ for $J = [t_0, t_0 + T]$, $t_0, T \in \mathbb{R}^+$ and $\Omega \subseteq E$.

Definition 2.1: $\alpha_0, \beta_0 \in C^1[J, E]$ are said to be the natural lower and upper solutions of (2.1), respectively, if the the following inequalities are satisfied

$$\alpha_0' \leq (N\alpha_0)(t),\ \alpha_0(t_0) \leq u_0 \text{ for } t \geq t_0 \tag{2.2}$$
$$\beta_0' \geq (N\beta_0)(t),\ \beta_0(t_0) \geq u_0 \text{ for } t \geq t_0 \tag{2.3}$$

respectively.

Definition 2.2: $\alpha_0, \beta_0 \in C^1[J, E]$ are said to be the coupled lower and upper solutions of (2.1), respectively, if the the following inequalities are satisfied

$$\alpha_0' \leq (N\beta_0)(t),\ \alpha_0(t_0) \leq u_0 \text{ for } t \geq t_0 \tag{2.4}$$
$$\beta_0' \geq (N\alpha_0)(t),\ \beta_0(t_0) \geq u_0 \text{ for } t \geq t_0 \tag{2.5}$$

respectively.

Definition 2.3: $N : E \to E$ is said to be semi nondecreasing in t for each x if

$$(Nx)(t_1) = (Ny)(t_1) \text{ and } (Nx)(t) \le (Ny)(t), t_0 \le t < t_1 < T + t_0 \tag{2.6}$$

for

$$x(t_1) = y(t_1), \ x(t) < y(t), \ t_0 \le t < t_1 < T + t_0. \tag{2.7}$$

Definition 2.4: Let $N \in C[J \times E, E]$. At $x \in E$

$$(N(x+h))(t) = (Nx)(t) + L(x,h)(t) + \|h\| \eta(x,h)(t) \tag{2.8}$$

where $\lim_{\|h\| \longrightarrow 0} \|\eta(x,h)(t)\| = 0$ and $L(x,\cdot)(t)$ is a linear operator. $L(x,h)(t)$ is said to be Fréchet derivative of N at x with the increment h for the remainder $\eta(x,h)(t)$.

3. Causal Functional Inequalities

We give some basic results in causal functional inequalities for the scalar case as follows [5].

Theorem 3.1: Assume that

(i) $N : E \to E$ is a continuous causal operator, $E = C[J, \mathbb{R}]$ for $J = [t_0, t_0 + T]$, $t_0, T \in \mathbb{R}^+$ and let α_0, $\beta_0 \in E$ satisfy

$$\alpha_0 < (N\alpha_0)(t) \text{ for } t_0 \le t \le T + t_0 \tag{3.1}$$

$$\beta_0 \ge (N\beta_0)(t) \text{ for } T + t_0 \ge t \ge t_0. \tag{3.2}$$

(ii) N is semi nondecreasing, i.e.

$$x(t_1) = y(t_1), x(t) < y(t), t_0 \le t < t_1 < T + t_0$$

implies

$$(Nx)(t_1) = (Ny)(t_1) \text{ and } (Nx)(t) \le (Ny)(t), t_0 \le t < t_1 < T + t_0.$$

Then

$$\alpha_0(t) < \beta_0(t) \text{ for } t_0 \le t \le T + t_0 \tag{3.3}$$

provided

$$\alpha_0(t_0) < \beta_0(t_0). \tag{3.4}$$

Proof [5]: Suppose that the conclusion (3.3) of the theorem is not true and $\alpha_0 < (N\alpha_0)(t)$. Then because of the continuity of the functions and (3.4), there would exist a $t_1 > t_0$ such that

$$\alpha_0(t_1) = \beta_0(t_1) \text{ and } \alpha_0(t) < \beta_0(t) \text{ for } t_0 \le t < t_1 < T + t_0. \tag{3.5}$$

Since N is assumed to be semi nondecreasing, we have

$$\alpha_0(t_1) < (N\alpha_0)(t_1) \le (N\beta_0)(t_1) \le \beta_0(t_1).$$

This is a contradiction since $\alpha_0(t_1) = \beta_0(t_1)$. Therefore, it proves the claim (3.3).

Theorem 3.2: Assume that

(i) $N : E \to E$ is a continuous causal operator, $E = C[J, \mathbb{R}]$ for $J = [t_0, t_0 + T]$, t_0, $T \in \mathbb{R}^+$ and let α_0, $\beta_0 \in E$ satisfy

$$\alpha_0 \le (N\alpha_0)(t) \text{ for } t_0 \le t \le T + t_0 \tag{3.6}$$

$$\beta_0 > (N\beta_0)(t) \text{ for } T + t_0 \ge t \ge t_0. \tag{3.7}$$

(ii) N is semi nondecreasing.

Then the conclusion of the Theorem 3.1 remains the same.

Proof [5]: Suppose that the conclusion of the theorem is not true and $\beta_0 > (N\beta_0)(t)$. Then because of the continuity of the functions and such that (3.4) is satisfied, there would exist a $t_1 > t_0$ such that (3.5) is satisfied. Since N is assumed to be semi nondecreasing, we have

$$\alpha_0(t_1) \le (N\alpha_0)(t_1) \le (N\beta_0)(t_1) < \beta_0(t_1).$$

This is a contradiction again since $\alpha_0(t_1) = \beta_0(t_1)$. Therefore, it proves the claim (3.3).

First we state the following theorem and prove it.

Theorem 3.3: Assume that

(i) $N : E \to E$ is a continuous causal operator, $E = C[J, \mathbb{R}]$ for $J = [t_0, t_0 + T]$, t_0, $T \in \mathbb{R}^+$ and let α_0, $\beta_0 \in E$ satisfy

$$\alpha_0 < (N\alpha_0)(t) \text{ for } t_0 \le t \le T + t_0 \tag{3.8}$$

$$\beta_0 \ge (N\beta_0)(t) \text{ for } T + t_0 \ge t \ge t_0. \tag{3.9}$$

or

$$\alpha_0 \le (N\alpha_0)(t) \text{ for } t_0 \le t \le T + t_0 \tag{3.10}$$

$$\beta_0 > (N\beta_0)(t) \text{ for } T + t_0 \ge t \ge t_0. \tag{3.11}$$

(ii) N is semi nondecreasing.

(iii) $(Nx)(t) - (Ny)(t) \le L \max_{t_0 \le s \le t} [x(s) - y(s)]$ whenever $x(s) \ge y(s)$ for $t_0 \le s \le t$ and $0 < L < 1$.

Then

$$\alpha_0(t) \le \beta_0(t) \text{ for } t_0 \le t \le t_0 + T \tag{3.12}$$

provided

$$\alpha_0(t_0) \leq \beta_0(t_0). \tag{3.13}$$

Proof [5]: Let us define $\beta_{0\varepsilon}(t) = \beta_0(t) + \varepsilon$ where $\varepsilon > 0$ is arbitrary small. Then we have

$$\beta_{0\varepsilon}(t_0) = \beta_0(t_0) + \varepsilon \geq \alpha_0(t_0) + \varepsilon > \alpha_0(t_0)$$

and

$$\beta_{0\varepsilon}(t) \geq \beta_0(t) \text{ for } t_0 \leq t \leq t_0 + T.$$

Now, by using the one sided Lipschitz condition in (iii), we get

$$\beta_{0\varepsilon}(t) = \beta_0(t) + \varepsilon \geq (N\beta_0)(t) + \varepsilon \geq (N\beta_{0\varepsilon})(t) - L\varepsilon + \varepsilon > (N\beta_{0\varepsilon})(t)$$

for $t_0 \leq t \leq t_0 + T$, since $0 < L < 1$. Now by applying Theorem 3.2 with (3.10) and (3.11) to $\alpha_0(t)$ and $\beta_{0\varepsilon}(t)$, we find that

$$\alpha_0(t) < \beta_{0\varepsilon}(t) \text{ for } t_0 \leq t \leq t_0 + T. \tag{3.14}$$

Since $\varepsilon > 0$ is arbitrary small, by taking $\varepsilon \to 0$ in (3.14), we get

$$\alpha_0(t) \leq \beta_0(t) \text{ for } t_0 \leq t \leq t_0 + T.$$

Therefore this completes the proof.

The proof of Theorem 3.3 can also be done by using (3.8), (3.9) and Theorem 3.1.

4. Causal Differential Inequalities

We give a basic result in causal differential inequalities for the scalar case as follows.

Theorem 4.1: Assume that

(i) $\alpha_0, \beta_0 \in C^1[J, \mathbb{R}]$ and $N : E \to E$ is a continuous causal operator, $E = C[J, \mathbb{R}]$ for $J = [t_0, t_0 + T]$, $t_0, T \in \mathbb{R}^+$ and let α_0, $\beta_0 \in E$ satisfy

$$\alpha_0' < (N\alpha_0)(t) \text{ for } t_0 \leq t \leq T + t_0 \tag{4.1}$$

$$\beta_0' \geq (N\beta_0)(t) \text{ for } T + t_0 \geq t \geq t_0. \tag{4.2}$$

or

$$\alpha_0' \leq (N\alpha_0)(t) \text{ for } t_0 \leq t \leq T + t_0 \tag{4.3}$$

$$\beta_0' > (N\beta_0)(t) \text{ for } T + t_0 \geq t \geq t_0. \tag{4.4}$$

(ii) the causal operator N is semi nondecreasing.

Then

$$\alpha_0(t_0) < \beta_0(t_0)$$

implies

$$\alpha_0(t) < \beta_0(t) \text{ for } t_0 \leq t \leq T + t_0. \tag{4.5}$$

Proof : Suppose that the conclusion (4.5) of Theorem 4.1 is false and $\alpha_0'(t) < (N\alpha_0)(t)$. Then the continuity of the $\alpha_0(t), \beta_0(t)$ and the fact that $\alpha_0(t_0) < \beta_0(t_0)$ yield that there exists a $t_1 > t_0$ such that

$$\alpha_0(t_1) = \beta_0(t_1), \ \alpha_0(t) < \beta_0(t) \ \text{ for } t_0 \le t < t_1. \tag{4.6}$$

The semi nondecreasing nature of N and (4.6) give

$$(N\alpha_0)(t_1) \le (N\beta_0)(t_1). \tag{4.7}$$

In view of (4.6), we get for small $h > 0$,

$$\alpha_0(t_1 - h) - \alpha_0(t_1) < \beta_0(t_1 - h) - \beta_0(t_1)$$

and hence (4.1) and (4.7) show that

$$(N\alpha_0)(t_1) \le (N\beta_0)(t_1) \le \beta_0'(t_1) \le \alpha_0'(t_1) < (N\alpha_0)(t_1).$$

This is a contradiction and therefore the claim (4.5) is valid. The proof is complete.

The proof of Theorem 4.1 can also be done by using (4.3) and (4.4).

As before, for nonstrict differential inequalities, we require a one-sided Lipschitz condition.

Theorem 4.2: Assume that

(i) $\alpha_0, \beta_0 \in C^1[J, \mathbb{R}]$ and $N : E \to E$ is a continuous causal operator, $E = C[J, \mathbb{R}]$ for $J = [t_0, t_0 + T]$, $t_0, T \in \mathbb{R}^+$ and let $\alpha_0, \beta_0 \in E$ satisfy

$$\alpha_0' < (N\alpha_0)(t) \ \text{ for } t_0 \le t \le T + t_0 \tag{4.8}$$
$$\beta_0' \ge (N\beta_0)(t) \ \text{ for } T + t_0 \ge t \ge t_0. \tag{4.9}$$

or

$$\alpha_0' \le (N\alpha_0)(t) \ \text{ for } t_0 \le t \le T + t_0 \tag{4.10}$$
$$\beta_0' > (N\beta_0)(t) \ \text{ for } T + t_0 \ge t \ge t_0. \tag{4.11}$$

(ii) N is semi nondecreasing.
(iii) $(Nx)(t) - (Ny)(t) \le L \max_{t_0 \le s \le t}[x(s) - y(s)]$ whenever $x(s) \ge y(s)$ for $t_0 \le s \le t$ and $0 < L < 1$.
Then

$$\alpha_0(t_0) \le \beta_0(t_0) \tag{4.12}$$

implies

$$\alpha_0(t) \le \beta_0(t) \ \text{ for } t_0 \le t \le T + t_0. \tag{4.13}$$

Proof: Let us set $\beta_{0\varepsilon}(t) = \beta_0(t) + \varepsilon \exp(2L(t - t_0))$ for small $\varepsilon > 0$. Then

$$\beta_{0\varepsilon}(t_0) > \beta_0(t_0) \ \text{ and } \beta_{0\varepsilon}(t) > \beta_0(t). \tag{4.14}$$

Now we use the one-sided Lipschitz condition

$$(N\beta_{0\varepsilon})(t) - (N\beta_0)(t) \leq L \max_{t_0 \leq s \leq t} [\beta_{0\varepsilon}(s) - \beta_0(s)] \leq L\varepsilon \exp(2L(t-t_0))$$

to obtain

$$\begin{aligned} \beta'_{0\varepsilon}(t) &= \beta'_0(t) + 2L\varepsilon \exp(2L(t-t_0)) \\ &\geq (N\beta_0)(t) + 2L\varepsilon \exp(2L(t-t_0)) \\ &\geq (N\beta_{0\varepsilon})(t) - L\varepsilon \exp(2L(t-t_0)) + 2L\varepsilon \exp(2L(t-t_0)) \\ &= (N\beta_{0\varepsilon})(t) + L\varepsilon \exp(2L(t-t_0)) \\ &> (N\beta_{0\varepsilon})(t). \end{aligned}$$

We will show that $\alpha_0(t) < \beta_{0\varepsilon}(t)$ for $t_0 \leq t \leq t_0 + T$. If this is not true, because of (4.13), there would exist a $t_1 > t_0$ such that

$$\alpha_0(t_1) = \beta_{0\varepsilon}(t_1) \text{ and } \alpha_0(t) < \beta_{0\varepsilon}(t),\ t_0 \leq t < t_1 < T. \tag{4.15}$$

Now we have

$$\begin{aligned} \beta'_{0\varepsilon}(t) &> (N\beta_{0\varepsilon})(t), \beta_{0\varepsilon}(t_0) \geq x_0 \text{ and } \alpha'_0(t) \leq (N\alpha_0)(t), \alpha_0(t_0) \leq x_0 \\ \text{for } t_0 &\leq t \leq t_0 + T. \end{aligned}$$

And semi-nondecreasing nature of N and (4.15) give

$$(N\alpha_0)(t_1) = (N\beta_{0\varepsilon})(t_1) \text{ and } (N\alpha_0)(t) \leq (N\beta_{0\varepsilon})(t)\ , t_0 \leq t < t_1 < T. \tag{4.16}$$

Also, in view of (4.15), we get for small $h > 0$

$$\alpha_0(t_1 - h) - \alpha_0(t_1) < \beta_{0\varepsilon}(t_1 - h) - \beta_{0\varepsilon}(t_1).$$

Hence the assumption (i), (4.11) and (4.16) show that

$$(N\alpha_0)(t_1) \geq \alpha'_0(t_1) \geq \beta'_{0\varepsilon}(t_1) > (N\beta_{0\varepsilon})(t_1) \geq (N\alpha_0)(t_1).$$

This leads to the contradiction because of (4.15). Then we have

$$\begin{aligned} \beta'_{0\varepsilon}(t) &> (N\beta_{0\varepsilon})(t), \beta_{0\varepsilon}(t_0) \geq \alpha_0(t_0) \text{ and } \alpha'_0(t) \leq (N\alpha_0)(t), \alpha_0(t_0) \leq \beta_{0\varepsilon}(t_0) \\ \text{for } t_0 &\leq t \leq t_0 + T \end{aligned}$$

and $\alpha_0(t) < \beta_{0\varepsilon}(t)$ for $t_0 \leq t \leq t_0 + T$. We get $\alpha_0(t) \leq \beta_0(t)$ as ε approaches to zero for $t_0 \leq t \leq t_0 + T$. This completes the proof.

The proof of Theorem 4.2 can also be done by using (4.10) and (4.11).

Theorem 4.3: Assume that

(i) $\alpha_0 \in C^1[[t_0, t_0 + T], E]$, $t_0, T > 0$, $\beta_0 \in C^1[[\tau_0, \tau_0 + T], E]$, $\tau_0 \geq 0$ and $N \in C[R^+ \times E, E]$, $\alpha'_0(t) \leq (N\alpha_0)(t)$, $\alpha_0(t_0) \leq x_0$ for $t_0 \leq t \leq t_0 + T$ and $\beta'_0(t) \geq (N\beta_0)(t)$, $x_0 \leq \beta_0(\tau_0)$ for $\tau_0 \leq t \leq \tau_0 + T$;

(*ii*) $(Nx)(t) - (Ny)(t) \leq L \max_{t_0 \leq s \leq t} [x(s) - y(s)]$ whenever $x(s) \geq y(s)$ for $t_0 \leq s \leq t$ and $0 < L < 1$.

(*iii*) $(Nu)(t)$ is semi-nondecreasing in u for each t.

(*iv*) $t_0 < \tau_0$, $(Nu)(t)$ is nondecreasing in t for each u.

Then (I) $\alpha_0(t) \leq \beta_0(t+\eta)$ for $t_0 \leq t \leq t_0+T$ where $\eta = \tau_0 - t_0$, (II) $\alpha_0(t-\eta) \leq \beta_0(t)$ for $\tau_0 \leq t \leq \tau_0 + T$ where $\eta = \tau_0 - t_0$.

Proof: Suppose that $\bar{\beta}_0(t) = \beta_0(t+\eta)$ so that $\bar{\beta}_0(t_0) = \beta_0(t_0+\eta) = \beta_0(\tau_0) \geq x_0 \geq \alpha_0(t_0)$, and

$$\bar{\beta}_0'(t) = \beta_0'(t+\eta) \geq (N\beta_0)(t+\eta) = (N\bar{\beta}_0)(t) \text{ for } t_0 \leq t \leq t_0 + T.$$

Let $\bar{\beta}_{0\varepsilon}(t) = \bar{\beta}_0(t) + \varepsilon \exp(2L(t-t_0))$ for small $\varepsilon > 0$. Then

$$\bar{\beta}_{0\varepsilon}(t) > \bar{\beta}_0(t) \text{ and } \bar{\beta}_{0\varepsilon}(t_0) > \bar{\beta}_0(t) \geq \alpha_0(t_0). \tag{4.17}$$

We will show that $\alpha_0(t) < \bar{\beta}_{0\varepsilon}(t)$ for $t_0 \leq t \leq t_0 + T$. If this is not true, because of (I), there would exist a $t_1 > t_0$ such that

$$\alpha_0(t_1) = \bar{\beta}_{0\varepsilon}(t_1) \text{ and } \alpha_0(t) < \bar{\beta}_{0\varepsilon}(t),\ t_0 \leq t < t_1 < T. \tag{4.18}$$

Now we use the one-sided Lipschitz condition

$$\left(N\bar{\beta}_{0\varepsilon}\right)(t) - \left(N\bar{\beta}_0\right)(t) \leq L \max_{t_0 \leq s \leq t} \left[\bar{\beta}_{0\varepsilon}(s) - \bar{\beta}_0(s)\right] \leq L\varepsilon \exp(2L(t-t_0))$$

to obtain

$$\begin{aligned}
\bar{\beta}_{0\varepsilon}'(t) &= \bar{\beta}_0'(t) + 2L\varepsilon \exp(2L(t-t_0)) \\
&\geq (N\bar{\beta}_0)(t) + 2L\varepsilon \exp(2L(t-t_0)) \\
&\geq (N\bar{\beta}_{0\varepsilon})(t) + 2L\varepsilon \exp(2L(t-t_0)) \\
&\geq \left(N\bar{\beta}_{0\varepsilon}\right)(t) - L\varepsilon \exp(2L(t-t_0)) + 2L\varepsilon \exp(2L(t-t_0)) \\
&> \left(N\bar{\beta}_{0\varepsilon}\right)(t).
\end{aligned}$$

Now we have

$$\begin{aligned}
\bar{\beta}_{0\varepsilon}'(t) &> \left(N\bar{\beta}_{0\varepsilon}\right)(t), \bar{\beta}_{0\varepsilon}(t_0) \geq x_0 \text{ and } \alpha_0'(t) \leq (N\alpha_0)(t), \alpha_0(t_0) \leq x_0 \\
\text{for } t_0 &\leq t \leq t_0 + T.
\end{aligned}$$

Semi-nondecreasing nature of N and (4.18) give

$$(N\alpha_0)(t_1) = (N\bar{\beta}_{0\varepsilon})(t_1) \text{ and } (N\alpha_0)(t) \leq (N\bar{\beta}_{0\varepsilon})(t)\ ,t_0 \leq t < t_1 < T. \tag{4.19}$$

Also, in view of (4.19), we get for small $h > 0$

$$\alpha_0(t_1 - h) - \alpha_0(t_1) < \bar{\beta}_{0\varepsilon}(t_1 - h) - \bar{\beta}_{0\varepsilon}(t_1). \tag{4.20}$$

Hence the assumption (i) and (4.20) show that

$$(N\alpha_0)(t_1) \geq \alpha_0'(t_1) \geq \bar{\beta}_{0\varepsilon}'(t_1) > (N\bar{\beta}_{0\varepsilon})(t_1) \geq (N\alpha_0)(t_1).$$

Since $t_0 < \tau_0$, assumption (iv), $(Nu)(t)$ being nondecreasing in t, leads to a contradiction because of (4.19).

By applying Theorem 4.1, we have

$$\bar{\beta}_{0\varepsilon}'(t) > \left(N\bar{\beta}_{0\varepsilon}\right)(t)\,, \bar{\beta}_{0\varepsilon}(t_0) \geq \alpha_0(t_0) \text{ and } \alpha_0'(t) \leq (N\alpha_0)(t), \alpha_0(t_0) \leq \bar{\beta}_{0\varepsilon}(t_0)$$
$$\text{for } t_0 \leq t \leq t_0 + T$$

and $\alpha_0(t) < \bar{\beta}_{0\varepsilon}(t)$ for $t_0 \leq t \leq t_0 + T$. We get $\alpha_0(t) \leq \bar{\beta}_0(t)$ as ε approaches to zero for $t_0 \leq t \leq t_0 + T$. This completes the proof.

To prove (II), we set $\bar{\alpha}_0(t) = \alpha_0(t-\eta)$ for $\tau_0 \leq t$ so that $\bar{\alpha}_0(\tau_0) = \alpha_0(\tau_0 - \eta) = \alpha_0(t_0) \leq x_0 \leq \beta_0(\tau_0)$, and

$$\bar{\alpha}_0'(t) = \alpha_0'(t+\eta) \leq (N\alpha_0)(t-\eta) = (N\bar{\alpha}_0)(t) \text{ for } \tau_0 \leq t \leq \tau_0 + T.$$

Setting $\bar{\alpha}_{0\varepsilon}(t) = \bar{\alpha}_0(t) - \varepsilon \exp\left(2L\left(t - t_0\right)\right)$ for some $\varepsilon > 0$ small. Then proceeding similarly, we derive the estimate $\alpha_0(t-\eta) \leq \beta_0(t)$ for $\tau_0 \leq t \leq \tau_0 + T$ where $\eta = \tau_0 - t_0$. Therefore the proof is completed.

If we know the existence of lower and upper solutions of (2.1) such that $\alpha_0\left(t\right) \leq \beta_0\left(t+\eta\right),\ t \in J$, then we can prove the existence of a solution of the initial value problem (2.1) in the closed set $\tilde{\tilde{\Omega}} = \{u \in E : \alpha_0\left(t\right) \leq u \leq \beta_0\left(t+\eta\right),\ t \in J\}$ where $\eta = \tau_0 - t_0$.

Theorem 4.4: Assume that

(i) $\alpha_0 \in C^1\left[\left[t_0, t_0 + T\right], E\right],\ t_0, T > 0,\ \beta_0 \in C^1\left[\left[\tau_0, \tau_0 + T\right], E\right],\ \tau_0 \geq 0$ and $N \in C\left[R^+ \times E, E\right],\ \alpha_0'(t) \leq (N\alpha_0)(t), \alpha_0(t_0) \leq x_0$ for $t_0 \leq t \leq t_0 + T$ and $\beta_0'(t) \geq (N\beta_0)(t), x_0 \leq \beta_0(\tau_0)$ for $\tau_0 \leq t \leq \tau_0 + T$;

(ii) $\left(Nx\right)(t) - \left(Ny\right)(t) \leq L \max_{t_0 \leq s \leq t}\left[x(s) - y(s)\right]$ whenever $x(s) \geq y(s)$ for $t_0 \leq s \leq t$ and $0 < L < 1$;

(iii) $(Nu)(t)$ is semi-nondecreasing in u for each t;

(iv) $t_0 < \tau_0$, $(Nu)(t)$ is nondecreasing in t for each u;

(v) the operator N is bounded on $\tilde{\tilde{\Omega}}$.

Then there exists a solution $u(t)$ of (2.1) in the closed set $\tilde{\tilde{\Omega}}$ with $u(t_0) = u_0$, satisfying $\alpha_0(t) \leq u \leq \beta_0(t+\eta)$ for $t_0 \leq t \leq t_0 + T$.

Proof: Let $P \in C(J, \mathbb{R})$ be defined by

$$(Pu)(t) = \max\left[\alpha_0(t), \min\left[u(t), \beta_0(t+\eta)\right]\right].$$

Then $(NPu)(t)$ defines a continuous extension of N on E which is also bounded since N is assumed to be bounded on $\tilde{\tilde{\Omega}}$. Therefore, there exists a solution of the initial value problem

$$u'(t) = (NPu)(t), u(t_0) = u_0$$

on J. For any $\varepsilon > 0$, and for $\bar{\beta}_0(t) = \beta_0(t+\eta)$ consider

$$\begin{aligned} \bar{\beta}_{0\varepsilon}(t) &= \bar{\beta}_0(t) + \varepsilon(1+t) \\ \bar{\alpha}_{0\varepsilon}(t) &= \alpha_0(t) + \varepsilon(1-t). \end{aligned}$$

Then we have $\bar{\alpha}_{0\varepsilon}(t_0) < u_0 < \bar{\beta}_{0\varepsilon}(t_0)$, since $\alpha_0(t_0) \leq u_0 \leq \bar{\beta}_0(t_0)$. We need to show that

$$\bar{\alpha}_{0\varepsilon}(t) < u(t) < \bar{\beta}_{0\varepsilon}(t), \text{ on } J. \tag{4.21}$$

If this is not true, then there exists a $t_1 \in (t_0, t_0+T]$ at which $u(t_1) = \bar{\beta}_{0\varepsilon}(t_1)$ and $\bar{\alpha}_{0\varepsilon}(t) < u(t) < \bar{\beta}_{0\varepsilon}(t)$, $t_0 \leq t < t_1$. Then

$$u(t_1) > \bar{\beta}_0(t_1) \text{ and } (Pu)(t_1) = \bar{\beta}_0(t_1). \tag{4.22}$$

Moreover,

$$\alpha_0(t_1) \leq (Pu)(t_1) \leq \beta_0(t_1+\eta).$$

Hence,

$$\bar{\beta}_0'(t_1) \geq (NPu)(t_1) = u'(t_1).$$

Since $\bar{\beta}_{0\varepsilon}'(t_1) > \bar{\beta}_0'(t_1)$, we have $\bar{\beta}_{0\varepsilon}'(t_1) > u'(t_1)$. However, we have $\bar{\beta}_{0\varepsilon}'(t_1) \leq u'(t_1)$ since $u(t_1) = \bar{\beta}_{0\varepsilon}(t_1)$ and $u(t) < \bar{\beta}_{0\varepsilon}(t)$, $t_0 \leq t < t_1$. This is a contradiction with $\bar{\beta}_{0\varepsilon}'(t_1) > u'(t_1)$. Hence for all $t \in J, u(t) < \bar{\beta}_{0\varepsilon}(t)$ and consequently (4.21) holds on J, *i.e.* $\bar{\alpha}_{0\varepsilon}(t) < u(t) < \bar{\beta}_{0\varepsilon}(t)$ on J. We get

$$\alpha_0(t) \leq u(t) \leq \bar{\beta}_0(t) \text{ on } J \text{ as } \varepsilon \to 0. \tag{4.23}$$

This completes the proof.

5. Main Results

In this section, we will prove the main theorem that gives several different conditions to apply the method of generalized quasilinearization to the nonlinear causal differential equations [5] with initial time difference and state remarks and corollaries for special cases.

Theorem 5.1: Assume that

(i) $N : E \to E$ is a continuous causal operator, $E = C[J, \mathbb{R}]$ for $J = [t_0, \tau_0 + T]$, $t_0, \tau_0, T \in \mathbb{R}^+$ and there exists a constant M such that $(Nu)(t) \le M$ on $J \times \Omega$;
(ii) $(Nu)(t)$ is semi nondecreasing in u for each $t \in J$;
(iii) $\alpha_0 \in C^1[[t_0, t_0 + T], E]$ and $\beta_0 \in C^1[[\tau_0, \tau_0 + T], E]$ for $\tau_0 \ge t_0 > 0$ and $T > 0$,

$$\begin{aligned} \alpha_0'(t) &\le (N\alpha_0)(t) \text{ for } t_0 \le t \le t_0 + T \\ \beta_0'(t) &\ge (N\beta_0)(t) \text{ for } \tau_0 \le t \le \tau_0 + T \end{aligned}$$

where $\alpha_0(t_0) \le \beta_0(\tau_0)$;
(iv) $t_0 < s_0 < \tau_0$, $(Nu)(t)$ is nondecreasing in t for each u;
$\alpha_0, \beta_0 \in C^1[J, \mathbb{R}]$ such that $\alpha_0'(t) \le (N\alpha_0)(t)), (N\beta_0)(t) \le \beta_0'(t)$ and $\alpha_0(t) \le \beta_0(t)$, $t \in J$;
(v) the Fréchet derivative; $(N_x x)(t)$ exists and is continuous and $(N_x x)(t) \le L_1$ for $(t, x) \in J \times \Omega$, for some $L_1 > 0$ and $(Ny)(t) \le (Nx)(t) - (N_x y)(t)(x - y)$ where $\alpha_0(t) \le y \le x \le \beta_0(t)$, $t \in J$;
(vi) $(N_x x)(t) - (N_x y)(t) \le L_2(x - y)^\gamma, t \in J$, where L_2 is a positive constant and $0 \le \gamma < 1$.

Then there exist monotone sequences $\left\{\tilde{\alpha}_n(t)\right\}$ and $\left\{\tilde{\beta}_n(t)\right\}$ which converge uniformly to the unique solution of (2.1) with $u(s_0) = x_0$ where s_0 is between initial time t_0 and τ_0 and the convergence is superlinear.

Proof: Since $\tilde{\beta}_0(t) = \beta_0(t + \eta_1)$, $\eta_1 = \tau_0 - t_0$ we get $\tilde{\beta}_0(t_0) = \beta_0(\tau_0) \ge \alpha_0(t_0) = \tilde{\alpha}_0(t_0)$ and $\tilde{\beta}_0'(t) \ge (N\tilde{\beta}_0)(t + \eta_1)$ for $t_0 \le t \le t_0 + T$. Using the assumptions *(iv)*, it is clear that $N(t, x)$ satisfies the Lipschitz condition in x for $(t, x) \in J \times \Omega$. Furthermore, we have the following inequalities

$$(Nx)(t) \ge (Ny)(t) + (N_x y)(t)(x - y) \text{ whenever } \tilde{\alpha}_0(t) \le y \le x \le \tilde{\beta}_0(t) \text{ on } J \quad (5.1)$$

and also by using *(iv)* we see that whenever $\tilde{\alpha}_0(t) \le y \le x \le \tilde{\beta}_0(t)$,

$$(Nx)(t) - (Ny)(t) \le L(x - y) \quad (5.2)$$

for some $L > 0$.

Consider the linear IVPs of causal differential equations for $\eta_2 = s_0 - t_0$

$$\widetilde{\alpha}_1' = (N\widetilde{\alpha}_0)(t+\eta_2) + (N_x\widetilde{\alpha}_0)(t+\eta_2)\left(\widetilde{\alpha}_1 - \widetilde{\alpha}_0\right),\ \widetilde{\alpha}_1(t_0) = u_0 \tag{5.3}$$

$$\widetilde{\beta}_1' = (N\widetilde{\beta}_0)(t+\eta_2) + (N_x\widetilde{\alpha}_0)(t+\eta_2)\left(\widetilde{\beta}_1 - \widetilde{\beta}_0\right),\ \widetilde{\beta}_1(t_0) = u_0 \tag{5.4}$$

where $\tilde{\alpha}_0(t_0) \le u_0 \le \widetilde{\beta}_0(t_0)$. We shall show that $\tilde{\alpha}_0 \le \tilde{\alpha}_1$ on J. To do this, let $p = \tilde{\alpha}_0(t) - \tilde{\alpha}_1(t)$, so that $p(t_0) \le 0$. Then

$$\begin{aligned} p' &= \tilde{\alpha}_0' - \tilde{\alpha}_1' \\ &\le (N\tilde{\alpha}_0)(t+\eta_2) - \left[(N\tilde{\alpha}_0)(t+\eta_2) + (N_x\tilde{\alpha}_0)(t+\eta_2)\left(\widetilde{\alpha}_1 - \tilde{\alpha}_0\right)\right] \\ &= (N_x\tilde{\alpha}_0)(t+\eta_2)p. \end{aligned}$$

Theorem 4.2 gives $p(t) \le 0$ on J proving that $\tilde{\alpha}_0(t) \le \tilde{\alpha}_1(t)$ on J. Now set $p = \widetilde{\alpha}_1 - \widetilde{\beta}_0$ and note that $p(t_0) \le 0$. Also, using (5.1)

$$\begin{aligned} p' &= \widetilde{\alpha}_1' - \widetilde{\beta}_0' \\ &\le (N\tilde{\alpha}_0)(t+\eta_2) + (N_x\tilde{\alpha}_0)(t+\eta_2)\left(\widetilde{\alpha}_1 - \tilde{\alpha}_0\right) - (N\widetilde{\beta}_0)(t+\eta_1) \\ &\le (N\widetilde{\beta}_0)(t+\eta_2) - (N_x\tilde{\alpha}_0)(t+\eta_2)\left(\widetilde{\beta}_0 - \tilde{\alpha}_0\right) + (N_x\tilde{\alpha}_0)(t+\eta_2)\left(\widetilde{\alpha}_1 - \tilde{\alpha}_0\right) \\ &\quad -(N\widetilde{\beta}_0)(t+\eta_2) \\ &\le (N_x\tilde{\alpha}_0)(t+\eta_2)p \end{aligned}$$

which again implies $\widetilde{\alpha}_1(t) \le \widetilde{\beta}_0(t)$ on J.

Similarly, we can obtain that $\tilde{\alpha}_0(t) \le \tilde{\beta}_1(t) \le \tilde{\beta}_0(t)$ on J. In order to prove that $\tilde{\alpha}_1(t) \le \tilde{\beta}_1(t)$ on J, we proceed as follows. Since $\tilde{\alpha}_0 \le \tilde{\alpha}_1 \le \tilde{\beta}_0$,using (5.1), we see that

$$\tilde{\alpha}_1(t) = (N\tilde{\alpha}_0)(t+\eta_2) + (N_x\tilde{\alpha}_0)(t+\eta_2)\left(\widetilde{\alpha}_1 - \tilde{\alpha}_0\right) \le (N\tilde{\alpha}_1)(t+\eta_2).$$

Similarly, $(N\widetilde{\beta}_1)(t+\eta_2) \le \widetilde{\beta}_1'(t)$ and therefore by Theorem 4.2 it follows that $\tilde{\alpha}_1(t) \le \widetilde{\beta}_1'(t)$ on J which shows that

$$\tilde{\alpha}_0(t) \le \tilde{\alpha}_1(t) \le \tilde{\beta}_1(t) \le \tilde{\beta}_0(t) \text{ on } J.$$

Assume that for some $n > 1$, $\widetilde{\alpha}_n' \le (N\tilde{\alpha}_n)(t+\eta_2), (N\widetilde{\beta}_n)(t+\eta_2) \le \widetilde{\beta}_n'$ and $\widetilde{\alpha}_n(t) \le \widetilde{\beta}_n(t)$, $t \in J$.

We must show that

$$\tilde{\alpha}_n(t) \le \tilde{\alpha}_{n+1}(t) \le \tilde{\beta}_{n+1}(t) \le \tilde{\beta}_n(t) \text{ on } J \tag{5.5}$$

where $\tilde{\alpha}_{n+1}(t)$ and $\tilde{\beta}_{n+1}(t)$ are the solutions of linear IVPs of casual differential equations as follows

$$\widetilde{\alpha}'_{n+1} = (N\tilde{\alpha}_n)(t+\eta_2) + (N_x\tilde{\alpha}_n)(t+\eta_2)(\widetilde{\alpha}_{n+1} - \tilde{\alpha}_n),\ \tilde{\alpha}_{n+1}(t_0) = u_0 \tag{5.6}$$

$$\widetilde{\beta}'_{n+1} = (N\widetilde{\beta}_n)(t+\eta_2) + (N_x\tilde{\alpha}_n)(t+\eta_2)(\widetilde{\beta}_{n+1} - \widetilde{\beta}_n),\ \widetilde{\beta}_{n+1}(t_0) = u_0. \tag{5.7}$$

Hence setting $p = \tilde{\alpha}_n(t) - \tilde{\alpha}_{n+1}(t)$, it follows as before $p' \leq (N_x\tilde{\alpha}_n)(t+\eta_2)p$ on J and hence $\tilde{\alpha}_n(t) \leq \tilde{\alpha}_{n+1}(t) \leq \tilde{\beta}_n(t)$ on J. In a similar manner, we can prove that $\tilde{\alpha}_n(t) \leq \tilde{\beta}_{n+1}(t) \leq \tilde{\beta}_n(t)$ on J.

Using (5.1), we obtain

$$\begin{aligned}
\alpha'_{n+1} &= (N\tilde{\alpha}_n)(t+\eta_2) + (N_x\tilde{\alpha}_n)(t+\eta_2)\left(\widetilde{\alpha}_{n+1} - \tilde{\alpha}_n\right) \\
&\leq (N\tilde{\alpha}_{n+1})(t+\eta_2) - (N_x\tilde{\alpha}_n)(t+\eta_2)\left(\widetilde{\alpha}_{n+1} - \tilde{\alpha}_n\right) \\
&\quad + (N_x\tilde{\alpha}_n)(t+\eta_2)\left(\widetilde{\alpha}_{n+1} - \tilde{\alpha}_n\right) \\
&= (N\tilde{\alpha}_{n+1})(t+\eta_2).
\end{aligned}$$

Similar arguments yield $\left(N\widetilde{\beta}_{n+1}\right)(t+\eta_2) \leq \widetilde{\beta}'_{n+1}$and hence Theorem 4.2 shows that $\tilde{\alpha}_{n+1}(t) \leq \tilde{\beta}_{n+1}(t)$ on J which proves (5.5) is true. So by using induction we obtain

$$\tilde{\alpha}_0 \leq \tilde{\alpha}_1 \leq \cdots \leq \tilde{\alpha}_n \leq \tilde{\alpha}_{n+1} \leq \tilde{\beta}_{n+1} \leq \tilde{\beta}_n \leq \cdots \leq \tilde{\beta}_1 \leq \tilde{\beta}_0 \text{ on } J.$$

Now using standard arguments (Arzela-Ascoli and Dini's Theorems, see[2]), it can be shown that the sequences $\{\tilde{\alpha}_n(t)\}$ and $\{\tilde{\beta}_n(t)\}$ converge uniformly and monotonically to the unique solution of $u(t)$ of (2.1) on J.

$$\widetilde{u}'(t) = (N\widetilde{u})(t+\eta_2),\ \widetilde{u}(t_0) = u_0 \tag{5.8}$$

But letting $s = t+\eta_2$ and changing the variable, we can show that (5.8) is equivalent to

$$u'(s) = (Nu)(s),\ u(s_0) = u_0.$$

Finally, to prove superlinear convergence, we let

$$p_n(t) = \widetilde{u}(t) - \tilde{\alpha}_n(t) \text{ and } q_n(t) = \tilde{\beta}_n(t) - \widetilde{u}(t).$$

Note that $p_n(t_0) = q_n(t_0) = 0$.

$$\begin{aligned}
p_n'(t) &= \widetilde{u}'(t) - \tilde{\alpha}_n'(t) \\
&= N\left(\widetilde{u}\right)(t+\eta_2) \\
&\quad -\left[(N\tilde{\alpha}_{n-1})(t+\eta_2) + (N_x\tilde{\alpha}_{n-1})(t+\eta_2)\left(\widetilde{\alpha}_n - \tilde{\alpha}_{n-1}\right)\right] \\
&= \int_0^1 \left(N_x\left(s\widetilde{u} + (1-s)\tilde{\alpha}_{n-1}\right)\right)(t+\eta_2)\left(\widetilde{u} - \tilde{\alpha}_{n-1}\right) ds \\
&\quad - N_x(\tilde{\alpha}_{n-1})(t+\eta_2)\left(\widetilde{\alpha}_n - \tilde{\alpha}_{n-1}\right) \\
&= \int_0^1 \left(N_x\left(s\widetilde{u} + (1-s)\tilde{\alpha}_{n-1}\right)\right)(t+\eta_2)\, p_{n-1} ds \\
&\quad - N_x(\tilde{\alpha}_{n-1})(t+\eta_2)(p_{n-1} - p_n) \\
&= \int_0^1 \left[\left(N_x(s\widetilde{u} + (1-s)\tilde{\alpha}_{n-1})\right)(t+\eta_2) - (N_x\tilde{\alpha}_{n-1})(t+\eta_2)\right] p_{n-1} ds \\
&\quad + N_x\tilde{\alpha}_{n-1})(t+\eta_2)\, p_n.
\end{aligned}$$

From (iv) and (v), it follows that

$$\begin{aligned}
\|p_n'(t)\| &\leq \int_0^1 L_2 \left\|s\widetilde{u} + (1-s)\tilde{\alpha}_{n-1} - \tilde{\alpha}_{n-1})\right\|^\gamma \|p_{n-1}\| ds + L_1\|p_n\| \\
&\leq \int_0^1 L_2 \left\|s\widetilde{u} - s\tilde{\alpha}_{n-1}\right\|^\gamma \|p_{n-1}\| ds + L_1\|p_n\| \\
&\leq \int_0^1 L_2 \|sp_{n-1}\|^\gamma \|p_{n-1}\| ds + L_1\|p_n\| \\
&= L_2\|p_{n-1}\|^{\gamma+1} + L_1\|p_n\|.
\end{aligned}$$

Then setting $a_n = \|p_n\|$, we find

$$a_n' \leq \|p_n'\| \leq L_2(a_{n-1})^{\gamma+1} + L_1 a_n.$$

Now Gronwall's inequality implies

$$0 \leq a_n(t) \leq L_2 \int_0^t \exp\left[L_1(t-s)\right](a_n(s))^{\gamma+1} ds \text{ on } J$$

which yields the estimate

$$\max_J \|p_n(t)\| \leq L_2 \frac{\exp(L_1 T)}{L_1} \max_J \|p_{n-1}(t)\|^{\gamma+1}.$$

Similarly,

$$\begin{aligned}
q_n'(t) &= \tilde{\beta}_n'(t) - \tilde{u}'(t) \\
&= \left(N\tilde{\beta}_{n-1}\right)(t+\eta_2) + \left(N_x\tilde{\alpha}_{n-1}\right)(t+\eta_2)\left(\tilde{\beta}_n - \tilde{\beta}_{n-1}\right) - N\left(\tilde{u}(t)\right)(t+\eta_2) \\
&= \int_0^1 \left(N_x(s\tilde{\beta}_{n-1} + (1-s)\tilde{u}(t))\right)(t+\eta_2)\left(\tilde{\beta}_{n-1} - \tilde{u}\right) ds \\
&\quad + \left(N_x\tilde{\alpha}_{n-1}\right)(t+\eta_2))\left(\tilde{\beta}_n - \tilde{\beta}_{n-1}\right) \\
&= \int_0^1 \left(N_x(s\tilde{\beta}_{n-1} + (1-s)\tilde{u}(t))\right)(t+\eta_2)\, q_{n-1} ds + \\
&\quad \left(N_x\tilde{\alpha}_{n-1}\right)(t+\eta_2)\left(q_n - q_{n-1}\right) \\
&= \int_0^1 \left[\left(N_x(s\tilde{\beta}_{n-1} + (1-s)\tilde{u}(t))\right)(t+\eta_2) - \left(N_x\tilde{u}\right)(t+\eta_2)\right] q_{n-1} ds \\
&\quad + \left[\left(N_x\tilde{u}\right)(t+\eta_2) - \left(N_x\tilde{\alpha}_{n-1}\right)(t+\eta_2)\right] q_{n-1} + \left(N_x\tilde{\alpha}_{n-1}\right)(t+\eta_2) q_n.
\end{aligned}$$

We find, using (iv) and (v), that

$$\begin{aligned}
\left\| q_n'(t) \right\| &\leq \int_0^1 L_2 \left\| s\tilde{\beta}_{n-1} + (1-s)\tilde{u} - \tilde{u} \right\|^\gamma \left\| q_{n-1} \right\| ds \\
&\quad + L_2 \left\| \tilde{u} - \tilde{\beta}_{n-1} \right\|^\gamma \left\| q_{n-1} \right\| + L_1 \left\| q_n \right\| \\
&\leq L_2 \left\| q_{n-1} \right\|^{\gamma+1} + L_2 \left\| p_{n-1} \right\|^\gamma \left\| q_{n-1} \right\| + L_1 \left\| q_n \right\|.
\end{aligned}$$

Setting $b_n = \|q_n\|$ and $a_{n-1} = \|p_{n-1}\|$, it easily follows that

$$b_n' \leq \|q_n'\| \leq L_2 (b_{n-1})^{\gamma+1} + L_2 (a_{n-1})^\gamma b_{n-1} + L_1 b_n.$$

Similarly, an application of Gronwall's inequality yields

$$0 \leq \|q_n\| \leq L_2 \int_0^t \exp\left[L_1 (t-s)\right] \left[\|q_{n-1}(s)\|^{\gamma+1} + \|p_{n-1}(s)\|^\gamma \|q_{n-1}(s)\|\right] ds \text{ on } J,$$

and hence

$$\max_J \|q_n(t)\| \leq L_2 \frac{\exp(L_1 T)}{L_1} \left[\max_J \|q_{n-1}(t)\|^{\gamma+1} + \max_J \|p_{n-1}(t)\|^\gamma \|q_{n-1}(t)\|\right].$$

This completes the proof.

Next we give the following remarks and corollaries for special cases.

Remark 5.1 Instead of assumption (v) in Theorem 5.1 if we assume that

$$\|(N_x x)(t) - (N_x y)(t)\| \leq L_2 \|x - y\|, t \in J$$

where L_2 is a positive constant, then we can see that the convergence is quadratic.

Remark 5.2 Let the assumption of Remark 5.1 be valid. If we assume that $(Nx)(t)$ is uniformly convex in x instead of the assumption (iv) in Theorem 5.1, then by Lemma 4.5.1 in [2], we have quadratic convergence as well. Moreover, the quasimonotonicity of $(Nx)(t)$ in x implies by Lemma 4.2.5 in [2] that $(N_x\alpha)(t)x$ is also quasimonotone in x.

Corollary 5.1: If the assumptions of the Theorem 5.1 hold with $s_0 = t_0$, then the conclusion of the theorem remains valid.

Proof: For the proof, we let $\tilde{\beta}_0(t) = \beta(t + \eta_1)$, $\tilde{\alpha}_0(t) = \alpha(t)$ and $\tilde{u}(t) = u(t)$ and proceed, as we did in Theorem 4.1.

Corollary 5.2: If the assumptions of the Theorem 5.1 hold with $s_0 = \tau_0$, then the conclusion of the theorem remains valid.

Proof: Similarly, we let $\tilde{\alpha}_0(t) = \alpha(t - \eta_1)$, $\tilde{\beta}_0(t) = \beta(t)$ and $\tilde{u}(t) = u(t)$ and proceed, as we did in Theorem 5.1.

References

[1] Jankowski, T.: Quadratic Convergence of monotone iterations for differential equations with initial time difference. Dynamic Systems and Applications 14, 245-252 (2005).

[2] Köksal, S. and Yakar, C.: Generalized Quasilinearization Method with Initial Time Difference, Simulation, an International Journal of Electrical, Electronic and other Physical Systems, 24 (5), (2002).

[3] Ladde, G.S, Lakshmikantham, V. and Vatsala, A.S.: Monotone Iterative Technique for Nonlinear Differential Equations, Boston. Pitman Publishing Inc. 1985.

[4] Lakshmikantham, V. and Leela, S.: Nonlinear Differential Equations in Abstract Spaces, New York. Pergamon Press 1981.

[5] Lakshmikantham, V., Leela, S. Drici Z. and McRae F.A. : Theory of Causal Differential Equations, Amsterdam,World Scientific 2009.

[6] Lakshmikantham, V., Leela, S. and Vasundhara Devi, J.: Another Approach to the Theory of Differential Inequalities Relative to Changes in the Initial Times. Journal of Inequalities and Applications. 4, 163-174 (1999).

[7] Lakshmikantham, V. and Vatsala, A.S.: Differential Inequalities with Initial Time Difference and Applications. Journal of Inequalities and Applications. 3, 233-244 (1999).

[8] Lakshmikantham, V. and Vatsala, A.S.: Generalized Quasilinearization for Nonlinear Problems, The Netherlands. Kluwer Academic Publisher 1998.

[9] Lakshmikantham, V. and Vatsala, A.S.: Theory of Differential and Integral Inequalities with Initial Time Difference and Applications. *Analytic and Geometric Inequalities and Applications* Kluwer Academic Publishers, , 191-203 (1999).

[10] Yakar, C. and S.G., Deo.: Variation of Parameters Formulae with Initial Time Difference for Linear Integrodifferential Equations. *Journal of Applicable Analysis.* Taylor & Francis.85, 333-343 (2006)

[11] Yakar, C. and Yakar, A.: An Extension of the Quasilinearization Method with Initial Time Difference. Dynamics of Continuous, Discrete and Impulsive Systems. Series A: Mathematical Analysis. Waterloo, Watam Press.14 (S2) 275-279 (2007).

[12] Yakar, C. and Yakar, A: Further Generalization of Quasilinearization Method with Initial Time Difference. Journal of Applied Functional Analysis. Eudoxus Press 4, 714-727. (2009).

[13] Yakar, C. Initial Time Difference Quasilinearization Method in Banach Space. Journal of Communications in Mathematics and Applications. Volume 2 (2011) Number2-3, pp.77-85.

Current address: Gebze Institute of Technology Faculty of Sciences, Department of Mathematics Applied Mathematics, Gebze-Kocaeli 141-41400, TURKEY

E-mail address: cyakar@gyte.edu.tr

URL: http://communications.science.ankara.edu.tr/index.php?series=A1