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SOLUTION OF NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS USING POLYNOMIAL-SINC APPROXIMATION

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Abstract. A new highly accurate algorithm for the solution of nonlinear singular boundary value problems for ordinary differential equations is presented. The algorithm uses a collocation technique based on polynomial approximation at Sinc points. The scheme is tested for some nonlinear singular boundary value problems showing an exponential convergence rate. The examples are of second and higher order singular, nonlinear boundary value problems. For each example the error formula of the approximation is discussed and verified in a comparison of the analytic solution.

1. Introduction

The topic of singular boundary value problems has been of rapidly growing interest in many applications in science and engineering. A singular problem is found in many important applications, e.g. boundary layer theory, the study of stellar interiors, and control and optimization theory. Some of these problems are linear and others are nonlinear. For the nonlinear singular boundary value problems, many numerical solutions have been discussed in literature [32, 33, 29]. One of the commonly used technique is the series solution [11]. Although this technique is a highly accurate solution it has to be modified and adopted for each single problem according to the type of singularity. On the other hand authors used the shooting method [18] which is based on root finding methods that need a large number of iterations to get a high accuracy solution. Also, many articles developed Adomian decomposition method [2, 4, 5, 24, 31], and the related modifications [21, 32, 33] to investigate various singular nonlinear models. The Adomian decomposition techniques have the same disadvantage as polynomial and series solutions that need to be adopted to each different model. One of the most accurate approximation methods is the use of Sinc approximation [29]. There are two techniques based on

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Sinc approximation to solve singular boundary value problems, Sinc-Galerkin [29] and Sinc-Collocation methods [9, 29]. Both of these techniques show exponential convergence rate. But, Sinc approximation has a problem in approximating the derivative of the analytic functions on a finite or semi-infinite intervals [29]. In this paper, we present a polynomial approximation that overcomes this problem. More over, it keeps the same advantage as Sinc approximation, that it shows an exponential convergence rate.

Now, let us consider the nonlinear differential equations of order $m \geq 2$:

$$\begin{aligned} L(u) \equiv u^{(m)} + l_1(x,u)u^{(m-1)} + l_2(x,u)u^{(m-2)} + ... + \\ l_{m-1}(x,u)u^{'} + l_m(x,u) = f(x),\, a \leq x \leq b \end{aligned} \tag{1.1}$$

With the boundary conditions:

$$u^{(j)}(a) = \alpha_j, u^{(k)}(b) = \beta_k,\, 0 \leq j \leq J,\, 0 \leq k \leq K \text{ and}, J + K = m - 2 \tag{1.2}$$

where the functions $l_i(x,u)$, $1 \leq i \leq m$ are functions of x and u which are analytic in $[a,b]$, nonlinear (or linear), and singular (or non-singular). The existence and uniqueness of the solution of (1.1-1.2) is discussed by Agarwal and Akrivis in [6].

2. Collocation of Poly-Sinc Method

In this section we introduce a collocation method based on the polynomial-like approximations at Sinc data. This collocation method reduce the boundary value problems to a system of nonlinear algebraic equations. Without any adaptation of the technique, the collocation method is able to deal with the singularity at the end-points.

It is well known that whitacker's cardinal function can be used to approximate almost every calculus operation [29]. All of these approximations posses an exceptionally fast rate of convergence [29]. One of the few shortcomings of Sinc approximation is that it delivers a poor accuracy in the neighborhood of the finite end-point when differentiating the interpolation Sinc formula on a finite or semi-infinite interval. This problem has been solved in [27] by introducing polynomial-like methods of approximation. The fortune of this approach is the fact that polynomials as well as their derivatives converge rapidly on a finite interval for functions that are analytic in a region containing this interval.

2.1. **Sinc points and Lagrange polynomial notations.** In Lagrange approximation different sets of points $\{x_k, f(x_k)\}_{k=0}^{n}$ are used as interpolation points. The most famous set of points are the equidistant points but it is well known that these points deliver bad results [30]. To improve the accuracy of Lagrange approximation other sets of points are used, like Chebychev points and modified Chebychev points [26]. Recently it has been shown that it is more effective to use Sinc points as interpolation points [28]. This sequence of points is created using a conformal map that redistribute the infinite equidistant points of the real line on a finite interval locating most of these points near the end-points of the finite interval. Also, it is proved that using Sinc-points as interpolation points will give a high accurate approximation and allows an accuracy similar to the classical Sinc approximation [27].

To define these interpolation points let $\mathbb{Z}$ denote the set of all integers. Let $\mathbb{R}$ be the real line, and $\mathbb{C}$ denote the complex plane. Let h denote a positive parameter and let $k \in \mathbb{Z}$, $z \in \mathbb{C}$.

Let d denote a positive number, and let ϕ denote the conformal map of a simply connected region $\mathbb{D} \subset \mathbb{C}$ onto the strip

$$\mathbb{D}_d = \{z \in \mathbb{C} : |Im(z)| < d\}.$$

Let $\Gamma = \phi^{-1}(\mathbb{R})$ be an arc and let $a = \phi^{-1}(-\infty)$ and $b = \phi^{-1}(\infty)$ denote the end points of Γ. Then we define the set of Sinc points by $z_k = \phi^{-1}(kh)$, and set $\rho = e^{\phi(z)}$.

Finally, let $\alpha \in (0, 1]$ and $\beta \in$(0,1] denote fixed positive numbers, let us restrict d introduced above to the interval $(0, \pi)$. Let $\mathcal{L}_{\alpha,\beta}(\mathbb{D})$ denote the family of all functions that are analytic in $\mathbb{D}$, such that for all $z \in \mathbb{D}$, we have

$$|u(z)| \leq c_1 \frac{|\rho(z)|^{\alpha}}{[1 + |\rho(z)|]^{\alpha+\beta}}.$$

The space of functions $M_{\alpha,\beta}(\mathbb{D})$ denotes the set of all functions g defined on $\mathbb{D}$ that have finite limits $g(a) = \lim_{z\to a} g(z)$ and $g(b) = \lim_{z\to b} g(z)$, where the limits are taken from within $\mathbb{D}$, and such that $u \in \mathcal{L}_{\alpha,\beta}(\mathbb{D})$, where,

$$u{=}g - \frac{g(a) + \rho g(b)}{1 + \rho}.,$$

now we are in position to define a family of polynomial-like approximations that interpolate given Sinc data of the form $\{x_k,\ u\,(x_k)\}_{k=-M}^{N}$ where the x_k are Sinc points. This novel family of Lagrange polynomials was recently derived in [27]. The approximation is accurate, provided that the function u with $u_k = u\,(x_k)$ belongs to a suitable space of analytic functions.

Generally, Lagrange polynomial approximation over the interval $[a, b]$ is defined in the following way.

Given a set of $n = M + N + 1$ distinct points $\{x_k\}_{k=-M}^{N}$on the interval $[a, b]$ and function values, $\{u\,(x_k)\}_{k=-M}^{N}$. Let x_k be the Sinc points that are defined using the conformal map $\phi(x) = \ln((x - a)/(b - x))$ and so the Sinc points can be given by:

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}},\ k = -M, -M + 1, ..., N - 1, N. \tag{2.1}$$

At these points $\{x_k,\ u(x_k)\}_{k=-M}^{N}$, there exists a unique polynomial $p(x)$ of degree at most $n-1$ satisfying,

$$p(x_k) = u_k,\ k = -M, ..., N.$$

In this case $p(x)$ can be expressed as:

$$p(x) = \sum_{k=-M}^{N} b_k(x) u_k, \tag{2.2}$$

with,

$$b_k(x) = \frac{g(x)}{(x - x_k)\, g'(x_k)}, \tag{2.3}$$

where,

$$g(x) = \prod_{j=-M}^{N} (x - x_j).$$

This approximation, like regular Sinc approximation, yields an exceptional accuracy in approximating the function that is known at Sinc points. Unlike Sinc approximation, it gives an exponential convergence rate when differentiating the interpolation formula given in (2.2), [27].

For the sake of simplicity of our results, we shall assume here that $M = N$.

Theorem 2.1. *Let $h = \frac{\pi}{\sqrt{N}}$, and let $\{x_k\}_{k=-N}^{N}$denote the Sinc points as defined in (2.1). Let u be in $M_{\alpha,\beta}(\mathbb{D})$, and let $p(x)$ be defined as in (2.2). Then there exist two constants $A > 0$ and $B > 0$, independent of N, such that*

$$|u(x) - p(x)| \le A \frac{\left(\sqrt{N}\right)}{B^{2N}}\ \exp\left(\frac{-\pi^2 N^{\frac{1}{2}}}{2}\right), \tag{2.4}$$

Proof. For the proof of (2.4), see [27]. □

The space $M_{\alpha,\beta}(\mathbb{D})$ is connected with a Hardy space $H^p(\mathbb{U})$, where $\mathbb{U}$ is the unit disk. In $H^p(\mathbb{U})$ the best approximation rate of the form $o\left(e^{-c\sqrt{N}}\right)$ has been obtained [7]. The constant $c = \frac{\pi}{2}$ in our own estimate is not large as this optimal upper bound of the error discussed in [7] as $c = \pi$, but it is still an excellent upper bound comparing with the upper bounds in Sinc approximations. The optimal distribution of the sequence of points to be chosen on the interval $[a, b]$ is still an open problem till now [7, 13], but one can see that the approximation we introduced here gives an exceptional upper bound of error. In fact getting an exponential decaying rate of convergence is not only the privilege of this approximation, but dealing effectively without any modification with the singularity at the end points.

2.2. **The Poly-Sinc Algorithm.** In the next step we set up the collocation method based on the use of Lagrange interpolation at Sinc points. We will replace $u(x)$ in equation (1.1) and (1.2) by the Lagrange polynomial defined in (2.2). This will reduce the problem to a system of nonlinear algebraic equations. In the approximation

$$u(x) \approx p(x) = \sum_{k=-N}^{N} b_k(x)\, u_k. \tag{2.5}$$

The $2N+1$, coefficients $\{u_k\}$ are determined by substituting $u(x)$ defined in (2.5) into equation (1.1) and (1.2) and evaluating the results at the Sinc points $x_j = \phi^{-}1(jh),\ j = -N+1,\ ..., N-2$. This delivers a nonlinear system of $2N-2$ equations.

If we use the approximation (2.5) to solve differential equations, not only the unknown function u is needed but also its derivatives. To define an approximation for the first and second order derivative, we have to differentiate formula (2.5) with respect to x twice. This results into two $n \times n$ matrices $A = [a_{j,k}]$ and $C = [c_{j,k}]\,, j = -N, ..., N$ and $k = -N, ..., N$, representing the first order derivative and the second order derivative, respectively. For the first order derivative,

$$u'\left(x_j\right) \approx p'\left(x_j\right) = \sum_{k=-N}^{N} a_{j,k} u_k, \tag{2.6}$$

where

$$a_{j,k} = b_k^{'}(x_j) = \begin{cases} \frac{g'(x_j)}{(x_j - x_k) g'(x_k)} & \text{if}\, k \neq j \\ \sum_{l=-N, l\neq j}^{N} \frac{1}{(x_j - x_k)} & \text{if}\, k = j. \end{cases} \tag{2.7}$$

Theorem 2.2. *Let $h = \pi/\sqrt{N}$, and let $\{x_k\}_{k=-N}^{N}$ denote the Sinc points as defined in (2.1). Let u be in $M_{\alpha,\beta}(\mathbb{D})$, and let $p'(x)$ be defined as in (2.6). Then there exist two constants $C > 0$ and $D > 0$ independent of N, such that*

$$\max_{j=-N,\ldots,N} |u'(x_j) - p'(x_j)| \leq C \frac{N}{D^{2N}} \exp\left(\frac{-\pi^2 N^{\frac{1}{2}}}{2}\right). \tag{2.8}$$

Proof. For the proof of (2.9), see [27]. □

For the second order derivative,

$$u''(x_j) \approx p''(x_j) = \sum_{k=-N}^{N} c_{j,k} u_k. \tag{2.9}$$

It is straight forward to verify that

$$c_{j,k} = b_k''(x_j) = \begin{cases} \frac{-2g'(x_j)}{(x_j-x_k)^2 g'(x_k)} + \frac{g''(x_j)}{(x_j-x_k)g'(x_k)} & \text{if } k \neq j \\ \\ \sum_{n=-N}^{N} \sum_{\substack{l=-N \\ l,n\neq j}}^{N} \frac{1}{(x_j-x_l)(x_j-x_n)} & \text{if } k = j. \end{cases} \tag{2.10}$$

Substituting Equations (2.5), (2.6), (2.8), and (2.10) in (1.1) and (1.2), the problem will be reduced to a nonlinear system of $2N+1$ algebraic equations for $2N+1$ unknowns that can be solved by using Newton's root finding method to get the coefficients u_k.

The substitution of the expansion coefficients u_k into (2.5) delivers the approximation of the BV problem (1.1) and (1.2).

Beside the given error formulas for the approximation in (6) and (10) we use for practical purposes the error norm relation to compare between two different solutions. This error formula can be given by:

$$\epsilon = \|u(x) - u_s(x)\| = [\int_a^b (u(x) - u_s(x))^2 \, dx]^{\frac{1}{2}}, \tag{2.11}$$

where $u(x)$ is the analytic solution and $u_s(x)$ is the approximation obtained from the Poly-Sinc algorithm.

3. Numerical Results

In this section, we demonstrate the effectiveness of the Poly-Sinc algorithm with several illustrative nonlinear singular examples. For comparison reason, some problems have homogeneous boundary conditions and some other problems have inhomogeneous boundary conditions but all of them have a known analytic solutions. For each test example, the absolute error between the exact solution or the analytic approximate solution obtained in one of the references and the results obtained by the Poly-Sinc method.

Example 3.1. Consider the following nonlinear singular boundary value problem,

$$e^{u(x)}u''(x) + \frac{1}{x(x-1)}u'(x) + \frac{1}{1-x}u^2(x) = f(x),\ 0 \leq x \leq 1, \tag{3.1}$$
$$u(0) = u(1) = 0,$$

where,

$$f(x) = \frac{1}{1-x}[\frac{2x^4 - 4x^3 + x^2 + 3x - 1}{x(x^2 - x + 1)} + log^2(x^2 - x + 1)].$$

The difficulty of this problem is not only the non linearity but also the existence of the singularity at the end points. This kind of singularity requires a specific distribution of interpolation points, to be clustered near the end points. This means

that using Sinc points as interpolation points in the collocation scheme might give a great help to overcome this singularity problem. Equation (3.1) has the exact solution $u(x) = log(x^2 - x + 1)$.
Fig. 1 represents the exact solution, solid line, and the approximate solution, dashed line. For the approximate solution we used $n = 2N + 1 = 7$ Sinc points.

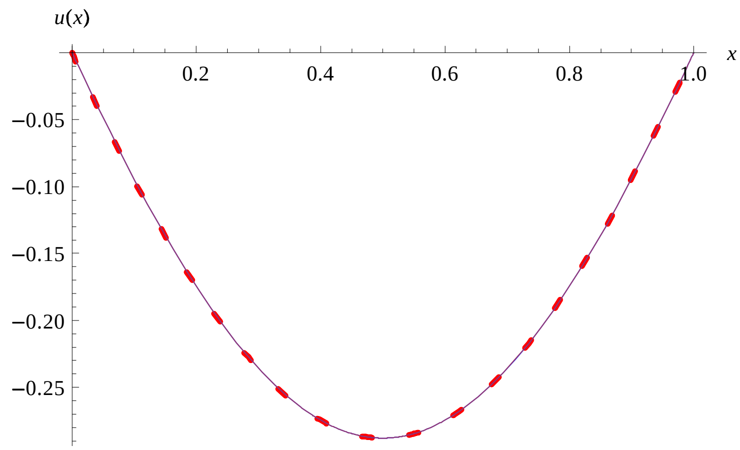

FIGURE 1. The exact and approximate solution of Eq. (14) using 7 Sinc points.

Example 3.2. using the square norm error defined in (13), we get an error of 10^{-4}. It is obvious from this result that with a small number of Sinc-points we get an excellent approximation with a small error. This property is one of the advantages of using Sinc-points as collocation points. To show the property of exponential convergence as regular Sinc method let us present the error function as a function of N only as, $C\sqrt{N}E^{-c\sqrt{N}}$, where C and c are constants. Fig. 2 shows that the norm error based on (13) calculated in this example for different number of Sinc-points $n = 2N + 1 = 5, 7, 9, 11$ is satisfying the exponential decaying error function.

If the singularities at the end points exist, it is well known that a large number of collocation points should be used to get an acceptable error. Using the collocation method defined in this paper we can see that using a small number of interpolation or collocation points delivers an excellent error. In addition, this error has the property of decaying exponentially as in the classical Sinc approximation.

As in example (3.1), example (3.2) will give another example of nonlinear equations that is defined on finite interval with singularities at the end points.

Example 3.3. Consider the following nonlinear singular boundary value problem,

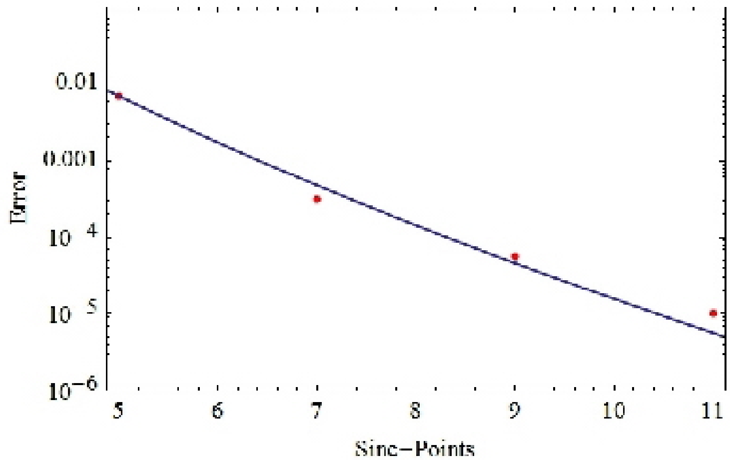

FIGURE 2. The error function $\alpha\sqrt{N}E^{-\beta\sqrt{N}}$ (solid) and the calculated norm error (dots) for $n = 5, 7, 9, 11$.

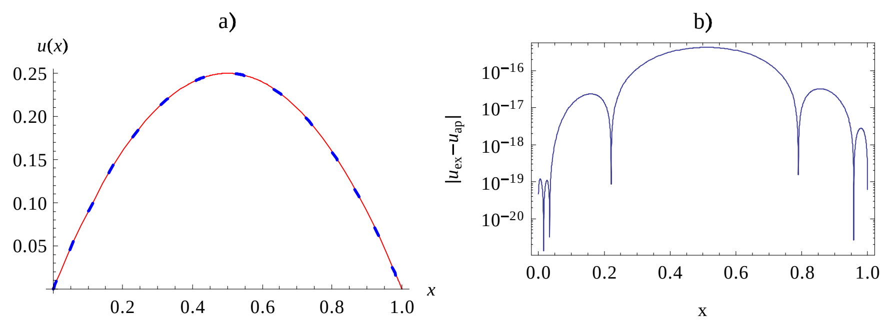

FIGURE 3. a. The exact and approximate solution of eq. (15) using $n = 9$, b. The local error $|u_{ex} - u_{ap}|$.

$$sin(u(x))u''(x) + \frac{1}{x\sqrt{1-x}}u'(x) + \frac{1}{1-x}u^3(x) = f(x), 0 \le x \le 1, \tag{3.2}$$
$$u(0) = u(1) = 0,$$

where,

$$f(x) = \frac{1-2x}{x\sqrt{1-x}} + \sqrt{\frac{x}{1-x}} - 2sin(x - x^2).$$

This nonlinear equation (3.2) has singularities at the end points, at $x = 0$ and $x = 1$. In addition it has the exact solution $u(x) = x - x^2$. As in example (3.1), we still claiming that using a few number of collocation points will deliver an exponentially small error. To do so, we use $n = 2N+1 = 9$ Sinc points as collocation points to get, using square norm error (13), an error of 10^{-16} . In Fig. 3 a., we represent the exact solution, solid line, and the approximate solution, dashed line. In Fig 3 b. the local error calculated as$|u_{ex} - u_{ap}|$, where u_{ex} represents the exact solution and u_{ap} represents the approximate one at $n = 9$.

Again in this example we show that using a few number of Sinc points guarantees a very efficient approximate solution, with an error approximately zero.

Next we consider another example of nonlinear boundary value problems over a finite interval with homogeneous Dirichlet boundary conditions.

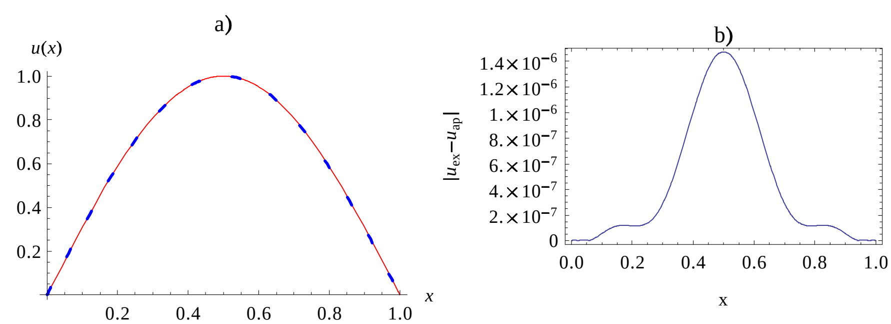

FIGURE 4. a. The exact and approximate solution of eq. (16) using $n = 9$, b. The local error $|u_{ex} - u_{ap}|$.

Example 3.4. Consider the following nonlinear singular boundary value problem [15],

$$\sqrt{u(x)}u''(x) + \frac{30}{sin^3(x)(1-x^2)^2(x-0.4)^2(x-0.6)}u'(x) + \frac{1}{1-x}\sqrt{u(x)} = f(x),\ 0 \leq x \leq 1 \,\text{and}\, u(0) = u(1) = 0, \tag{3.3}$$

where $f(x)$ is compatible to the exact solution $u(x) = sin(\pi x)$. This equation contains a singularity at one of the end points, $x = 1$. To handle such kind of one sided singularity, we need to shift more collocation points near by the end point that includes the singularity. This can be done by using the Sinc points $x_k = e^{kh}/(e^{kh} + q)$ where $q = (1 + c)/(1 - c)$ and c is an arbitrary constant in $(-1, 1)$ [27]. To cluster more Sinc points near by the end point $x = 0$ we simply choose $c \in (0, 1)$ and to dense them around $x = 1$ we choose $c \in (-1, 0)$.

Fig. 4 a. represents the exact solution, solid line, and the approximate solution, dashed line. Fig 4 b. the local error calculated as$|u_{ex} - u_{ap}|$, where u_{ex} represents the exact solution and u_{ap} represents the approximate one at $n = 9$. In [15], Geng used a reproducing kernel spaces to solve this singular nonlinear boundary value problems. Table 1 gives a comparison between the obtained error in [15] and the recent Poly-Sinc algorithm with the number of used collocation points.

Method	Error	n
Poly-Sinc	$(10)^{-7}$	9
Reproducing kernel, [15]	$(10)^{-5}$	26

Table 1: Error in example 3.3.

In the last three examples, we were dealing with homogeneous Dirichlet boundary conditions. The next example examines a nonlinear boundary value problems with inhomogeneous boundary conditions.

Example 3.5. Consider the following nonlinear singular boundary value problem [23]

$$u''(x) + \frac{1+5x}{2x(x+1)}u'(x) + u(x)ln(u(x)) = f(x), 0 \leq x \leq 1 \tag{3.4}$$
$$u(0) = 1, u(1) = e,$$

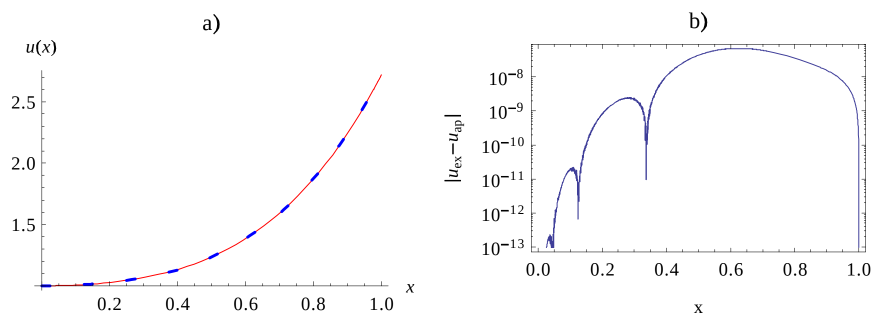

FIGURE 5. a. The exact and approximate solution of eq. (16) using $n = 15$, b. The local error $|u_{ex} - u_{ap}|$.

Example 3.6. where $f(x)$ is compatible to the exact solution $u(x) = (1-x+x^2)e^x$. For the inhomogeneous boundary conditions, we can collocate the equation (3.4) using x_k points where $k : -N + 1, ..., N + 1$. This collocation will deliver $2N + 1$ algebraic equations. Then we can add the two boundary equations represented by the polynomial approximation:

$$p(0) = 1, p(1) = e,$$

to have a system of $2N + 1$ equations in $2N + 1$ coefficients.

Fig 5 a. represents the exact solution, solid line, and the approximate solution, dashed line. For the approximate solution we used 15 Sinc points. Fig 5 b. gives the local error calculated at $n = 15$. For the global error, we use the square norm error defined in eq. (13) to get an error of 10^{-7}.

4. Higher Order BVPs

Higher order Boundary Value Problems arise in the mathematical modeling of many physical problems. For example, for a two-dimensional channel with porous walls, viscoelastic and inelastic flows, deformation of beams, plate deflection theory, beam element theory and a number of other engineering and applied mathematics applications. Solving such type of boundary value problems analytically is possible only in very rare cases. Many authors worked on the numerical solutions of higher order boundary value problems [11, 15, 25, 16]. Some numerical methods such as finite difference method, differential transformation method, Adomian's decomposition method, homotopy perturbation method, variational iteration method, spline methods have been developed for solving such boundary value problems [25, 14, 23, 1]. For Sinc methods, there is not much information available in literature for higher order boundary value problems solved by Sinc methods. For a sixth-order BV problem El-Gamel et al [12] discussed a Sinc-Galerkin method and for fourth-order BV problems Hajii discusses a nonlinear BV problem [19]. Both groups demonstrated the application to a specific problem and discuss the convergence properties for this specific type of equation. Bialecki in his paper [9] comments on higher order BV problems and states a general formula but does not discuss details of convergence or calculations.

Here we will use the already known algorithm for second order equations and extend this algorithm to an arbitrary higher order algorithm for 1D equations. The theoretical procedure follows the same line as presented for second order equations. However, the method for higher order is different from the second order method with respect to the incorporation of the boundaries and the discretization of derivatives.

Mainly we will discuss four examples. The first example is Blasuis equation that is originally introduced in the study of the laminar flow of a fluid. The second example is the fourth order beam equation. The third example is a nonlinear fourth order boundary value problem. Finally, we discuss the sixth order boundary value problem from [12]. In this example we compare the Poly-Sinc method with the Glarekin Sinc method that is introduced in [12]. This comparison will give an evidence that Poly-Sinc is better than some techniques based on Sinc approximations.

Example 4.1. *Blasius Equation* [11]
The Blasius equation is a nonlinear third order differential equation which is given by,

$$u'''(x) + u(x)u''(x) = \beta(u'^2(x) - 1) \tag{4.1}$$
$$u(0) = u'(0) = 0, u'(x) \to k \,\text{as}\, x \to \infty,$$

where β is an arbitrary constant and k is a constant. When $\beta \geq 0$, Eq. (4.1) has a unique solution. The original problem with $\beta = 0$ was solved by H. Blasius [10] who introduced it in the study of the laminar flow of a fluid. He found an exact solution of boundary layer equation over a flat plate. A more general case $\beta \neq 0$ has been solved by many authors, for example Howarth [22] solved it by means of numerical methods and many others like Hartree, Falkner, and Skan [11] . Recently, Asaithambi [8] used a finite difference method to solve Blasius equation. Some other techniques have been adopted to solve the Blasius equation, like Laplace transform and homotopy perturbation method [21] and Adomian decomposition method [1].

Since this equation occupies an important place in the boundary layer problem of hydrodynamics we will give a short background how the original equation of Blasius is constructed.

For a two-dimensional incompressible flow with zero pressure gradient over a flat plate the stationary equations for the velocity fields u and v are giving by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.2}$$
$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = v\frac{\partial^2 u}{\partial y^2},$$

where x is the direction of flow, y the direction of the normal to the plate, u and v the component of the velocity in the directions x and y respectively. The boundary conditions are:

$$u(x, y = 0) = v(x, y = 0) = 0,\, u(x, y = \infty) = u_\infty$$

where U is the velocity of the fluid. Define the following,

$$z = y\sqrt{\frac{U_\infty}{v\,x}},\, \phi = \sqrt{v\,x\,U_\infty}\, f(z),$$

where $f(z)$ is the dimensionless stream function. Now for the velocity component u,

$$u = \frac{\partial \phi}{\partial y} = \sqrt{v\,x\,U_\infty}\, f'(z)\sqrt{\frac{U_\infty}{}v\,x} = U_\infty\, f'(z). \tag{4.3}$$

Also the transverse velocity component can be expressed as,

$$v = -\frac{\phi}{\partial x} = 12\sqrt{\frac{v\,U_\infty}{x}}(zf'(z) - f(z)). \tag{4.4}$$

Now inserting (4.3) and (4.4) in (4.2) to get,

$$f'''(z) + f(z)f''(z) = 0, \tag{4.5}$$

with the boundary conditions:

$$f(0) = 0, f'(0) = 0, f'(z) \to k \, as \, z \to \infty.$$

Blasius provided an exact solution in the form,

$$f(z) = \sum_{i=0}^{\infty} (-1)^i \sigma^{i+1} \frac{z^{3i+2}}{(3i+2)!} A_i, \tag{4.6}$$

where A_i is defined as:

$$A_0 = A_1 = 1, A_i = \sum_{r=0}^{i-1} \genfrac{}{}{0pt}{}{3i-1}{3r} A_r A_{i-r-1}, i \geq 2,$$

and $\sigma = f''(0)$. For $k = 1$, Blasius evaluated $\sigma = 0.332$ [10]. Later on Hartree [20] gained a more accurate value $\sigma = 0.4696$. For $k = 2$ Howarth [22] computed a five decimal value of $\sigma = 1.32824$.

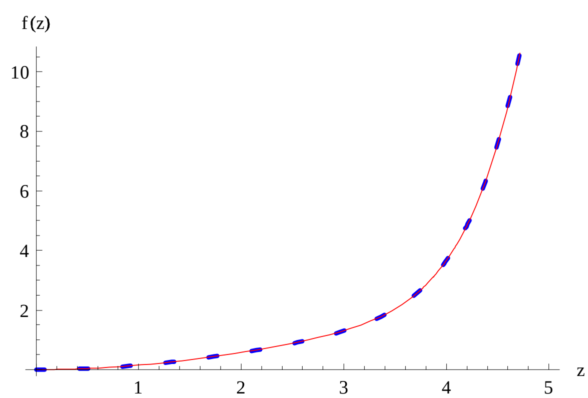

FIGURE 6. The approximate(dashed) and analytic solution (solid) of eq. (22) using $n = 9$.

Example 4.2. Now using Lagrange-Sinc collocation to solve eq. (22) with $k = 1$. Fig. 6 represents the analytic solution, solid line, in [11] and the approximate solution, dashed line.

Now using equation (13) to calculate the error using the exact solution given in (4.6) as a reference to get an error 10^{-3} with 9 Sinc points.

Example 4.3. *Beam problem*

The beam equation which appears in the study of deformation of elastic beams on elastic bearing is a fourth order equations with nonlinear boundary conditions involving third order derivatives. This problem is given by,

$$\begin{aligned} u^{(4)}(x) &= h(x, u(x)), 0 \leq x \leq 1, \\ u(0) = u'(0) = u''(0) = 0, u'''(0) &= g(u(1)). \end{aligned} \tag{4.7}$$

The difficulty in this problem is to obtain its numerical solution due to the appearance of third order nonlinear boundary conditions. The existence and uniqueness of the solution for eq. (4.7) has been discussed by Gupta in [17]. Recently, Geng [14] and Silva [25] introduced two different iterative methods for solving eq (4.7). We will solve this problem using the algorithm proposed in section 2 and showing, numerically, that using Poly-Sinc collocation method will produce much better approximation than these methods exist in [14] and [25]. To solve eq. (4.7) consider $h(x, u)$ and $g(u)$ as [14]:

$$h(x, u) = u^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48, g(u) = 12u. \tag{4.8}$$

Eq. (4.7) together with eq. (4.8) has the exact solution $u(x) = x^5 - 2x^4 + 2x^2$ [14]. In Fig 7, we represent the exact solution, solid line, and the approximate solution, dashed. For the approximate solution we used 11 Sinc points.

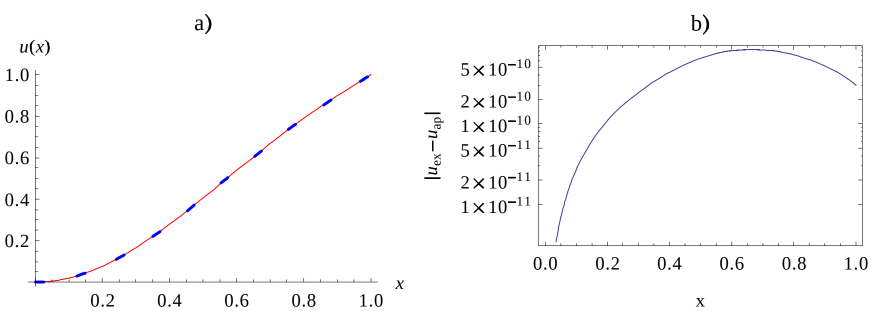

FIGURE 7. a. The exact and approximate solution of eq. (24) using $n = 11$, b. The local error $|u_{ex} - u_{ap}|$.

Example 4.4. Using the square norm error defined in eq (13), we get error of 10^{-10}. Table 2 includes the errors obtained using Poly-Sinc and other techniques in [25] and [14].

Method	Error
Poly-Sinc	$(10)^{-10}$
Contraction principle, [25]	$(10)^{-2}$
Reproducing kernel, [14]	$(10)^{-3}$

Table 2: Error in example 4.2.

Example 4.5. Consider the following fourth order boundary value problems [16]

$$u^4(x) + 4u(x) = 1,\ -1 \leq x \leq 1, \tag{4.9}$$

with the boundary conditions,

$$u(-1) = u(1) = 0,$$

$$u'(-1) = -u'(1) = \frac{\sinh 2 - \sin 2}{4(\cosh 2 + \cos 2)}$$

Gupta used cubic B-spline to solve this fourth order boundary value problems [16]. Here we will use the Poly-Sinc algorithm to solve (25).

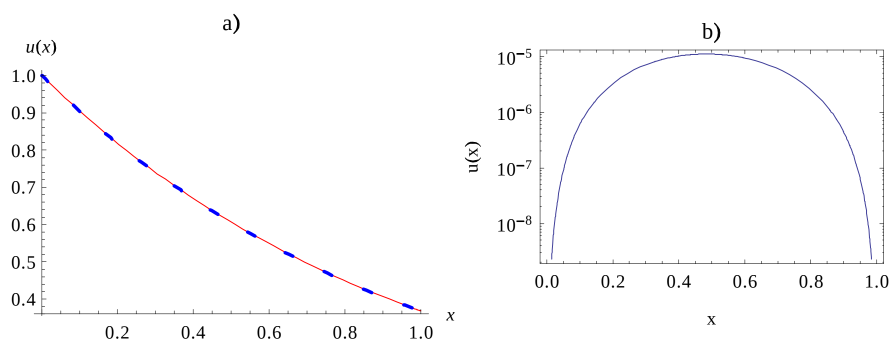

FIGURE 8. a. The exact and approximate solution of eq. (27) using $n = 13$, b. The local error $|u_{ex} - u_{ap}|$.

Example 4.6. Table 3 includes the obtained error using Poly-Sinc and the error obtained in [16].

Method	Error
Poly-Sinc	$(10)^{-10}$
Cubic B-spline, [16]	$(10)^{-6}$

Table 3: Error in example 4.3.

Example 4.7. Consider the following six order boundary value problem [12]

$$\begin{aligned} u^{(6)}(x) + e^{-x}u^2(x) = e^{-x} + e^{x},\ 0 \le x \le 1, \\ u(0) = 1,\ u'(0) = -1,\ u''(0) = 1, \\ u(1) = \frac{1}{e},\ u'(1) = -\frac{1}{e},\ u''(1) = \frac{1}{e}. \end{aligned} \tag{4.10}$$

this equation has the exact solution $u(x) = e^{-x}$. This Problem was discussed by Zayed and Gamel [12]. They presented a Sinc-Glarekin method to solve nonlinear boundary value problems with different types of boundary conditions. For this example Zayed and Gamel used 33 Sinc-points to achieve an error 10^{-3}. Here we will use the Poly-Sinc technique to show that with much smaller number of Sinc-points we can achieve a better approximation.

Fig 9 represents the exact solution, solid line, and the approximate solution, dashed. For the approximate solution we used 13 Sinc points. Using the square norm error defined in eq (13), we get an error of 10^{-6}. This means that we used less than half of the points that Zayed used to double the accuracy of the approximate solution.

5. Conclusion

The Poly-Sinc collocation technique is used to obtain accurate numerical solutions of nonlinear boundary value problems with homogeneous and inhomogeneous boundary conditions. This technique needs very little computational effort concerning only the solution of an algebraic nonlinear system of equations. Two advantages of this technique have been introduced here. The first advantage is that the technique can deal easily with different types of singularity at the boundaries without any modification of the technique. The second advantage is the exponential convergence of this solution that is reaching the optimal upper bound of that are discussed in all Sinc approximations. Finally, we demonstrated that this technique is much more effective than the other techniques that are used to solve the equations discussed in the examples.

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