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ON CONVERGENCE PROPERTIES OF FIBONACCI-LIKE CONDITIONAL SEQUENCES

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Abstract. It is well-known that the ratios of successive terms of Fibonacci numbers $\left\{\frac{F_{n+1}}{F_n}\right\}$ converge to the golden ratio $\frac{1+\sqrt{5}}{2}$, so it is natural to ask if analogous results exist for the generalizations of the Fibonacci sequence. In this paper, we consider the generalization of the Fibonacci sequence, which is called Fibonacci-like conditional sequences and we investigate the convergence properties of this sequences.

1. INTRODUCTION

The sequence F_n of Fibonacci numbers are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$.

One can find many applications of this numbers in various branches of science like pure and applied mathematics, in biology, among many others. Especially, this numbers are relatives of the golden section, which itself appears in the study of nature and of art. See also [3] and [6] for additional references and history.

This sequence has been generalized in many ways. In [5], authors introduced the *Fibonacci-like conditional sequence* which is defined by for $n \geq 2$

$$v_n = \begin{cases} a_0 v_{n-1} + b_0 v_{n-2}, & \text{if } n \equiv 0 \pmod{k} \\ a_1 v_{n-1} + b_1 v_{n-2}, & \text{if } n \equiv 1 \pmod{k} \\ \vdots & \\ a_{k-1} v_{n-1} + b_{k-1} v_{n-2}, & \text{if } n \equiv k-1 \pmod{k} \end{cases} \tag{1.1}$$

with arbitrary initial conditions v_0, v_1 and $a_0, a_1, ..., a_{k-1}, b_0, b_1, ..., b_{k-1}$ are non zero numbers. Taking $a_0 = a_1 = ... = a_{k-1} = 1$ and $b_0 = b_1 = ... = b_{k-1} = 1$ with initial conditions $v_0 = 0$ and $v_1 = 1$, it turns out the classical Fibonacci sequence.

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In [5], it was also given the Binet's formulas for the sequence $\{v_n\}$ in terms of a generalized continuant;

$$v_{nk+r} = (-1)^{k(n+1)} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} v_{k+r} - B \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} v_r \right) \tag{1.2}$$

where $\alpha = \frac{(-1)^k A + \sqrt{A^2 - 4(-1)^k B}}{2}$ and $\beta = \frac{(-1)^k - \sqrt{A^2 - 4(-1)^k B}}{2}$ are the roots of the polynomial $p(z) = z^2 - (-1)^k Az + (-1)^k B$,

$$A := K_1 + b_1 K_2 \qquad B := \prod_{r=0}^{k-1} b_r. \tag{1.3}$$

Here, K_1 and K_2 are generalized continuants which are defined in [5].

As $\left\{ \frac{F_{n+1}}{F_n} \right\}$ converges to the golden ratio $\frac{1+\sqrt{5}}{2}$, so it is natural to ask if analogous results exist for the generalizations of the Fibonacci sequence. In [2], it is investigated the convergence of the ratios of the terms of the *k-periodic Fibonacci sequence*, which is defined by taking $b_0 = b_1 = ... = b_{k-1} = 1$ in (1.1), with initial conditions $v_0 = 0$ and $v_1 = 1$. Also in [1], for the k-periodic Fibonacci sequence in the case of $k = 2$, authors show that successive terms of the sequence do not converge, though convergence of ratios of terms when increasing by two's or ratios of even or odd terms.

Following [1] and [2], here we investigate the convergence properties of the Fibonacci-like conditional sequences in (1.1). Our results generalize the former results.

2. Main Results

Assume that $A \neq 0$ and $A^2 > (-1)^k 4B$. Hence, either $\left|\frac{\beta}{\alpha}\right| < 1$ or $\left|\frac{\alpha}{\beta}\right| < 1$.

Theorem 2.1. *For $n \geq 1$, the ratios of successive terms of the subsequence $\{v_{nk+r}\}$ converge to*

$$\begin{cases} (-1)^k \alpha, & \text{if } \left|\frac{\beta}{\alpha}\right| < 1 \\ (-1)^k \beta, & \text{if } \left|\frac{\alpha}{\beta}\right| < 1 \end{cases} \tag{2.1}$$

Proof. From (1.2), we get

$$\begin{aligned} \frac{v_{(n+1)k+r}}{v_{nk+r}} &= \frac{(-1)^{k(n+2)} \left[\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) v_{k+r} - B \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) v_r \right]}{(-1)^{k(n+1)} \left[\left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) v_{k+r} - B \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \right) v_r \right]} \\ &= (-1)^k \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) v_{k+r} - B \left(\alpha^n - \beta^n\right) v_r}{\left(\alpha^n - \beta^n\right) v_{k+r} - B \left(\alpha^{n-1} - \beta^{n-1}\right) v_r}. \end{aligned}$$

If $\left|\frac{\beta}{\alpha}\right| < 1$, we have

$$\frac{v_{(n+1)k+r}}{v_{nk+r}} = (-1)^k \alpha \frac{\left(1-\left(\frac{\beta}{\alpha}\right)^{n+1}\right) v_{k+r} - \frac{B}{\alpha}\left(1-\left(\frac{\beta}{\alpha}\right)^{n}\right) v_r}{\left(1-\left(\frac{\beta}{\alpha}\right)^{n}\right) v_{k+r} - \frac{B}{\alpha}\left(1-\left(\frac{\beta}{\alpha}\right)^{n-1}\right) v_r}$$

and

$$\lim_{n\to\infty} \frac{v_{(n+1)k+r}}{v_{nk+r}} = (-1)^k \alpha.$$

Similarly, if $\left|\frac{\alpha}{\beta}\right| < 1$, we have

$$\frac{v_{(n+1)k+r}}{v_{nk+r}} = (-1)^k \beta \frac{\left(\left(\frac{\alpha}{\beta}\right)^{n+1} - 1\right) v_{k+r} - \frac{B}{\beta}\left(\left(\frac{\alpha}{\beta}\right)^{n} - 1\right) v_r}{\left(\left(\frac{\alpha}{\beta}\right)^{n} - 1\right) v_{k+r} - \frac{B}{\beta}\left(\left(\frac{\alpha}{\beta}\right)^{n-1} - 1\right) v_r}$$

and

$$\lim_{n\to\infty} \frac{v_{(n+1)k+r}}{v_{nk+r}} = (-1)^k \beta$$

□

Theorem 2.2. *For $n \geq 1$ and $r = 1, 2, ..., k-1$, $\frac{v_{nk+r}}{v_{nk+r-1}}$ converge to*

$$\begin{cases} \frac{v_{k+r} + (-1)^{k+1}\beta v_r}{v_{k+r-1} + (-1)^{k+1}\beta v_{r-1}}, & \text{if } \left|\frac{\beta}{\alpha}\right| < 1 \\ \frac{v_{k+r} + (-1)^{k+1}\alpha v_r}{v_{k+r-1} + (-1)^{k+1}\alpha v_{r-1}}, & \text{if } \left|\frac{\alpha}{\beta}\right| < 1 \end{cases} \tag{2.2}$$

as $n \to \infty$.

Proof. From (1.2) and $B = (-1)^k \alpha\beta$, we get

$$\begin{aligned} \frac{v_{nk+r}}{v_{nk+r-1}} &= \frac{(-1)^{k(n+1)}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta} v_{k+r} - B\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} v_r\right)}{(-1)^{k(n+1)}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta} v_{k+r-1} - B\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} v_{r-1}\right)} \\ &= \frac{(\alpha^n - \beta^n) v_{k+r} - B\left(\alpha^{n-1} - \beta^{n-1}\right) v_r}{(\alpha^n - \beta^n) v_{k+r-1} - B\left(\alpha^{n-1} - \beta^{n-1}\right) v_{r-1}} \end{aligned}$$

If $\left|\frac{\beta}{\alpha}\right| < 1$ we have

$$\frac{v_{nk+r}}{v_{nk+r-1}} = \frac{\left(1-\left(\frac{\beta}{\alpha}\right)^{n}\right) v_{k+r} - \frac{B}{\alpha}\left(1-\left(\frac{\beta}{\alpha}\right)^{n-1}\right) v_r}{\left(1-\left(\frac{\beta}{\alpha}\right)^{n}\right) v_{k+r-1} - \frac{B}{\alpha}\left(1-\left(\frac{\beta}{\alpha}\right)^{n-1}\right) v_{r-1}}$$

and

$$\lim_{n\to\infty}\frac{v_{nk+r}}{v_{nk+r-1}}=\frac{v_{k+r}+(-1)^{k+1}\beta v_r}{v_{k+r-1}+(-1)^{k+1}\beta v_{r-1}}.$$

Similarly, since $\left|\frac{\alpha}{\beta}\right|<1$ we have

$$\frac{v_{nk+r}}{v_{nk+r-1}}=\frac{\left(\left(\frac{\alpha}{\beta}\right)^n-1\right)v_{k+r}-\frac{B}{\beta}\left(\left(\frac{\alpha}{\beta}\right)^{n-1}-1\right)v_r}{\left(\left(\frac{\alpha}{\beta}\right)^n-1\right)v_{k+r-1}-\frac{B}{\beta}\left(\left(\frac{\alpha}{\beta}\right)^{n-1}-1\right)v_{r-1}}$$

and

$$\lim_{n\to\infty}\frac{v_{nk+r}}{v_{nk+r-1}}=\frac{v_{k+r}+(-1)^{k+1}\alpha v_r}{v_{k+r-1}+(-1)^{k+1}\alpha v_{r-1}}.$$

□

In the case of $k=2$ in $\{v_n\}$, with the initial conditions $v_0=0$ and $v_1=1$, Teorem 2.1 and Teorem 2.2 reduce the following results.

Corollary 1. *For $n\geq 1$, the ratios of successive even terms of the sequence $\{v_n\}$ converge to*

$$\begin{cases}\frac{A+\sqrt{A^2-4B}}{2}, & \text{if } \left|\frac{\beta}{\alpha}\right|<1\\ \frac{-A+\sqrt{A^2-4B}}{2}, & \text{if } \left|\frac{\alpha}{\beta}\right|<1\end{cases}\tag{2.3}$$

where $A=a_0a_1+b_0+b_1$ and $B=b_0b_1$.

Corollary 2. *For $n\geq 1$, the ratios $\frac{v_{2n+1}}{v_{2n}}$ converge to*

$$\begin{cases}\frac{\alpha-b_0}{a_0}, & \text{if } \left|\frac{\beta}{\alpha}\right|<1\\ \frac{\beta-b_0}{a_0}, & \text{if } \left|\frac{\alpha}{\beta}\right|<1\end{cases}\tag{2.4}$$

as $n\to\infty$.

Corollary 3. *For $n\geq 1$, the ratios $\frac{v_{2n+2}}{v_{2n+1}}$ converge to*

$$\begin{cases}\frac{\alpha a_0}{\alpha-b_0}, & \text{if } \left|\frac{\beta}{\alpha}\right|<1\\ \frac{\beta a_0}{\beta-b_0}, & \text{if } \left|\frac{\alpha}{\beta}\right|<1\end{cases}\tag{2.5}$$

as $n\to\infty$.

Proof. From (1.2) and using $\alpha\beta = b_0 b_1$, we have

$$\begin{aligned}\frac{v_{2n+2}}{v_{2n+1}} &= \frac{a_0\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)}{\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right) - b_0\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)} \\ &= \frac{a_0\left(\alpha^{n+1}-\beta^{n+1}\right)}{\left(\alpha^{n+1}-\beta^{n+1}\right) - b_0\left(\alpha^n-\beta^n\right)}.\end{aligned}$$

If $\left|\frac{\beta}{\alpha}\right| < 1$, we have

$$\frac{v_{2n+2}}{v_{2n+1}} = \frac{a_0\left(1-\left(\frac{\beta}{\alpha}\right)^{n+1}\right)}{\left(1-\left(\frac{\beta}{\alpha}\right)^{n+1}\right) - \frac{b_0}{\alpha}\left(1-\left(\frac{\beta}{\alpha}\right)^{n}\right)}$$

and

$$\lim_{n\to\infty}\frac{v_{2n+2}}{v_{2n+1}} = \frac{\alpha a_0}{\alpha - b_0}.$$

Similarly, if $\left|\frac{\alpha}{\beta}\right| < 1$, then

$$\frac{v_{2n+2}}{v_{2n+1}} = \frac{a_0\left(\left(\frac{\alpha}{\beta}\right)^{n+1}-1\right)}{\left(\left(\frac{\alpha}{\beta}\right)^{n+1}-1\right) - \frac{b_0}{\alpha}\left(\left(\frac{\alpha}{\beta}\right)^{n}-1\right)}$$

and

$$\lim_{n\to\infty}\frac{v_{2n+2}}{v_{2n+1}} = \frac{\beta a_0}{\beta - b_0}.$$

□

Now, we consider the sequences $\left\{\frac{v_{nk+r}}{v_{nk+r-1}}\right\}$ for $r = 1, 2, ..., k$.

In [4], it is shown that, for k-periodic Fibonacci sequence

$$\left\{\frac{v_{nk+r}}{v_{nk+r-1}}\right\} \to L_{r-1}$$

where

$$L_{i+1} = \frac{a_{i+2}L_i + 1}{L_i},\ i \in \{0, 1, ..., k-2\}.$$

It is surprising that when we consider the Fibonacci-like conditional sequences, we can also get the similar results.

By using the definition of the sequence $\{v_n\}$, we have

$$v_{nk+i} = a_i v_{nk+i-1} + b_i v_{nk+i-2},\ i = 0, 1, ..., k-1. \tag{2.6}$$

If $\left|\frac{\beta}{\alpha}\right| < 1$, from (2.2)

$$\frac{v_{nk+1}}{v_{nk}} \to \frac{v_{k+1} + (-1)^{k+1}\beta v_1}{v_k + (-1)^{k+1}\beta v_0}.$$

Let $\frac{v_{k+1}+(-1)^{k+1}\beta v_1}{v_k+(-1)^{k+1}\beta v_0} = L_0$.

Similarly, one can show that

$$\frac{v_{nk+2}}{v_{nk+1}} \to \frac{v_{k+2} + (-1)^{k+1}\beta v_2}{v_{k+1} + (-1)^{k+1}\beta v_1} = \frac{a_2 L_0 + b_2}{L_0} = L_1.$$

Analogously,

$$\frac{v_{nk+3}}{v_{nk+2}} \to \frac{a_3 L_1 + b_3}{L_1} = L_2.$$

More generally,

$$\frac{v_{nk+(i+2)}}{v_{nk+(i+1)}} \to \frac{a_{i+2} L_i + b_{i+2}}{L_i} = L_{i+1},\ i \in \{0, 1, ..., k-2\}.$$

And note that $L_{k+i} = L_i$, for all non negative integers i. Hence, the set of the limits of the sequences $\left\{\frac{v_{nk+r}}{v_{nk+r-1}}\right\}$ is

$$\{L_0, L_1, ..., L_{k-1}\}. \tag{2.7}$$

If $\left|\frac{\alpha}{\beta}\right| < 1$, following the same ideas, one can show that

$$\begin{aligned} \frac{v_{nk+1}}{v_{nk}} &\to \widetilde{L}_0, \\ \frac{v_{nk+2}}{v_{nk+1}} &\to \frac{a_2 \widetilde{L}_0 + b_2}{\widetilde{L}_0} = \widetilde{L}_1 \end{aligned}$$

and more generally,

$$\frac{v_{nk+(i+2)}}{v_{nk+(i+1)}} \to \frac{a_{i+2} \widetilde{L}_i + b_{i+2}}{\widetilde{L}_i} = \widetilde{L}_{i+1}, i \in \{0, 1, ..., k-2\}.$$

3. Examples

Example 3.1. Let $k = 3$. The sequence $\{v_n\}$ satisfies with initial conditions $v_0 = 0, v_1 = 1$ and for $n \geq 2$,

$$v_n = \begin{cases} a_0 v_{n-1} + b_0 v_{n-2}, & \text{if } n \equiv 0 \pmod 3 \\ a_1 v_{n-1} + b_1 v_{n-2}, & \text{if } n \equiv 1 \pmod 3 \\ a_2 v_{n-1} + b_2 v_{n-2}, & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{3.1}$$

Using definition of generalized continuant in [5], we have

$$A = a_0 a_1 a_2 + a_1 b_0 + a_0 b_2 + a_2 b_1 \quad \text{and} \quad B = b_0 b_1 b_2.$$

Since

$$\begin{aligned} v_2 &= a_2,\ v_3 = a_0a_2 + b_0,\ v_4 = a_0a_1a_2 + a_1b_0 + a_2b_1, \\ v_5 &= a_2\left(a_0a_1a_2 + a_1b_0 + a_2b_1\right) + b_2\left(a_0a_2 + b_0\right) \end{aligned}$$

and taking $a_0 = 1, a_1 = 2, a_2 = 1, b_0 = 2, b_1 = -1, b_2 = 1,$ the terms of the sequence $\{v_n\}$ are listed in the following table

n	1	2	3	4	5	6	7	8	9	10	11	12	...
v_n	1	1	3	5	8	18	28	46	102	158	260	576	...

.

Since

$$\alpha = -3 + \sqrt{7} \quad \text{and} \quad \beta = -3 - \sqrt{7},$$

then $\left|\frac{\alpha}{\beta}\right| < 1.$

For $r = 1, 2, 3,$ the limit of the terms of the sequence $\left\{\frac{v_{3n+r}}{v_{3n+r-1}}\right\}$ are

$$\begin{aligned} \frac{v_{3n+1}}{v_{3n}} &\to \frac{v_4 + \alpha v_1}{v_3 + \alpha v_0} = \frac{2+\sqrt{7}}{3} = \widetilde{L}_0 \\ \frac{v_{3n+2}}{v_{3n+1}} &\to \frac{v_5 + \alpha v_2}{v_4 + \alpha v_1} = -1 + \sqrt{7} \\ &= \frac{a_2\widetilde{L}_0 + b_2}{\widetilde{L}_0} = \frac{\widetilde{L}_0 + 1}{\widetilde{L}_0} = \widetilde{L}_1 \\ \frac{v_{3n+3}}{v_{3n+2}} &\to \frac{\widetilde{L}_1 + 2}{\widetilde{L}_1} = \frac{4+\sqrt{7}}{3} = \widetilde{L}_2. \end{aligned}$$

Example 3.2. Let $k = 2$. The sequence $\{v_n\}$ satisfies with initial conditions $v_0 = 0$, $v_1 = 1$ and for $n \geq 2$,

$$v_n = \begin{cases} a_0v_{n-1} + b_0v_{n-2}, & \text{if } n \equiv 0 \pmod 2 \\ a_1v_{n-1} + b_1v_{n-2}, & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{3.2}$$

$$A = a_0a_1 + b_0 + b_1 \quad \text{and} \quad B = b_0b_1.$$

Taking $a_0 = 1, a_1 = 2, b_0 = 3, b_1 = 4$, the terms of the sequence $\{v_n\}$ are listed in the following table

n	1	2	3	4	5	6	7	8	9	10	11	12	...
v_n	1	1	6	9	42	69	306	513	2250	3789	16578	27945	...

.

Since

$$\alpha = \tfrac{9+\sqrt{33}}{2} \quad \text{and} \quad \beta = \tfrac{9-\sqrt{33}}{2},$$

then $\left|\frac{\beta}{\alpha}\right| < 1.$

The ratios of successive even terms of $\{v_n\}$ converge

$$\frac{v_{2n+2}}{v_{2n}} \to \alpha = \frac{9+\sqrt{33}}{2}.$$

For $r = 1, 2$, the limit of the sequence $\left\{\frac{v_{2n+r}}{v_{2n+r-1}}\right\}$ are

$$\begin{aligned} \frac{v_{2n+1}}{v_{2n}} &\to \frac{\alpha - b_0}{a_0} = \frac{3+\sqrt{33}}{2} = L_0 \\ \frac{v_{2n+2}}{v_{2n+1}} &\to \frac{\alpha a_0}{\alpha - b_0} = \frac{1+\sqrt{33}}{4} \\ &= \frac{a_0 L_0 + b_0}{L_0} = L_1. \end{aligned}$$

Example 3.3. In (3.2), by taking $a_0 = 3, a_1 = -5, b_0 = 2, b_1 = 1$, the terms of the sequence $\{v_n\}$ are listed in the following table

n	1	2	3	4	5	6	7	8	9	10	11	12	...
v_n	1	3	-14	-36	166	426	-1964	-5040	23 236	59628	-274 904	-705 456	...

.

Since

$$\alpha = -6 + \sqrt{34} \quad \text{and} \quad \beta = -6 - \sqrt{34},$$

then $\left|\frac{\alpha}{\beta}\right| < 1$.

The ratios of successive even terms of the sequence $\{v_n\}$ converge

$$\frac{v_{2n+2}}{v_{2n}} \to \beta = -6 - \sqrt{34}.$$

For $r = 1, 2$, the limit of the sequence $\left\{\frac{v_{2n+r}}{v_{2n+r-1}}\right\}$ are

$$\begin{aligned} \frac{v_{2n+1}}{v_{2n}} &\to \frac{\beta - b_0}{a_0} = -\frac{8+\sqrt{34}}{3} = \widetilde{L}_0 \\ \frac{v_{2n+2}}{v_{2n+1}} &\to \frac{\beta a_0}{\beta - b_0} = \frac{7+\sqrt{34}}{5} \\ &= \frac{a_0 \widetilde{L}_0 + b_0}{\widetilde{L}_0} = \widetilde{L}_1. \end{aligned}$$

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