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COMPLEX FACTORIZATION OF SOME TWO-PERIODIC LINEAR RECURRENCE SYSTEMS

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Abstract. In this paper, we define the generalized two-periodic linear recurrence systems and find the factorizations of this recurrence systems. We also solve an open problem given in [3] under certain conditions.

1. Introduction

Definition 1.1. Let $a_0,\ a_1,\ b_0,\ b_1$ are real numbers. The two-periodic second order linear recurrence system $\{v_n\}$ is defined by $v_0 := 0$, $v_1 \in \mathbb{R}$ and for $n \geq 1$

$$v_{2n} := a_0 v_{2n-1} + b_0 v_{2n-2}$$
$$v_{2n+1} := a_1 v_{2n} + b_1 v_{2n-1}.$$

Also, let $A := a_0 a_1 + b_0 + b_1$, $B := b_0 b_1$, and assume $A^2 - 4B \neq 0$.

Heleman studied two periodic second order linear recurrence systems and called it as $\{f_n\}$ in [2]. Curtis and Parry also worked on the same linear recurrence systems in [3]. If we take $v_0 = 0$, $v_1 = 1$ then we get the sequence $\{f_n\}$, so here we study more general case.

We need the following results of Theorem 6 and Theorem 9 in [1], in the case $r = 2$.

The generating function of the sequence $\{v_n\}$ is

$$G(x) = \frac{v_1 x + a_0 v_1 x^2 - b_0 v_1 x^3}{1 - Ax^2 + Bx^4}$$

and the terms of the sequences $\{v_n\}$ satisfy

$$v_{2n} = \frac{\alpha^n - \beta^n}{\alpha - \beta} a_0 v_1 \tag{1.1}$$

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where

$$\alpha = \frac{A+\sqrt{A^2-4B}}{2} \quad , \quad \beta = \frac{A-\sqrt{A^2-4B}}{2}$$

that is, α and β are the roots of the polynomial $p(z) = z^2 - Az + B$. Since $A^2 - 4B \neq 0$ thus α and β are distinct.

We also need to define, the following matrix, for a positive integer n,

$$T(n) := \begin{bmatrix} v_1 & b_0 & & & & & \\ 0 & a_0 & b_1 & & & & \\ & -1 & a_1 & b_0 & & & \\ & & -1 & a_0 & b_1 & & \\ & & & -1 & a_1 & b_0 & \\ & & & & -1 & a_0 & \ddots \\ & & & & & \ddots & \ddots \\ & & & & & & \\ & & & & & & \end{bmatrix}_{n\times n}.$$

It is easily seen by induction that for $n \geq 1$,

$$v_n = \det(T(n)). \tag{1.2}$$

2. The Factorization of v_{2n}

We give two lemmas to prove our main results, Theorem 2.3 and Theorem 2.4.

Lemma 2.1. *Let $n \geq 2$, then*

$$\det(T(2n)) = 0 \iff a_0 = 0 \text{ or } v_1 = 0 \text{ or } a_0a_1 + b_0 + b_1 = 2\sqrt{b_0b_1}\cos\left(\frac{k\pi}{n}\right)$$

where $1 \leq k \leq n-1$.

Proof. By 1.1 and 1.2

$$\begin{aligned} \det(T(2n)) &= 0 \iff v_{2n} = \frac{\alpha^n - \beta^n}{\alpha - \beta} a_0 v_1 = 0 \\ &\iff a_0 = 0 \text{ or } v_1 = 0 \text{ or } \alpha^n - \beta^n = 0 \\ \alpha^n - \beta^n &= 0 \iff \left(\frac{\alpha}{\beta}\right)^n = 1 \end{aligned}$$

Hence, for some $0 \leq k \leq n-1$ we have

$$\begin{aligned} \left(\frac{\alpha}{\beta}\right)^n &= e^{2k\pi i} \\ &\iff \frac{\alpha}{\beta} = e^{\frac{2k\pi i}{n}}. \end{aligned}$$

We note here that $k \neq 0$ since $\alpha \neq \beta$.

Let

$$\theta := \frac{2k\pi i}{n}$$

for some $1 \leq k \leq n-1$. Then,

$$\begin{aligned} \frac{\alpha}{\beta} &= \frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}} = e^{i\theta} \\ \iff & \quad A+\sqrt{A^2-4B} = e^{i\theta}\left(A-\sqrt{A^2-4B}\right). \end{aligned}$$

Next,

$$\sqrt{A^2-4B}e^{i\theta}+\sqrt{A^2-4B} = Ae^{i\theta}-A.$$

Then,

$$\sqrt{A^2-4B} = A\frac{e^{i\theta}-1}{e^{i\theta}+1} = A\frac{e^{i\theta}-1}{e^{i\theta}+1}\frac{e^{-i\theta}+1}{e^{-i\theta}+1} = A\frac{e^{i\theta}-e^{-i\theta}}{2+e^{i\theta}+e^{-i\theta}}.$$

Now, since

$$e^{i\theta} = \cos(\theta)+i\sin(\theta) \text{ and } \sin(-\theta) = -\sin(\theta)\ ,\ \cos(-\theta) = \cos(\theta)$$

we have

$$\sqrt{A^2-4B} = A\frac{e^{i\theta}-e^{-i\theta}}{2+e^{i\theta}+e^{-i\theta}} = A\frac{i\sin(\theta)}{1+\cos(\theta)} = Ai\tan\left(\frac{\theta}{2}\right).$$

Squaring both sides of this equality and after some simplifications we have

$$A = 2\sqrt{B}\cos\left(\frac{\theta}{2}\right). \tag{2.1}$$

Now, substituting the values of A, B and θ in 2.1, we get

$$a_0a_1+b_0+b_1 = 2\sqrt{b_0b_1}\cos\left(\frac{k\pi}{n}\right)$$

for some $1 \leq k \leq n-1$. This is what we wanted prove. □

Lemma 2.2. *Let $n \geq 2$. The eigenvalues of $T(2n)$ are*

$$a_0,\ v_1 \text{ and } \frac{a_0+a_1}{2} \pm \sqrt{\left(\frac{a_0-a_1}{2}\right)^2-(b_0+b_1)+2\sqrt{b_0b_1}\cos\left(\frac{k\pi}{n}\right)},\quad 1 \leq k \leq n-1.$$

Proof. Let $g_0 := 0$, $g_1 := v_1 - t$ and for $n \geq 1$

$$\begin{aligned} g_{2n} &:= (a_0-t)\,g_{2n-1}+b_0g_{2n-2} \\ g_{2n+1} &:= (a_1-t)\,g_{2n}+b_1g_{2n-1}. \end{aligned}$$

The eigenvalues of $T(2n)$ are the solutions of $\det(T(2n)-tI_{2n}) = g_{2n} = 0$. By Lemma 2.1,

$$g_{2n} = 0 \iff a_0-t = 0 \text{ or } g_1 = v_1-t = 0 \text{ or } (a_0-t)(a_1-t)+b_0+b_1 = 2\sqrt{b_0b_1}\cos\left(\frac{k\pi}{n}\right)$$

for some $1 \leq k \leq n-1$. Therefore, the eigenvalues of $T(2n)$ are a_0, v_1 and the solutions of the quadratic equation

$$t^2 - (a_0 + a_1)\,t + a_0 a_1 + b_0 + b_1 = 2\sqrt{b_0 b_1}\cos\left(\frac{k\pi}{n}\right)$$

for some $1 \leq k \leq n-1$. Completing the square we have

$$t^2 - (a_0 + a_1)\,t + \left(\frac{a_0 + a_1}{2}\right)^2 = \left(\frac{a_0 + a_1}{2}\right)^2 - a_0 a_1 - b_0 - b_1 + 2\sqrt{b_0 b_1}\cos\left(\frac{k\pi}{n}\right).$$

Therefore, the eigenvalues of $T(2n)$ are a_0, v_1 and

$$\frac{a_0 + a_1}{2} \pm \sqrt{\left(\frac{a_0 - a_1}{2}\right)^2 - (b_0 + b_1) + 2\sqrt{b_0 b_1}\cos\left(\frac{k\pi}{n}\right)}$$

for some $1 \leq k \leq n-1$. □

Theorem 2.3. *Let $\{v_n\}$ be the two-periodic second order linear recurrence system, and $n \geq 2$. Then*

$$v_{2n} = a_0 v_1 \prod_{k=1}^{n-1}\left(\frac{a_0 + a_1}{2} \pm \sqrt{\left(\frac{a_0 - a_1}{2}\right)^2 - (b_0 + b_1) + 2\sqrt{b_0 b_1}\cos\left(\frac{k\pi}{n}\right)}\right).$$

Proof. The result follows from Lemma 2.2, $v_{2n} = \det(T(2n))$ and the fact that the determinant of a matrix is the product of the eigenvalues of the matrix. □

Theorem 2.4. *Let $\{v_n\}$ be the two-periodic second order linear recurrence system, $n \geq 2$ and $b_1 := 0$. Then*

$$v_{2n+1} = a_0 a_1 v_1 (a_0 a_1 + b_0)^{n-1}.$$

Proof. If we take $b_1 = 0$ in Definition 1, we get $v_0 = 0,\ v_1 \in \mathbb{R}$ and for $n \geq 1$

$$\begin{aligned} v_{2n} &= a_0 v_{2n-1} + b_0 v_{2n-2} \\ v_{2n+1} &= a_1 v_{2n}. \end{aligned}$$

By Theorem 2.3, we have

$$\begin{aligned} v_{2n} &= a_0 v_1 \prod_{k=1}^{n-1}\left(\frac{a_0 + a_1}{2} \pm \sqrt{\left(\frac{a_0 - a_1}{2}\right)^2 - b_0}\right) \\ &= a_0 v_1 \prod_{k=1}^{n-1}(a_0 a_1 + b_0) \\ &= a_0 v_1 (a_0 a_1 + b_0)^{n-1}. \end{aligned}$$

Hence, by the definition of $\{v_n\}$, we get the result

$$v_{2n+1} = a_1 v_{2n} = a_0 a_1 v_1 (a_0 a_1 + b_0)^{n-1}.$$

□

Example 2.5. Let $v_0 = 0,\ v_1 = 1$ and for $n \geq 1$

$$\begin{aligned} v_{2n} &= a_0 v_{2n-1} + b_0 v_{2n-2} \\ v_{2n+1} &= a_1 v_{2n} + b_1 v_{2n-1}. \end{aligned}$$

Then $\{v_n\}$ is added one term to beginning of $\{f_n\}$ sequences in [3]. Namely,

$$f_n = v_{n+1}, \qquad n \geq 0.$$

Hence

$$f_{2n+1} = v_{2n} = a_0 \prod_{k=1}^{n-1} \left(\frac{a_0 + a_1}{2} \pm \sqrt{\left(\frac{a_0 - a_1}{2}\right)^2 - (b_0 + b_1) + 2\sqrt{b_0 b_1} \cos\left(\frac{k\pi}{n}\right)} \right).$$

Therefore this factorization is the same as Theorem 11 in [3].

They give several open questions for future work. One of this question is a complex factorization of the terms f_{2n}. We have solved in the following way at condition $b_1 = 0$ of this question by Theorem 2.4,

$$f_{2n} = v_{2n+1} = a_0 a_1 v_1 \left(a_0 a_1 + b_0\right)^{n-1}.$$

2.1. **Special Cases:**

Case 1. *The case* $v_0 := 0,\ v_1 := 1,\ a_0 := 1,\ a_1 := 1,\ b_0 := 1,\ b_1 := 1,$ *then* $\{v_n\}$ *becomes the sequence of Fibonacci numbers. Therefore, we get*

$$F_{2n} = \prod_{k=1}^{n-1} \left(3 - 2\cos\left(\frac{k\pi}{n}\right)\right).$$

that is the equation 4.1 in [4].

Case 2. *The case* $v_0 := 0\ ,\ v_1 := 1,\ a_0 := 2,\ a_1 := 2,\ b_0 := 1,\ b_1 := 1,$ *then* $\{v_n\}$ *becomes the sequence of Pell numbers. Therefore,*

$$P_{2n} = 2\prod_{k=1}^{n-1} \left(6 - 2\cos\left(\frac{k\pi}{n}\right)\right) = 2^n \prod_{k=1}^{n-1} \left(3 - \cos\left(\frac{k\pi}{n}\right)\right).$$

Case 3. *The case* $v_0 := 0,\ v_1 := 1,\ a_0 := 1,\ a_1 := 1,\ b_0 := 2,\ b_1 := 2,$ *then* $\{v_n\}$ *becomes the sequence of Jacobsthal numbers. Therefore,*

$$J_{2n} = \prod_{k=1}^{n-1} \left(5 - 4\cos\left(\frac{k\pi}{n}\right)\right).$$

Case 4. *The case* $v_0 := 0,\ v_1 := 1,\ a_0 := 1,\ a_1 := 1,\ b_0 := -1,\ b_1 := 1,$ *then* $\{v_n\}$ *becomes the sequence of A053602 on [5]. Then* $\{v_{2n}\}$ *becomes the sequence of Fibonacci numbers. Therefore, we get*

$$F_n = \prod_{k=1}^{n-1} \left(1 - 2i\cos\left(\frac{k\pi}{n}\right)\right).$$

that is the equation 1.1 in [4].

Case 5. *The case* $v_0 := 0,\ v_1 := 1,\ a_0 := 3,\ a_1 := 3,\ b_0 := -2,\ b_1 := -2$*, then* $\{v_n\}$ *becomes the sequence of Mersenne numbers. Therefore,*

$$M_{2n} = 3\prod_{k=1}^{n-1}\left(5 - 4\cos\left(\frac{k\pi}{n}\right)\right) = 3J_{2n}.$$

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