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INVESTIGATION OF TREND BY GRAPHICAL METHODS IN COUNTING PROCESSES

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Abstract. A data set $\{X_1, X_2, ..., X_n\}$ comes from a counting process in many of maintenance, repair and replacement problems commonly includes systematical changes. Systematical changes in the data set indicate a trend which suggests that $X_1, X_2, ..., X_n$ are not identically distributed. In this case renewal process cannot be used as a model. In modelling of a data set the existence of a trend has to be searched firstly. A simple way to detect a trend is to apply a graphical method. In this study the cumulative number of failures versus the cumulative time plot, estimating the rate of occurrence of failures in successive time periods plot, Nelson-Aalen plot and the total time on test plot will be described and evaluated on three real data sets.

1. INTRODUCTION

Let $(X_k)_{k=1,2,...}$be a sequence of nonnegative random variables representing the successive lifetimes. Define, $S_0 = 0$, $S_k = \sum_{j=1}^{k} X_j, k = 1, 2, \ldots$ and $N(t) = sup\{k : S_k \leq t\}, t \geq 0$. The stochastic process $\{N(t), t \geq 0\}$ is called a counting process. The $(X_k)_{k=1,2,...}$ and S_k are often called the interarrival times and kth arrival time, respectively.

A reasonable model should reflect the nature of the data at the hand. Hence, it is important to detect possible systematic changes in a data set observed from a counting process. The purpose of this study is to present some graphical methods to identify the trend in the data with systematical changes.

A data set $\{X_1, X_2, ..., X_n\}$ comes from a counting process includes systematic changes frequently. A trend in the data set $\{X_1, X_2, ..., X_n\}$ indicates that the random variables $X_1, X_2, ..., X_n$ are not identically distributed. A trend in the data can be either monotonic or non-monotonic. Monotonic trend denotes that intrearrival times tend to get longer (decreasing trend or improving system) or

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shorter (increasing trend or deteriorating system). In the case of a non-monotonic trend it can be cyclic or bathtube trend. The most common types of trend in applications are monotone trend and bathtube trend.

If there is no trend in the data a renewal process can be used as a model. In particular, if the interarrival times are independent and identically distributed (iid) having exponential distribution, one must use a homogeneous Poisson process. If there is a trend in the data, a non-homogeneous Poisson process or a stochastic monotone process can be used.

A simple informative way is to apply graphical methods for visual examination and qualification of a data set coming from a counting process. In general, the data without trend will come up with a pattern which has spread around a linear straight line. The deviation from linearity will lead to doubt that there is a trend. There are some graphical methods to detect the trend and its nature.

2. SOME COUNTING PROCESSES

Failure times of a system are usually modelled by a counting process. In this section, the most common stochastic models will be reminded.

Renewal process. Let $\{N(t), t \geq 0\}$ be a counting process. If the interarrival times $X_1, X_2, ...$ are iid with a distribution function F, the counting process $\{N(t), t \geq 0\}$ is called a renewal process (RP).

Homogeneous Poisson process. A counting process $\{N(t), t \geq 0\}$ is said to be a homogeneous Poisson process (HPP) having rate λ, $\lambda > 0$, if the process satisfies the following conditions. (1) $N(0) = 0$, (2) $\{N(t), t \geq 0\}$ has independent and stationary increments, (3) $P(N(h) = 1) = \lambda h + o(h)$, (4) $P(N(h) \geq 2) = o(h)$.

According to a HPP the interarrival times $X_1, X_2, ...$ are independent and identically exponential distributed with mean $1/\lambda$. Hence, HPP is a special RP.

Non-homogeneous Poisson process. Let $\{N(t), t \geq 0\}$ be a counting process and $\lambda(t)$ is a function of t for $t \geq 0$. If $\{N(t), t \geq 0\}$ satisfies the following conditions, it is called a non-homogeneous Poisson process (NHPP) with intensity function $\lambda(t)$.
(1) $N(0) = 0$, (2) $\{N(t), t \geq 0\}$ has independent increments,
(3) $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$, (4) $P(N(t+h) - N(t) \geq 2) = o(h)$

Let $\{N(t), t \geq 0\}$ be a NHPP with intensity function $\lambda(t)$. Then, $N(t+s) - N(s) \sim$Poisson $(M(t+s) - M(s))$, where $M(t) = \int_0^t \lambda(s)ds$ [7]. Thus, $N(t)$ has Poisson distribution with mean $M(t)$ for all $t > 0$.

The most common types of $\lambda(t)$ are powerlaw and loglinear functions [2]. These are given as $\lambda(t) = \alpha\beta t^{(\beta-1)}, t \geq 0 \; ; \alpha, \beta > 0$ and $\lambda(t) = e^{\alpha+\beta t} \;, t \geq 0 \; ; \alpha, \beta \in R$, respectively.

The NHPP with constant intensity $\lambda(t) = \lambda$ is a HPP with intensity λ.

Geometric process. Let $\{N(t), t \geq 0\}$ be a counting process. The $\{N(t), t \geq 0\}$ is said to be a geometric process (GP) with ratio parameter a if there exists a real number $a > 0$ such that $Y_i = a^{i-1}X_i,\ i = 1, 2, \ldots$ are iid random variables with distribution function F.

It can be easily shown that GP is stochastically increasing if $0 < a < 1$ and stochastically decreasing if $a > 1$. If $a = 1$, GP reduces to a RP [3].

3. GRAPHICAL METHODS

In the literature there are many graphical methods to detect the presence of trend and its nature. Some of them are given below.

3.1. Cumulative number of failures versus cumulative time plot. The interarrival times tend to become larger (smaller) when the structure of a system is improving (deteriorating). Thus plotting the cumulative number of failure versus cumulative time will tend to be concave (convex) for improving (deteriorating) system.

3.2. Estimating ROCOF in successive time periods plot. Local variations will tend to be hidden because of the monotonically increasing nature of cumulative failures. In this situation, an alternative method is to divide the total observation interval into three or more subintervals and to estimate the rate of occurrence of failures (ROCOF) for all subintervals. If the system is improving (deteriorating) the estimates of ROCOF in successive time periods will tend to be decreasing (increasing).

Let $\{N(t), t \geq 0\}$ be a counting process. The expected value of $N(t)$ is defined by $M(t)$, that is, $M(t) = E(N(t))$. It is assumed that $M(t)$ is absolutely continuous. Then, ROCOF is defined by $\lambda(t) = \frac{dM(t)}{dt}$. The ROCOF has the following simple interpretation: $\lambda(t)\Delta t$ is the probability that a failure, not necessarily the first, occurs in $(t, t + \Delta t]$[1].

Assume that a data set $\{X_1, X_2, ..., X_n\}$ comes from a counting process $\{N(t), t \geq 0\}$. Since $\lambda(t) = \frac{dM(t)}{dt}$, a natural estimator of $\lambda(t)$ with an appropriate Δt is as follows.

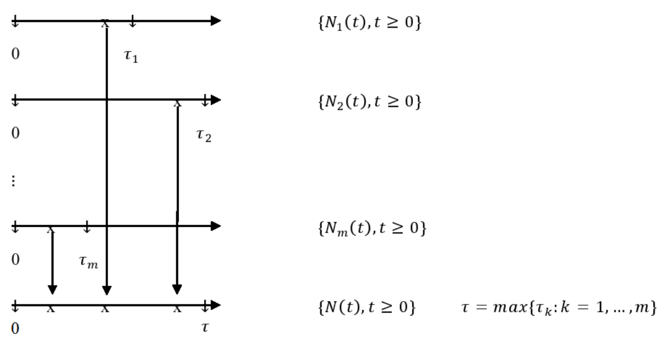
$$\hat{\lambda}(t) = \frac{1}{\Delta t} \sum_{i=1}^{n} I(t < X_1 + X_2 + ... + X_i < t + \Delta t) \tag{3.1}$$

where $I(.)$ is an indicator function and the summation in (3.1) denotes the number of the failures in the interval $(t, t + \Delta t]$ based on the data set $\{X_1, X_2, ..., X_n\}$.

If the total observation interval $(0, S_n]$ is divided into k subinterval as $(0, a_1]$, $(a_1, a_2], \ldots, (a_{k-1}, a_k]$, an estimator of $\lambda(\frac{1}{2}(a_{j-1} + a_j))$ for $j = 1, 2, \ldots, k$ is $\hat{\lambda}(\frac{1}{2}(a_{j-1} + a_j)) = \frac{N(a_j) - N(a_{j-1})}{a_j - a_{j-1}}$ [1].

Thus, it can be reached that a plot of $\hat{\lambda}(b_j)$ versus b_j where $b_j = \frac{1}{2}(a_{j-1} + a_j)$, $j = 1, 2, \ldots, k$. This plot will give a plausible evidence for the shape of intensity $\lambda(t)$.

3.3. Nelson-Aalen plot. Assume that m systems are observed and the processes related to systems will be independent and identical with probability one. Suppose further that the kth process is observed on the time interval $(0, \tau_k]$ and let $y(t)$ denote the number of the processes under observation or running at time t. It is clear that $y(t) = \sum_{k=1}^{m} I(\tau_k \geq t)$.



For $t \geq 0$, let us write $N(t) = \sum_{k=1}^{m} N_k(t)$, where $\{N_k(t), t \geq 0\}$, $k = 1, 2 \ldots, m$, is the individual failure process. The counting process $\{N(t), t \geq 0\}$ is called a superposed process. Let S_k denote the kth arrival time in the superposed process, hence $0 < S_1 \leq \cdots \leq S_N \leq \tau$, where $N = N(\tau)$ and $\tau = \max\{\tau_k : k = 1, \ldots, m\}$. Define the mean value function of a single failure process to be $M(t) = E(N_1(t))$. $M(t)$ has a nonparametric Nelson-Aalen estimator as $\hat{M}(t) = \sum_{S_k \leq t} \frac{1}{y(S_k)}$. Thus the plot of this estimator versus cumulative time is called Nelson-Aalen plot [4]. The deviation from linearity in this plot will lead to doubt that there is a trend.

3.4. TTT plot. Assume that m systems are observed and the processes related to systems will be independent and identical NHPP with a common intensity function $\lambda(t)$. For each $t \geq 0$, $y(t) = \sum_{k=1}^{m} I(\tau_k \geq t)$ is the number of the processes under observation at time t, where $\tau_k, k = 1, \ldots, m$, is the observation period for the kth failure process $\{N_k(t), t \geq 0\}$. It is clear that the function $y(t)$ is independent from the kth arrival time S_k of the superposed process $\{N(t), t \geq 0\}$. Since the failure processes $\{N_1(t), t \geq 0\}, \ldots, \{N_m(t), t \geq 0\}$ are independent NHPP with the same intensity function $\lambda(t)$, the superposed process $\{N(t), t \geq 0\}$ will be a NHPP with intensity function $\phi(t) = y(t)\lambda(t)$.

Defining the total time on test (TTT) as $r(t) = \int_0^t y(u)du$, the TTT plot is given by the points $(\frac{k}{N}, \frac{r(S_k)}{r(\tau)})$ $k = 1, \ldots, N$ where $N = N(\tau)$, that is, N is the total number of failures in the time interval $(0, \tau]$ [4]. It is obvious that $r(\tau) = \sum_{k=1}^{m} \tau_k$.

The TTT plot will be located near the main diagonal line on the unit square if there is no trend in the data set, that is, $\lambda(t)$ is a constant. Under the alternatives of decreasing, increasing and bathtube shaped intensity$\lambda(t)$, the TTT plot appears to be convex, concave and S-shaped, respectively.

4. APPLICATIONS OF THE GRAPHICAL METHODS ON THREE REAL DATA SETS

In this section, the graphical methods will be evaluated by using three real data sets.

Example 4.1. The cumulative failure times in operating hours to unscheduled maintenance actions for the USS Halfbeak No.3 main propulsion diesel engine are given in the following Table 4.1 [1]. It is assumed that the system was observed until the 71st failure at 25518 hours.

Table 4.1. Cumulative failure times to unscheduled maintenance actions for the USS Halfbeak No.3 main propulsion diesel engine [1].

1382	17632	21309	21943	23791
2990	18122	21310	21946	23822
4124	19067	21378	22181	24006
6827	19172	21391	22311	24286
7472	19299	21456	22634	25000
7567	19360	21461	22635	25010
8845	19686	21603	22669	25048
9450	19940	21658	22691	25268
9794	19944	21688	22846	25400
10848	20121	21750	22947	25500
11993	20132	21815	23149	25518
12300	20431	21820	23305	
15413	20525	21822	23491	
16497	21057	21888	23526	
17352	21061	21930	23774	

The cumulative number of failures versus the cumulative time is plotted in Figure 4.1. It is fairly apparent that the plot is convex. Thus, the data show very strong evidence for increasing trend.

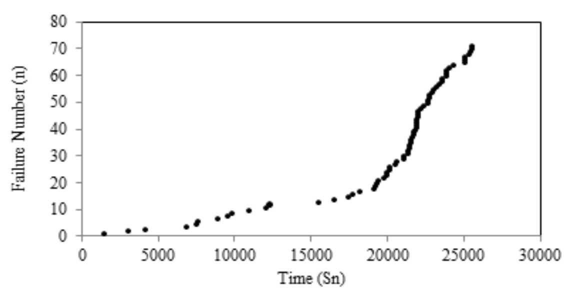


Figure 4.1. Cumulative failures versus cumulative times for unscheduled maintenance actions for the USS Halfbeak No.3 main propulsion diesel engine.

The estimates of ROCOF $\lambda(t)$ are presented in Table 4.2 by taking $\Delta t = 2835$ (or 9 subintervals) for USS Halfbeak No.3 main propulsion diesel engine data.

Table 4.2. Number of the failures for unscheduled maintenance actions for the USS Halfbeak No.3 main propulsion diesel engine and estimated $\lambda(t)$ values

Subintervals	b_j	Number of the Failures	$\hat{\lambda}(b_j)$
(0,2835]	1418	1	1/2835
(2835,5670]	4253	2	2/2835
(5670,8505]	7088	3	3/2835
(8505,11340]	9924	4	4/2835
(11340,14175]	12759	2	2/2835
(14175,17010]	15594	2	2/2835
(17010,19845]	18430	8	8/2835
(19845,22680]	21265	30	30/2835
(22680,25518]	24100	19	19/2835

The illustration of $\hat{\lambda}(t)$ calculated in Table 4.2 is given by Figure 4.2.

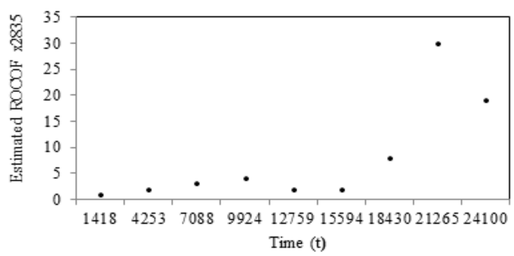


Figure 4.2. Plot of 2835 times the estimated ROCOF against time for unscheduled maintenance actions for the USS Halfbeak No.3 main propulsion diesel engine data.

Looking at the Figure 4.2, the USS Halfbeak No.3 main propulsion diesel engine data show apparent evidence for trend as revealed by cumulative failures versus cumulative times plot in Figure 4.1.

Example 4.2. The interarrival times in operating hours between the failures for a Boeing 720 aircraft ventilation system are given in the following Table 4.3 [6].

Table 4.3. Interarrival times between the failures for a Boeing 720 aircraft ventilation system.

413	9	118	57	14	169	34	62	58	447	31	7
37	184	18	22	100	36	18	34	65	201	67	

At first glance, it can easily be seen that towards the end the failures tend to become more frequent. This brings to mind the idea that there is a trend in the data set $\{X_1, X_2, \ldots, X_{23}\}$ so these random variables are not identically distributed.

The cumulative number of failures versus the cumulative time is plotted in Figure 4.3.

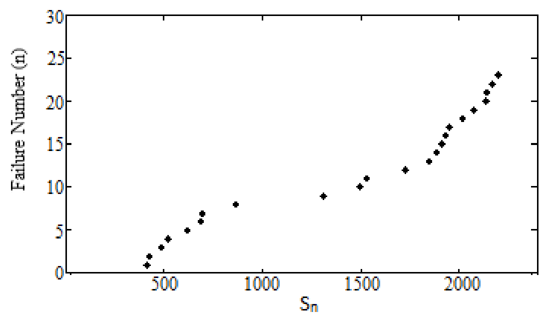


Figure 4.3. Cumulative failures versus cumulative times for a Boeing 720 aircraft ventilation system.

From Figure 4.3 it can be concluded that there is a non-monotonic trend in the data.

The estimates of ROCOF $\lambda(t)$ are presented in Table 4.4 by taking $\Delta t = 440$ (or 5 subintervals) for a Boeing 720 aircraft ventilation system data.

Table 4.4. Number of the failures for a Boeing 720 aircraft ventilation system data and estimated $\lambda(t)$ values.

Subintervals	b_j	Number of the Failures	$\hat{\lambda}(b_j)$
(0,440]	220	2	2/440
(440,880]	660	6	6/440
(880,1320]	1100	1	1/440
(1320,1760]	1540	3	3/440
(1760,2201]	1980.5	11	11/440

The illustration of $\hat{\lambda}(t)$ calculated in Table 4.4 is given by Figure 4.4.

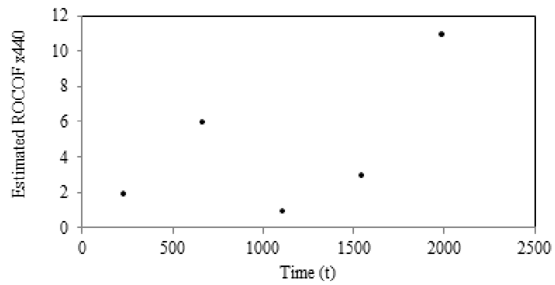


Figure 4.4. Plot of 440 times the estimated ROCOF against time for the Boeing 720 aircraft data.

In Figure 4.4, it is clear that there is a trend in the pattern of the failures, and also the shape of the trend is non-monotonic. Thus, two graphical methods give us the same conclusion for trend with this data set.

Example 4.3. The valve-seat replacement times for 41 diesel engines are given in Table 4.5 [5].

Table 4.5. Valve-seat replacement times in days for 41 diesel engines.

No.	Times	τ_k	No.	Times	τ_k	No.	Times	τ_k
1	-	761	15	635	641	29	-	601
2	-	759	16	349,404,561	649	30	410,581	601
3	98	667	17	-	631	31	-	611
4	326, 653, 654	667	18	-	596	32	-	608
5	-	665	19	120,479	614	33	-	587
6	84	667	20	323,449	582	34	367	603
7	87	663	21	139,140	589	35	202,563,570	585
8	646	653	22	-	593	36	-	587
9	92	653	23	573	589	37	-	578
10	-	651	24	165,408,604	606	38	-	578
11	258,328,377,621	650	25	249	594	39	-	586
12	61,539	648	26	344,497	613	40	-	585
13	254,276,298,640	644	27	265,586	595	41	-	582
14	76,538	642	28	166,206,348	389			

The values of $\hat{M}(t)$ are meaningful for $t \leq \tau$, since $\hat{M}(t)$ is constant for $t > \tau$ and, thus, meaningless. The following figure is drawn by using the calculated nonparametric estimates of $M(t)$ for the valve-seat data set. As an example, the value of $\hat{M}(t)$ is 0.15 for $t = 98$.

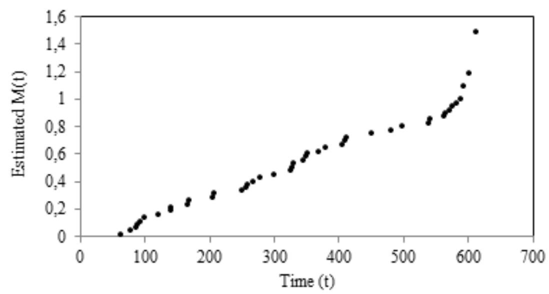


Figure 4.5. Nelson-Aalen plot.

In Figure 4.5, the ROCOF can be considered to be constant for the first 550 days, and then increases as revealed from the convex shape of the plot at the right end. Thus, it can be said that the data exhibit a trend.

Let us consider again the valve-seat replacement data set. Using the calculated values given in Table 4.6, the TTT plot is displayed in Figure 4.6.

Table 4.6. The values of $(\frac{k}{N}, \frac{r(S_k)}{r(\tau)})$ in TTT plot with $N = 48$ and $k = 1, \ldots, 48$.

$\frac{k}{N}$	$\frac{r(S_k)}{r(\tau)}$	$\frac{k}{N}$	$\frac{r(S_k)}{r(\tau)}$	$\frac{k}{N}$	$\frac{r(S_k)}{r(\tau)}$
0,0208	0,0986	0,3541	0,4283	0,6875	0,7991
0,0416	0,1228	0,375	0,4461	0,7083	0,8638
0,0625	0,1357	0,3958	0,4817	0,7291	0,8653
0,0833	0,1406	0,4166	0,5221	0,75	0,9
0,1041	0,1487	0,4375	0,5269	0,7708	0,9032
0,125	0,1584	0,4583	0,5302	0,7916	0,9142
0,1458	0,1939	0,4791	0,556	0,8125	0,919
0,1666	0,2246	0,5	0,5625	0,8333	0,9313
0,1875	0,2263	0,5208	0,5641	0,8541	0,9384
0,2083	0,2667	0,5416	0,5932	0,875	0,9576
0,2291	0,2683	0,5625	0,6094	0,8958	0,9702
0,25	0,3265	0,5833	0,6524	0,9166	0,9794
0,2708	0,333	0,6041	0,6587	0,9375	0,9826
0,2916	0,4025	0,625	0,6619	0,9583	0,986
0,3125	0,4105	0,6458	0,7234	0,9791	0,989
0,3333	0,417	0,6666	0,7707	1	0,9893

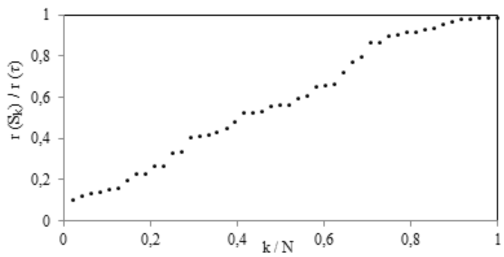


Figure 4.6. The TTT plot.

The TTT plot in Figure 4.6 shows that the ROCOF is fairly constant, but it increases with a slightly concave shape near the end. Hence it can be concluded that the data have a trend as revealed by the Nelson-Aalen plot in Figure 4.5.

5. DISCUSSION

In this paper, some of graphical methods are reviewed to detect the presence of trend and its nature. The plot of cumulative failures versus cumulative time is simple to apply. Nevertheless, local variations will tend to be hidden because of the monotonically increasing nature of cumulative number of failures. An alternative method is to plot of estimated ROCOF [1]. However, this method is sensitive for selected values of k and a_j. Some different inferences can be obtained based on the selection of k or a_j. The choice of k or a_j is similar to the choice of the number of intervals or the interval width for a histogram. The Nelson-Aalen plot can be considered as a generalization of the plot of cumulative failures versus cumulative times. When m equals to 1 the Nelson-Aalen plot is a plot of cumulative failures versus cumulative times. The TTT plot is often used to test whether the counting process $\{N(t), t \geq 0\}$ is a HPP with $\lambda = 1$, as well as to detect trend in the data set $\{X_1, X_2, \ldots, X_n\}$ [4].

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