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A FRACTIONAL ORDER MODEL ON BILINGUALISM

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Abstract. A fractional order model of one unilingual component and one bilingual component of a population is developed. Equilibrium points are found and their stability is investigated. Also, numerical solutions are obtained for an example of the system.

1. Introduction

Models for the growth of a population with one unilingual and one bilingual component are developed by Baggs and Freedman in which the models consist of a system of two ordinary differential equations are studied. In the Baggs-Freedman model and in most of the work following them, integer order differential equations has been used [1].

In recent years, it has turned out that many phenomena in different fields can be described very succesfully by the models using fractional order differential equations [2].The derivative of an arbitrary order or fractional derivative has been introduced almost 300 years ago with a query posed by L'Hospital to Leibnitz. The fractional calculus was reasonably developed by 19th century. It was realized, only in the past few decades that these derivatives are better models to study physical phenomenon in transient state [3]. Also using fractional order differential equations can help us reduce the errors arising from the neglected parameters in modelling real life phenomena [4]. These are some of the reasons that motivate us to use fractional order model instead of integer order one.

In this paper, we will develop a fractional order model of one unilingual component and one bilingual component of a population and we will investigate the dynamics of the interactions of the population in a closed environment and we apply and get the numerical results of the model to the bilingual population of Turkey.

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2. Modelling

First, we need to recall the definition of fractional order derivative. Altough there are several definitions of fractional order derivatives and integrals, we use Caputo's definition in this paper. The main advantage of Caputo's definition is that the initial conditions for fractional differential equations with Caputo derivatives are in the same form as for integer-order differential equations [3].

Definition 2.1. The Caputo-type fractional derivative of order $q > 0$ for a function $f : (0, \infty) \to R$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} f^{(n)}(\tau) d\tau,$$

where $n = \lfloor q \rfloor + 1$ and $\lfloor q \rfloor$ is the integer part of q.

Here and elsewhere Γ denotes the gamma function.

Alongside of Turkish speaking unilingual majority in Turkey, there are several bilingual groups which speak one of the languages such as Kurdish, Arabic, Lazish etc., as their mother tongue together with Turkish. In our model, the population which speaks only Turkish is considered as unilingual component and the population which also speaks mother tongue is considered as bilingual component. The interaction between these two components is given by an autonomous system which consists of two differential equations of fractional order. The concentrations of the components at time $t \geq 0$ are represented by $x(t)$ and $y(t)$, respectively.

As being the situation in Turkey, since the movement is assumed to be in steady-state, the effect of the emigration from the environment and immigration to the environment are not considered. We also assume that conversion from unilingual to bilingual doesn't exist. In addition, we assume that children born to bilingual parents may enter the population as bilinguals or unilinguals, but children of unilingual parents enter the population as unilinguals.

Considering all these assumptions, we can write

$$\begin{cases} D^q x(t) = (B_1 - D_1)x(t) - L_1 x^2(t) + P_1 B_2 y(t) \\ D^q y(t) = (P_2 B_2 - D_2)y(t) - L_2 y^2(t) \end{cases} \tag{2}$$

where $0 < q \leq 1$, $B_i > D_i$ $(i = 1, 2)$, $0 < P_1 < 1$ and $P_2 = 1 - P_1$. P_1 represents the rate of the children of y, which enter the population as unilinguals. Also, B_1 and $D_1 + L_1 x$ are, respectively, the specific birth rate and death rate of x, while similarly B_1 and $D_1 + L_1 x$ are of y. Carrying capasities of the environment of x and y will be denoted by K_1 and K_2, respectively, where $K_1 = \frac{B_1 - D_1}{L_1}$, $K_2 = \frac{B_2 - D_2}{L_2}$.

3. Stability of the System

Setting $D^q x(t) = 0$ and $D^q y(t) = 0$, we have $D^q x(t) + D^q y(t) = 0$. We find the equilibrium points by intersecting the curves that represent these equations. Denoting the curves given by the first and the third equailities by γ_1, γ_2, respectively, we get

$$\gamma_1 \quad : \quad y = \frac{x}{P_1 B_2}\left[L_1 x - (B_1 - D_1)\right]$$

$$\gamma_2 \quad : \quad L_1\left[x - \frac{B_1 - D_1}{2L_1}\right]^2 + L_2\left[y - \frac{B_2 - D_2}{2L_2}\right]^2 = \frac{(B_1 - D_1)^2}{4L_1} + \frac{(B_2 - D_2)^2}{4L_2}.$$

γ_1 is a parabola passing through the points $(0,0)$ and $(K_1, 0)$, while γ_2 is an ellipse with centre at $(\frac{K_1}{2}, \frac{K_2}{2})$, passing through the points $(0,0)$, $(K_1, 0)$, $(0, K_1)$ and (K_1, K_2) etc. Thus the equilibrium points with nonnegative components are $E_0(0,0)$, $E_1(K_1, 0)$ and $E^*\left(\frac{1}{2}K_1 + \frac{1}{2}\sqrt{K_1^2 + \frac{4P_1B_2(P_2B_2 - D_2)}{L_1L_2}}, \frac{P_2B_2 - D_2}{L_2}\right)$.

The Jacobian matrix $J(x^*, y^*)$ for the system given in (2) is

$$J(x^*, y^*) = \begin{bmatrix} B_1 - D_1 - 2L_1x^* & P_1B_2 \\ 0 & P_2B_2 - D_2 - 2L_2y^* \end{bmatrix}.$$

The Jacobian matrix at E_0 is

$$J(E_0) = J(0,0) = \begin{bmatrix} B_1 - D_1 & P_1B_2 \\ 0 & P_2B_2 - D_2 \end{bmatrix}.$$

We know that, if all of the eigenvalues, λ_i $(i = 1, 2)$ of $J(E_0)$ satisty the condition $|\arg \lambda_i| > \frac{q\pi}{2}$, then E_0 is asymptotically stable [5],[6]. Solving the characteristic equation $\det(J(E_0) - \lambda I) = 0$, we obtain the equation $(B_1 - D_1 - \lambda)(P_2B_2 - D_2 - \lambda) = 0$, which has the roots $\lambda_1 = B_1 - D_1$ and $\lambda_2 = P_2B_2 - D_2$. Since we assume that $B_1 - D_1 > 0$, E_0 is unstable. Following the same procedure as in the investigation of stability for E_0, we get the Jacobian matrix at E_1 as

$$J(E_1) = \begin{bmatrix} -(B_1 - D_1) & P_1B_2 \\ 0 & P_2B_2 - D_2 \end{bmatrix}$$

and the eigenvalues of the characteristic equation $\det(J(E_1) - \lambda I) = 0$ as $\lambda_1 = -(B_1 - D_1)$ and $\lambda_2 = P_2B_2 - D_2$. Since $\lambda_1 = -(B_1 - D_1) < 0$, the condition that makes E_1 asymptotically stable can be defined as $\lambda_2 = P_2B_2 - D_2 < 0$. But the points $E_0(0,0)$ and $E_1(K_1, 0)$ are of little interests, since the first one means that there is no population and the second one means that there is no bilingual population. Thus we are interested in analysing the equilibrium point E^* which has positive components.

Now, we investigate the existence, uniqueness and stability for E^*.

Theorem 3.1. *If* $\frac{-L_1K_1^2}{4B_2P_1} < \frac{K_2}{2} - \frac{1}{2}\sqrt{\frac{L_1K_1^2 + L_2K_2^2}{L_2}}$, *then* E^* *exists and is unique. If* $\frac{-L_1K_1^2}{4B_2P_1} \geq \frac{K_2}{2} - \frac{1}{2}\sqrt{\frac{L_1K_1^2 + L_2K_2^2}{L_2}}$, *then* E^* *doesn't exist.*

Proof. The vertex of the parabola γ_1 is $A(\frac{K_1}{2}, \frac{-L_1K_1^2}{4B_2P_1})$ and the image of $x = \frac{K_1}{2}$ on the ellipse γ_2 is $B(\frac{K_1}{2}, \frac{K_2}{2} - \frac{1}{2}\sqrt{\frac{L_1K_1^2 + L_2K_2^2}{L_2}})$.There are three cases in the context

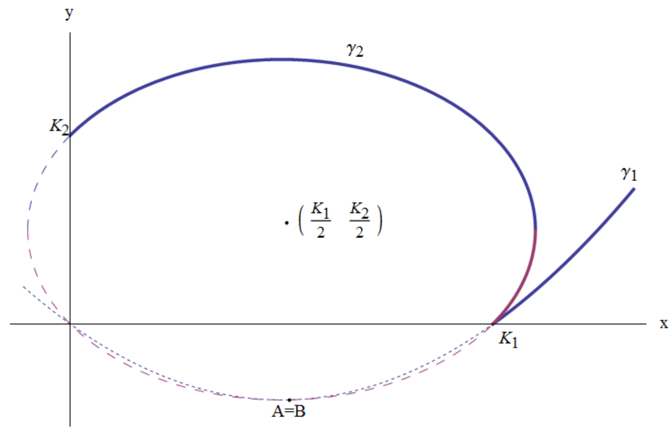

FIGURE 1. Case i

of the relationship betweeen the points A and B.

Case i) $A = B$. In this case, the second component of each point must equal, i.e., $\frac{-L_1K_1^2}{4B_2P_1} = \frac{K_2}{2} - \frac{1}{2}\sqrt{\frac{L_1K_1^2+L_2K_2^2}{L_2}}$. Thus, the only nonnegative point where the curves $\gamma_1$and γ_2 intersect is $(K_1, 0)$ (see Figure 1). Since we try to find an equilibrium E^* which has nonnegative components and since the second component of $(K_1, 0)$ is not positive, E^* doesn't exist.

Case ii) The point A is above B. In this case $\frac{-L_1K_1^2}{4B_2P_1} > \frac{K_2}{2} - \frac{1}{2}\sqrt{\frac{L_1K_1^2+L_2K_2^2}{L_2}}$ holds and as in the Case i, E^* doesn't exist (see Figure 2).

Case iii) The point A is below B. That is $\frac{-L_1K_1^2}{4B_2P_1} < \frac{K_2}{2} - \frac{1}{2}\sqrt{\frac{L_1K_1^2+L_2K_2^2}{L_2}}$. In this case E^* exists and is unique (see Figure 3).

□

Theorem 3.2. *If* $P_2B_2 - D_2 > 0$*, then* E^* *is asymptotically stable.*

Proof. The Jacobian matrix at E^* is

$$J(E^*) = \begin{bmatrix} -L_1\sqrt{K_1^2 + \frac{4P_1B_2(P_2B_2-D_2)}{L_1L_2}} & P_1B_2 \\ 0 & -(P_2B_2 - D_2) \end{bmatrix}$$

and the roots of the characteristic equation $\det(J(E^*) - \lambda I) = 0$ are $\lambda_1 = -L_1 \times \sqrt{K_1^2 + \frac{4P_1B_2(P_2B_2-D_2)}{L_1L_2}}$ and $\lambda_2 = -(P_2B_2 - D_2)$. It is easy to see that $\lambda_2 < 0$,

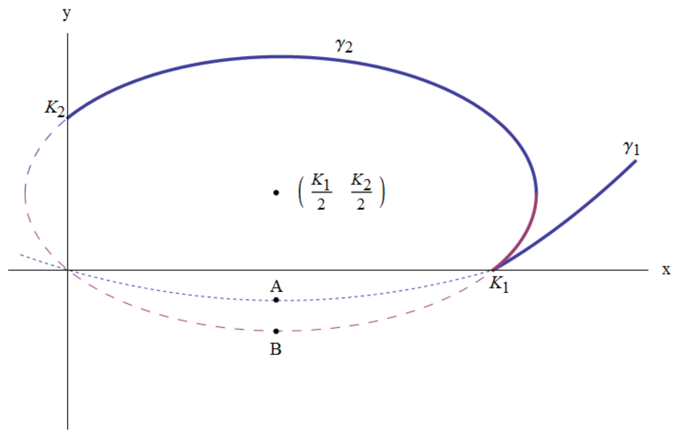

FIGURE 2. Case ii

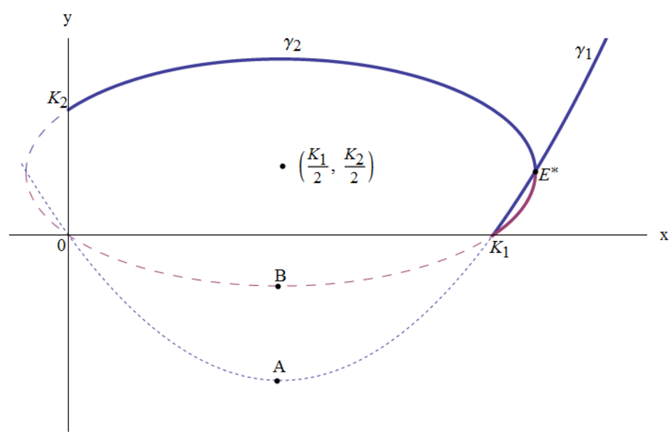

FIGURE 3. Case iii

since $P_2 B_2 - D_2 > 0$. Therefore, E^* is asymptotically stable.

□

4. SIMULATION

The following two theorems, which are given in [7], will be used for finding the solution of fractional differential system.

Theorem 4.1. *Let* $\|.\|$ *denote any convenient norm on* R^n. *Assume that* $f \in C[R_1, R^n]$, *where* $R_1 = [(t,X) : 0 \leq t \leq a$ *and* $\|X - X_0\| \leq b]$, $f = (f_1, f_2, ..., f_n)^T$, *and* $X = (x_1, x_2, ..., x_n)^T$, *and let* $\|f(t,X)\| \leq M$ *on* R_1. *Then there exists at least one solution for the system of fractional differential equation given by*

$$D^q X(t) = f(t, X(t)) \tag{3}$$

with the initial condition

$$X(0) = X_0 \tag{4}$$

on $0 \leq t \leq \beta$, *where* $\beta = \min(a, \left[\frac{b}{M}\Gamma(q+1)\right]^{1/q}),\ 0 < q < 1.$

Theorem 4.2. *Consider IVP given by (3)-(4) of order* q $(0 < q < 1)$. *Let* $g(v, X_*(v)) = f(t - (t^q - v\Gamma(q+1))^{1/q}, X(t - (t^q - v\Gamma(q+1))^{1/q}))$ *and assume that the condition of Theorem 4.1 hold. Then a solution* $X(t)$ *of*

$$\begin{aligned} D^q x(t) &= f(t, x(t)) \\ x(0) &= x_0 \end{aligned}$$

can be given by $X(t) = X_*(t^q/\Gamma(q+1))$, *where* $X_*(v)$ *is a solution of the system of integer order differetial equations*

$$\frac{d(X_*(v))}{dv} = g\left(v, X_*(v)\right)$$

with the initial condition

$$X_*(0) = X_0.$$

Substituting $B_1 = 0.14$, $B_2 = 0.2$, $D_1 = D_2 = 0.02$, $P_1 = 0.6$, $P_2 = 0.4$, $L_1 = L_2 = 0.002$, which are expected values for Turkey, and $q = 0.9$ in system (2), we obtain

$$\begin{aligned} D^{0.9}x &= 0.12x - 0.002x^2 + 0.12y \\ D^{0.9}y &= 0.06y - 0.002y^2 \end{aligned} \tag{5}$$

and the positive equilibrium point is $E^*(81.9615,\ 30)$. Letting the initial conditions

$$x(0) = 60,\ y(0) = 15 \tag{6}$$

and using Theorem 4.2 to solve IVP (5)-(6), the corresponding integer order system becomes

$$\begin{aligned} \frac{dx_*(v)}{dv} &= 0.12x_*(v) - 0.002x_*^2(v) + 0.12y_*(v) \\ \frac{dy_*(v)}{dv} &= 0.06y_*(v) - 0.002y_*^2(v) \end{aligned} \tag{7}$$

with the initial conditions

$$x_*(0) = 60,\ y_*(0) = 15. \tag{8}$$

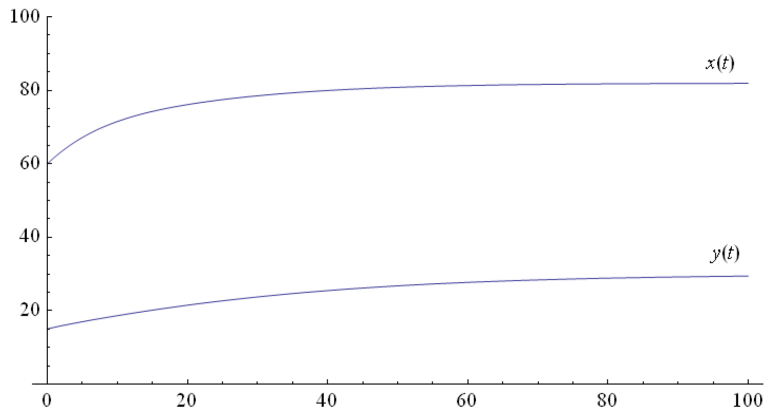

FIGURE 4. Solutions x and y of the system (5)-(6)

Since we have transformed the fractional order system (5)-(6) into the integer order system (7)-(8), we can apply any convenient method improved for finding solution of integer order systems. The second equation in system (7) is a seperable first order differential equation and its solution with the initial condition $y_*(0) = 15$ can easily be found as

$$y_*(v) = \frac{0.9e^{0.06v}}{0.03 + 0.03e^{0.06v}}. \tag{9}$$

Substituting $y_*(v)$ in the first equation of system (7), we have

$$\frac{dx_*(v)}{dv} = 0.12x_*(v) - 0.002x_*^2(v) + \frac{3.6e^{0.06v}}{1 + e^{0.06v}}. \tag{10}$$

We apply the Adams-Bashforth-Moulton method to get the numerical solution of the integer order differential equation (10) with the initial condition $x_*(0) = 60$, and then we take $v = t^{0.9}/\Gamma(1.9)$, both in (9) and (10), to turn back to the fractional order system (5)-(6) according to Theorem 4.2. It means that the solution $(x_*(v), y_*(v))$ of the integer order system corresponds the solution $(x_*(t^{0.9}/\Gamma(1.9)), y_*(t^{0.9}/\Gamma(1.9)))$ of the fractional order system (5)-(6). Therefore, we obtain the solution of the fractional order system (5)-(6) as shown in Figure 4. It is seen in this figure that $x(t)$ and $y(t)$ approximate to points 81.9615 and 30, respectively, which are the components of positive equilibrium E^*. It means that, under the assumptions which construct the model, unilingual and bilingual populations in Turkey (along with expected values) are steady-state at points 81.9615 and 30 (in millions).

We also solve the system (5)-(6), directly by using the method given in [8]. The method is a predictor-corrector type numerical method to solve fractional order initial value problems of the form (3)-(4). A statement in a pseudo-code notation for the algorithm, in which necessary changes are made to adapt our system, is as follows.

$h := T/N$
$m := \lceil q \rceil$
FOR $k := 1$ TO N DO BEGIN
$b_k := k^q - (k-1)^q$
$a_k := (k+1)^q - 2k^{q+1} + (k-1)^{q+1}$
END
FOR $j := 1$ TO N DO BEGIN

$$p_1 := x_0 + \frac{h^q}{\Gamma(q+1)} \sum_{k_1=0}^{j-1} b_{j-k_1}\ f_1(x_{k_1}, y_{k_1})$$

$$p_2 := y_0 + \frac{h^q}{\Gamma(q+1)} \sum_{k_2=0}^{j-1} b_{j-k_2}\ f_2(x_{k_2}, y_{k_2})$$

$$x_j := x_0 + \frac{h^q}{\Gamma(q+1)} \left(f_1(p_1, p_2) + \left((j-1)^{q+1} - (j-1-q)j^q\right) f_1(x_0, y_0) + \sum_{k=1}^{j} a_{j-k+1} f_1(x_k, y_k) \right)$$

$$y_j := y_0 + \frac{h^q}{\Gamma(q+1)} \left(f_2(p_1, p_2) + \left((j-1)^{q+1} - (j-1-q)j^q\right) f_2(x_0, y_0) + \sum_{k=1}^{j} a_{j-k+1} f_2(x_k, y_k) \right)$$

END

Here T is the upper bound of the interval where the solution is to be approximated and N is the number of time steps that the algorithm is to take. Both T and N are positive integers and f_i $(i = 1, 2)$ is the right-hand side of the ith equation of the system (5). In a programming language of our preference, the method is applied to the system and the result is shown in Figure 4.

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