Commun.Fac.Sci.Univ.Ank.Series A1
Volume 63, Number 2, Pages 135–145 (2014)
ISSN 1303–5991

REPRESENTATION NUMBER FORMULAE FOR SOME OCTONARY QUADRATIC FORMS

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ABSTRACT. We find formulae for the number of representation of a positive integer n by each of the quadratic forms

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 + 2x_6^2 + 6x_7^2 + 6x_8^2,$$
$$x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2 + 2x_5^2 + 2x_6^2 + 3x_7^2 + 3x_8^2,$$
$$x_1^2 + x_2^2 + 3x_3^2 + 3x_4^2 + 6x_5^2 + 6x_6^2 + 6x_7^2 + 6x_8^2,$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 + 2x_6^2 + 3x_7^2 + 3x_8^2,$$
$$x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2 + 2x_5^2 + 2x_6^2 + 6x_7^2 + 6x_8^2,$$
$$x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 + 2x_5^2 + 4x_6^2 + 6x_7^2 + 6x_8^2,$$
$$2x_1^2 + 2x_2^2 + 3x_3^2 + 6x_4^2 + 6x_5^2 + 6x_6^2 + 6x_7^2 + 12x_8^2,$$

by using some known convolution sums of divisor functions and known representation formulae for quaternary quadratic forms. Formulae for some other octonary quadratic forms of these type are given before in [4, 5, 6, 11, 17].

1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the set of natural numbers, non-negative integers, integers and complex numbers so that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}$ we set

$$\sigma_k(n) := \begin{cases} \sum_{\substack{d\in\mathbb{N} \\ d|n}} d^k & , \quad n \in \mathbb{N}, \\ 0 & , \quad n \notin \mathbb{N}. \end{cases}$$

We write $\sigma(n)$ for $\sigma_1(n)$. For $a_1, \ldots, a_8 \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we define

$$N(a_1, \ldots, a_8; n) := card\left\{(x_1, \ldots, x_8) \in \mathbb{Z}^k : n = a_1x_1^2 + \cdots + a_8x_8^2\right\}. \quad (1.1)$$

2010 *Mathematics Subject Classification.* 11A25, 11E25.
Key words and phrases. Quadratic forms; Representation numbers.
Received by the editors: 21.11.2013, accepted: 03.07.2014.

*The main results of this paper presented in part at the conference *Algerian-Turkish International Days on Mathematics 2013 (ATIM' 2013)* to be held September 12–14, 2013 in İstanbul at the Fatih University.

If l of $a_1, \ldots, a_8$ are equal, say,

$$a_i = a_{i+1} = \cdots = a_{i+l-1} = a, \tag{1.2}$$

we indicate this in $N(a_1, \ldots, a_8; n)$ by writing a^l for $a_i, a_{i+1}, \ldots, a_{i+l-1}$ as in [4, 6, 11].

It is clear that the number $N(1^8; n)$ is just the number of representations of n as the sum of eight squares. The problem of evaluating $N(1^8; n)$ was considered by earlier mathematicians. An Implicit formula for $N(1^8; n)$ is given by Jacobi [10]. Explicit formula are obtained in [5, 17]. The representation numbers $N(1^1, 4^7; n)$, $N(1^2, 4^6; n)$, $N(1^3, 4^5; n)$, $N(1^4, 4^4; n)$, $N(1^5, 4^3; n)$, $N(1^6, 4^2; n)$, $N(1^7, 4^1; n)$ are evaluated in [6]. Alaca and Williams [4] have found formulae for $N(1^2, 3^6; n)$, $N(1^4, 3^4; n)$, $N(1^6, 3^2; n)$. Alaca, Alaca and Williams [5] have determined $N(1^8; n)$, $N(1^6, 2^2; n)$, $N(1^5, 2^2, 4^1; n)$, $N(1^4, 2^4; n)$, $N(1^4, 2^2, 4^2; n)$, $N(1^4, 4^4; n)$, $N(1^3, 2^4, 4^1; n)$, $N(1^3, 2^2, 4^3; n)$, $N(1^2, 2^6; n)$, $N(1^2, 2^4, 4^2; n)$, $N(1^2, 2^2, 4^4; n)$, $N(1^1, 2^6, 4^1; n)$, $N(1^1, 2^4, 4^3; n)$, and $N(1^1, 2^2, 4^5; n)$. The author has derived formulae before for $N(1^4, 6^4; n)$, $N(2^4, 3^4; n)$ and $N(1^4, 3^4; n)$ in [11] and for $N\ (1^2, 2^2, 3^2, 6^2; n)$, $N(1^4, 3^2, 6^2; n)$, $N(1^2, 2^2, 3^4; n)$, $N(2^4, 3^2, 6^2; n)$ and $N(1^2, 2^2, 6^4; n)$ in [12].

In the present paper, we use the representation number formulae for the quaternary quadratic forms $x_1^2+x_2^2+x_3^2+x_4^2$, $x_1^2+x_2^2+2x_3^2+2x_4^2$, $x_1^2+x_2^2+3x_3^2+3x_4^2$ and some known convolution sums to derive formulae for seven octonary quadratic forms. We explicitly find formulae for any of $N(1^4, 2^2, 6^2; n)$, $N(1^2, 2^4, 3^2; n)$, $N(1^2, 3^2, 6^4; n)$, $N(1^4, 2^2, 3^2; n)$, $N(1^2, 2^4, 6^2; n)$, $N(1, 2^4, 4, 6^2; n)$ and $N(2^2, 3, 6^4, 12; n)$.

The formulae are given in terms of the function $\sigma_3(n)$ and the numbers $c_{1,6}(n)$, $c_{1,8}(n), c_{1,12}(n)$, $c_{3,4}(n), c_{1,24}(n), c_{3,8}(n)$, which are defined in next section. After some preliminaries we prove the following theorem in section 3. In our calculations we use the software Pari GP.

Theorem 1.1. *Let $n \in N$ then,*

(i)

$$\begin{aligned} N(1^4, 2^2, 6^2; n) \quad = \quad & \frac{7}{5}\sigma_3(n) - \frac{7}{5}\sigma_3(\frac{n}{2}) - \frac{27}{5}\sigma_3(\frac{n}{3}) + \frac{28}{5}\sigma_3(\frac{n}{4}) + \frac{27}{5}\sigma_3(\frac{n}{6}) - \frac{448}{5}\sigma_3(\frac{n}{8}) \\ & - \frac{108}{5}\sigma_3(\frac{n}{12}) + \frac{1728}{5}\sigma_3(\frac{n}{24}) + \frac{4}{5}c_{1,6}(n) - \frac{16}{5}c_{1,6}(\frac{n}{2}) - \frac{64}{5}c_{1,6}(\frac{n}{4}) \\ & - 2c_{1,8}(n) - \frac{22}{5}c_{1,12}(n) + \frac{61}{5}c_{1,24}(n), \end{aligned}$$

(ii)

$$\begin{aligned} N(1^2, 2^4, 3^2; n) \quad = \quad & \frac{7}{5}\sigma_3(n) - \frac{7}{5}\sigma_3(\frac{n}{2}) - \frac{27}{5}\sigma_3(\frac{n}{3}) + \frac{28}{5}\sigma_3(\frac{n}{4}) + \frac{27}{5}\sigma_3(\frac{n}{6}) - \frac{448}{5}\sigma_3(\frac{n}{8}) \\ & - \frac{108}{5}\sigma_3(\frac{n}{12}) + \frac{1728}{5}\sigma_3(\frac{n}{24}) + \frac{4}{5}c_{1,6}(n) + \frac{16}{5}c_{1,6}(\frac{n}{2}) - \frac{64}{5}c_{1,6}(\frac{n}{4}) \\ & + 2c_{1,8}(n) + \frac{8}{5}c_{3,4}(\frac{n}{2}) - \frac{1}{5}c_{3,8}(n), \end{aligned}$$

(iii)

$$\begin{aligned} N(1^2,3^2,6^4;n) &= \frac{1}{5}\sigma_3(n) - \frac{1}{5}\sigma_3(\frac{n}{2}) - \frac{21}{5}\sigma_3(\frac{n}{3}) + \frac{4}{5}\sigma_3(\frac{n}{4}) + \frac{21}{5}\sigma_3(\frac{n}{6}) - \frac{64}{5}\sigma_3(\frac{n}{8}) \\ &\quad -\frac{84}{5}\sigma_3(\frac{n}{12}) + \frac{1344}{5}\sigma_3(\frac{n}{24}) - \frac{4}{15}c_{1,6}(n) - \frac{16}{15}c_{1,6}(\frac{n}{2}) + \frac{64}{15}c_{1,6}(\frac{n}{4}) \\ &\quad -6c_{1,8}(\frac{n}{3}) - \frac{88}{15}c_{1,12}(\frac{n}{2}) + \frac{61}{15}c_{1,24}(n), \end{aligned}$$

(iv)

$$\begin{aligned} N(1^4,2^2,3^2;n) &= \frac{14}{15}\sigma_3(n) - \frac{14}{15}\sigma_3(\frac{n}{2}) - \frac{54}{5}\sigma_3(\frac{n}{3}) + \frac{28}{5}\sigma_3(\frac{n}{4}) + \frac{54}{5}\sigma_3(\frac{n}{6}) \\ &\quad -\frac{448}{5}\sigma_3(\frac{n}{8}) - \frac{108}{5}\sigma_3(\frac{n}{12}) + \frac{1728}{5}\sigma_3(\frac{n}{24}) - \frac{6}{5}c_{1,6}(n) - \frac{16}{5}c_{1,6}(\frac{n}{2}) \\ &\quad -\frac{64}{5}c_{1,6}(\frac{n}{4}) + 2c_{1,8}(n) + \frac{22}{5}c_{1,12}(n) + \frac{1}{5}c_{3,4}(n) + \frac{8}{5}c_{3,4}(\frac{n}{2}) \\ &\quad -\frac{1}{5}c_{3,8}(n), \end{aligned}$$

(v)

$$\begin{aligned} N(1^2,2^4,6^2;n) &= \frac{7}{10}\sigma_3(n) - \frac{7}{10}\sigma_3(\frac{n}{2}) - \frac{27}{10}\sigma_3(\frac{n}{3}) + \frac{28}{5}\sigma_3(\frac{n}{4}) + \frac{27}{10}\sigma_3(\frac{n}{6}) \\ &\quad -\frac{448}{5}\sigma_3(\frac{n}{8}) - \frac{108}{5}\sigma_3(\frac{n}{12}) + \frac{1728}{5}\sigma_3(\frac{n}{24}) + \frac{2}{5}c_{1,6}(n) + \frac{8}{5}c_{1,6}(\frac{n}{2}) \\ &\quad +\frac{48}{5}c_{1,6}(\frac{n}{4}) - c_{1,8}(n) - \frac{11}{5}c_{1,12}(n) - \frac{22}{5}c_{1,12}(\frac{n}{2}) - \frac{4}{5}c_{3,4}(\frac{n}{2}) \\ &\quad +\frac{61}{10}c_{1,24}(n), \end{aligned}$$

(vi)

$$\begin{aligned} N(1,2^4,4,6^2;n) &= \frac{7}{20}\sigma_3(n) - \frac{7}{20}\sigma_3(\frac{n}{2}) - \frac{27}{20}\sigma_3(\frac{n}{3}) + \frac{27}{20}\sigma_3(\frac{n}{6}) + \frac{28}{5}\sigma_3(\frac{n}{8}) \\ &\quad -\frac{448}{5}\sigma_3(\frac{n}{16}) - \frac{108}{5}\sigma_3(\frac{n}{24}) + \frac{1728}{5}\sigma_3(\frac{n}{48}) + \frac{1}{5}c_{1,6}(n) \\ &\quad +\frac{2}{5}c_{1,6}(\frac{n}{2}) - \frac{8}{5}c_{1,6}(\frac{n}{4}) - \frac{64}{5}c_{1,6}(\frac{n}{8}) - \frac{1}{2}c_{1,8}(n) + 2c_{1,8}(\frac{n}{2}) \\ &\quad -\frac{11}{10}c_{1,12}(n) - \frac{11}{5}c_{1,12}(\frac{n}{2}) + \frac{1}{5}c_{3,4}(\frac{n}{2}) + \frac{8}{5}c_{3,4}(\frac{n}{4}) \\ &\quad +\frac{61}{20}c_{1,24}(n) - \frac{1}{5}c_{3,8}(\frac{n}{2}), \end{aligned}$$

(vii)

$$\begin{aligned} N(2^2,3,6^4,12;n) &= \frac{1}{20}\sigma_3(n)-\frac{1}{20}\sigma_3(\frac{n}{2})-\frac{21}{20}\sigma_3(\frac{n}{3})+\frac{21}{20}\sigma_3(\frac{n}{6})+\frac{4}{5}\sigma_3(\frac{n}{8}) \\ &-\frac{64}{5}\sigma_3(\frac{n}{16})-\frac{84}{5}\sigma_3(\frac{n}{24})+\frac{1344}{5}\sigma_3(\frac{n}{48})-\frac{1}{15}c_{1,6}(n) \\ &-\frac{2}{15}c_{1,6}(\frac{n}{2})+\frac{8}{15}c_{1,6}(\frac{n}{4})+\frac{64}{15}c_{1,6}(\frac{n}{8})+\frac{3}{2}c_{1,8}(\frac{n}{3})-6c_{1,8}(\frac{n}{6}) \\ &-\frac{11}{15}c_{1,12}(\frac{n}{2})-\frac{88}{15}c_{1,12}(\frac{n}{4})+\frac{1}{30}c_{3,4}(n)+\frac{1}{15}c_{3,4}(\frac{n}{2}) \\ &+\frac{61}{15}c_{1,24}(\frac{n}{2})-\frac{1}{60}c_{3,8}(n). \end{aligned}$$

2. Some Known Formulae for Convolution Sums of Divisor Functions

In this section we give a short history and a list of necessary convolution sums of divisor function which will be required to find the representation numbers of the above mentioned quadratic forms. Most of the known convolutions sums are given by the authors Alaca, Alaca and Williams. To see the detailed calculations of the convolution sums see the cited articles. For $r, s, n \in \mathbb{N}$ with $r \leq s$ we define the convolution sum $W_{r,s}(n)$ by

$$W_{r,s}(n) := \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ rl+sm=n}} \sigma(l)\sigma(m).$$

The convolution sum

$$W_{1,1}(n) = \frac{5}{12}\sigma_3(n) - \frac{n}{2}\sigma(n) + \frac{1}{12}\sigma(n) \tag{2.1}$$

first appeared in a letter from Besge to Liouville [5]. It has been evaluated also in [8], [9] and [15].

The following three sums are given in [9]:

$$W_{1,2}(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(\frac{n}{2}) - \frac{n}{8}\sigma(n) - \frac{n}{4}\sigma(\frac{n}{2}) + \frac{1}{24}\sigma(n) + \frac{1}{24}\sigma(\frac{n}{2}), \tag{2.2}$$

$$W_{1,3}(n) = \frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3(\frac{n}{3}) - \frac{1}{12}n\sigma(n) - \frac{1}{4}n\sigma(\frac{n}{3}) + \frac{1}{24}\sigma(n) + \frac{1}{24}\sigma(\frac{n}{3}), \tag{2.3}$$

$$\begin{aligned} W_{1,4}(n) &= \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(\frac{n}{2}) + \frac{1}{3}\sigma(\frac{n}{4}) - \frac{n}{16}\sigma(n) - \frac{n}{4}\sigma(\frac{n}{4}) \\ &+\frac{1}{24}\sigma(n) + \frac{1}{24}\sigma(\frac{n}{4}). \end{aligned} \tag{2.4}$$

The convolution sum $W_{1,3}(n)$ was also evaluated in [13], [14], [16]. Alaca and Williams [3] have proved that

$$\begin{aligned} W_{1,6}(n) &= \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3(\frac{n}{2}) + \frac{3}{40}\sigma_3(\frac{n}{3}) + \frac{3}{10}\sigma_3(\frac{n}{6}) + (\frac{1}{24} - \frac{n}{24})\sigma(n) \\ &\quad +(\frac{1}{24} - \frac{n}{4})\sigma(\frac{n}{6}) - \frac{1}{120}c_{1,6}(n), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} W_{2,3}(n) &= \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3(\frac{n}{2}) + \frac{3}{40}\sigma_3(\frac{n}{3}) + \frac{3}{10}\sigma_3(\frac{n}{6}) + (\frac{1}{24} - \frac{n}{12})\sigma(\frac{n}{2}) \\ &\quad +(\frac{1}{24} - \frac{n}{8})\sigma(\frac{n}{3}) - \frac{1}{120}c_{1,6}(n), \end{aligned} \tag{2.6}$$

where

$$\sum_{n=1}^{\infty} c_{1,6}(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})^2(1-q^{3n})^2(1-q^{6n})^2. \tag{2.7}$$

It was shown in [18] that

$$\begin{aligned} W_{1,8}(n) &= \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3(\frac{n}{2}) + \frac{1}{16}\sigma_3(\frac{n}{4}) + \frac{1}{3}\sigma_3(\frac{n}{8}) + (\frac{1}{24} - \frac{n}{32})\sigma(n) \\ &\quad +(\frac{1}{24} - \frac{n}{4})\sigma(\frac{n}{8}) - \frac{1}{64}c_{1,8}(n), \end{aligned} \tag{2.8}$$

where

$$\sum_{n=1}^{\infty} c_{1,8}(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4. \tag{2.9}$$

Evaluation of the following two convolution sums are due to Alaca, Alaca and Williams [1].

$$\begin{aligned} W_{1,12}(n) &= \frac{1}{480}\sigma_3(n) + \frac{1}{160}\sigma_3(\frac{n}{2}) + \frac{3}{160}\sigma_3(\frac{n}{3}) + \frac{1}{30}\sigma_3(\frac{n}{4}) + \frac{9}{160}\sigma_3(\frac{n}{6}) \\ &\quad +\frac{3}{10}\sigma_3(\frac{n}{12}) + (\frac{1}{24} - \frac{n}{48})\sigma(n) + (\frac{1}{24} - \frac{n}{4})\sigma(\frac{n}{12}) - \frac{11}{480}c_{1,12}(n), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} & 11\sum_{n=1}^{\infty} c_{1,12}(n)q^n \\ = \; & 10q\prod_{n=1}^{\infty}(1-q^n)^{-1}(1-q^{2n})^2(1-q^{3n})^3(1-q^{4n})^3(1-q^{6n})^2(1-q^{12n})^{-1} \\ & +q\prod_{n=1}^{\infty}(1-q^n)^{-2}(1-q^{2n})^8(1-q^{3n})^{-2}(1-q^{4n})^{-2}(1-q^{6n})^8(1-q^{12n})^{-2} \end{aligned} \tag{2.11}$$

and

$$W_{3,4}(n) = \frac{1}{480}\sigma_3(n) + \frac{1}{160}\sigma_3(\frac{n}{2}) + \frac{3}{160}\sigma_3(\frac{n}{3}) + \frac{1}{30}\sigma_3(\frac{n}{4}) + \frac{9}{160}\sigma_3(\frac{n}{6}) + \frac{3}{10}\sigma_3(\frac{n}{12}) + (\frac{1}{24} - \frac{n}{16})\sigma(\frac{n}{3}) + (\frac{1}{24} - \frac{n}{12})\sigma(\frac{n}{4}) - \frac{1}{480}c_{3,4}(n), \quad (2.12)$$

where

$$\sum_{n=1}^{\infty} c_{3,4}(n)q^n \quad (2.13)$$
$$= 10q^2 \prod_{n=1}^{\infty}(1-q^n)^3(1-q^{2n})^2(1-q^{3n})^{-1}(1-q^{4n})^{-1}(1-q^{6n})^2(1-q^{12n})^3$$
$$+q \prod_{n=1}^{\infty}(1-q^n)^{-2}(1-q^{2n})^8(1-q^{3n})^{-2}(1-q^{4n})^{-2}(1-q^{6n})^8(1-q^{12n})^{-2}.$$

Here $11c_{1,12}(n)$ and $c_{3,4}(n)$ are integers, see [1].

Recently Alaca, Alaca and Williams [2] have shown that

$$W_{1,24}(n) = \frac{1}{1920}\sigma_3(n) + \frac{1}{640}\sigma_3(\frac{n}{2}) + \frac{3}{640}\sigma_3(\frac{n}{3}) + \frac{1}{160}\sigma_3(\frac{n}{4}) + \frac{9}{640}\sigma_3(\frac{n}{6}) + \frac{1}{30}\sigma_3(\frac{n}{8}) + \frac{9}{160}\sigma_3(\frac{n}{12}) + \frac{3}{10}\sigma_3(\frac{n}{24}) + (\frac{1}{24} - \frac{n}{96})\sigma(n) + (\frac{1}{24} - \frac{n}{4})\sigma(\frac{n}{24}) - \frac{61}{1920}c_{1,24}(n), \quad (2.14)$$

where

$$61\sum_{n=1}^{\infty} c_{1,24}(n)q^n \quad (2.15)$$
$$= 34q \prod_{n=1}^{\infty}(1+q^n)(1-q^{2n})(1-q^{3n})^3(1-q^{4n})^3(1-q^{6n})(1-q^{12n-6})$$
$$+30q \prod_{n=1}^{\infty}(1+q^n)^3(1-q^{2n})^2(1-q^{3n})(1-q^{4n})^2(1-q^{6n})^3(1-q^{12n-6})^2$$
$$-3q \prod_{n=1}^{\infty}(1-q^{2n-1})^2(1+q^{3n})^6(1-q^{4n})^2(1-q^{6n})^6(1-q^{12n-6})^6$$
$$+4q^2 \prod_{n=1}^{\infty}(1+q^n)(1+q^{2n})^2(1-q^{3n})^3(1-q^{4n})^4(1+q^{6n})(1-q^{12n})$$
$$-2q^2 \prod_{n=1}^{\infty}(1+q^n)^2(1-q^{2n})^3(1+q^{3n})^2(1-q^{4n})(1-q^{6n})^3(1-q^{12n}),$$

and

$$\begin{aligned} W_{3,8}(n) &= \frac{1}{1920}\sigma_3(n)+\frac{1}{640}\sigma_3(\frac{n}{2})+\frac{3}{640}\sigma_3(\frac{n}{3})+\frac{1}{160}\sigma_3(\frac{n}{4})+\frac{9}{640}\sigma_3(\frac{n}{6}) \\ &+\frac{1}{30}\sigma_3(\frac{n}{8})+\frac{9}{160}\sigma_3(\frac{n}{12})+\frac{3}{10}\sigma_3(\frac{n}{24})+(\frac{1}{24}-\frac{n}{32})\sigma(\frac{n}{3}) \\ &+(\frac{1}{24}-\frac{n}{12})\sigma(\frac{n}{8})-\frac{1}{1920}c_{3,8}(n), \end{aligned} \tag{2.16}$$

where

$$\sum_{n=1}^{\infty} c_{3,8}(n)q^n = \tag{2.17}$$

$$\begin{aligned} & q\prod_{n=1}^{\infty}(1-q^{2n-1})^2(1+q^{3n})^6(1-q^{4n})^2(1-q^{6n})^6(1-q^{12n-6})^6 \\ & +2q^2\prod_{n=1}^{\infty}(1-q^n)^2(1+q^{2n})^5(1+q^{3n})^6(1-q^{6n})^6(1-q^{12n-6})^3 \\ & +42q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1+q^{3n})^3(1-q^{6n})^6 \\ & -30q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})^3(1+q^{3n})^3(1-q^{4n-2})^2(1-q^{6n})^2(1-q^{12n})^2 \\ & +4q^3\prod_{n=1}^{\infty}(1+q^n)(1-q^{4n})^2(1-q^{6n-3})^3(1-q^{12n})^6 \\ & -52q^3\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q^{4n-2})^2(1-q^{12n})^6 \end{aligned}$$

It is obvious that $61c_{1,24}(n)$ and $c_{3,8}(n)$ are integers. In any formula $n \in \mathbb{N}$ and $q \in \mathbb{C}$.

3. Proof of Theorem 1.1

We just prove part (i) in details. The proofs of the remaining part are similar. We firstly consider the following three quaternary quadratic forms.

$$f_1 := x_1^2 + x_2^2 + x_3^2 + x_4^2 \tag{3.1}$$

$$f_2 := x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2 \tag{3.2}$$

$$f_3 := x_1^2 + x_2^2 + 3x_3^2 + 3x_4^2 \tag{3.3}$$

$$f_4 := x_1^2 + 2x_2^2 + 2x_3^2 + 4x_4^2 \tag{3.4}$$

For $l \in \mathbb{N}_0$ we set $r_i(l) = card\left\{(x_1, \ldots, x_4) \in \mathbb{Z}^4 : l = f_i(x_1, x_2, x_3, x_4)\right\}$. Obviously $r_i(0) = 1$ for $i \in \{1, 2, 3, 4\}$. It is known (see for example [7]) that

$$r_1(l) = 8\sigma(l) - 32\sigma(\frac{l}{4}),\ l \in \mathbb{N}, \tag{3.5}$$

$$r_2(l) = 4\sigma(l) - 4\sigma(\frac{l}{2}) + 8\sigma(\frac{l}{4}) - 32\sigma(\frac{l}{8}),\ l \in \mathbb{N}, \tag{3.6}$$

$$r_3(l) = 4\sigma(l) - 8\sigma(\frac{l}{2}) - 12\sigma(\frac{l}{3}) + 16\sigma(\frac{l}{4}) + 24\sigma(\frac{l}{6}) - 48\sigma(\frac{l}{12}),\ l \in \mathbb{N}, \tag{3.7}$$

$$r_4(l) = 2\sigma(l) - 2\sigma(\frac{l}{2}) + 8\sigma(\frac{l}{8}) - 32\sigma(\frac{l}{16}),\ l \in \mathbb{N}. \tag{3.8}$$

Proof. (i) The form $f := x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 + 2x_6^2 + 6x_7^2 + 6x_8^2$ can be obtained from the sum of the quaternary quadratic forms $f_1 := x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $f_3 := x_1^2 + x_2^2 + 3x_3^2 + 3x_4^2$. It is clear that

$$N(1^4, 2^2, 6^2; n) = \sum_{\substack{l,m \in \mathbb{N}_0 \\ l+2m=n}} r_1(l) r_3(m) = r_1(0) r_3(\frac{n}{2}) + r_1(n) r_3(0) + \sum_{\substack{l,m \in \mathbb{N} \\ l+2m=n}} r_1(l) r_3(m).$$

Thus by using the equations (3.5) and (3.7) we have

$$N(1^4,2^2,6^2;n)-(4\sigma(\frac{n}{2})-8\sigma(\frac{n}{4})-12\sigma(\frac{n}{6})+16\sigma(\frac{n}{8})+24\sigma(\frac{n}{12})-48\sigma(\frac{n}{24})$$
$$+8\sigma(n)-32\sigma(\frac{n}{4}))$$
$$=\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}(8\sigma(l)-32\sigma(\frac{l}{4}))(4\sigma(m)-8\sigma(\frac{m}{2})-12\sigma(\frac{m}{3})+16\sigma(\frac{m}{4})+24\sigma(\frac{m}{6})$$
$$-48\sigma(\frac{m}{12}))$$
$$=32\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(l)\sigma(m)-64\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(l)\sigma(\frac{m}{2})-96\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(l)\sigma(\frac{m}{3})$$
$$+128\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(l)\sigma(\frac{m}{4})+192\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(l)\sigma(\frac{m}{6})-384\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(l)\sigma(\frac{m}{12})$$
$$-128\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(\frac{l}{4})\sigma(m)-256\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(\frac{l}{4})\sigma(\frac{m}{2})+384\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(\frac{l}{4})\sigma(\frac{m}{3})$$
$$-512\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(\frac{l}{4})\sigma(\frac{m}{4})-768\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(\frac{l}{4})\sigma(\frac{m}{6})+1536\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(\frac{l}{4})\sigma(\frac{m}{12})$$
$$=32W_{1,2}(n)-64W_{1,4}(n)-96W_{1,6}(n)+128W_{1,8}(n)+192W_{1,12}(n)-384W_{1,24}(n)$$
$$-128W_{1,2}(\frac{n}{2})-256W_{1,1}(\frac{n}{4})+384W_{2,3}(\frac{n}{2})-512W_{1,2}(\frac{n}{4})-768W_{1,3}(\frac{n}{4})+1536W_{1,6}(\frac{n}{4}).$$

Using (2.1)-(2.6), (2.8), (2.10) and (2.14) and adding $4\sigma(\frac{n}{2})-8\sigma(\frac{n}{4})-12\sigma(\frac{n}{6})+16\sigma(\frac{n}{8})+24\sigma(\frac{n}{12})-48\sigma(\frac{n}{24})+8\sigma(n)-32\sigma(\frac{n}{4})$ to both sides we obtain the asserted formula.

(ii) $N(1^2,2^4,3^2;n)=\sum_{\substack{l,m\in\mathbb{N}_0\\ 2l+m=n}}r_1(l)r_3(m)$. Using equations (3.5), (3.7) and then (2.1)-(2.6), (2.8), (2.12) and (2.16) we obtain the asserted formula.

(iii) $N(1^2,3^2,6^4;n)=\sum_{\substack{l,m\in\mathbb{N}_0\\ 6l+m=n}}r_1(l)r_3(m)$. Using equations (3.5), (3.7) and then (2.1)-(2.6), (2.8), (2.10) and (2.14) we obtain the asserted formula.

(iv) $N(1^4,2^2,3^2;n)=\sum_{\substack{l,m\in\mathbb{N}_0\\ l+m=n}}r_2(l)r_3(m)$. Using equations (3.6), (3.7) and then (2.1)-(2.6), (2.8)), (2.10), (2.12) and (2.16) we obtain the asserted formula.

(v) $N(1^2,2^4,6^2;n)=\sum_{\substack{l,m\in\mathbb{N}_0\\ l+2m=n}}r_2(l)r_3(m)$. Using equations (3.6), (3.7) and then (2.1)-(2.6), (2.8), (2.10), (2.12) and (2.14) we obtain the asserted formula.

(vi) Clearly $N(1,2^4,4,6^2;n)=\sum_{\substack{l,m\in\mathbb{N}_0\\ 2l+m=n}} r_3(l)r_4(m)$. Using equations (3.7), (3.8) and then (2.1)-(2.6), (2.8), (2.10), (2.12), (2.14) and (2.16) we obtain the asserted formula.

(vii) $N(2^2,3,6^4,12;n)=\sum_{\substack{l,m\in\mathbb{N}_0\\ 2l+3m=n}} r_3(l)r_4(m)$. Using equations (3.7), (3.8) and then (2.1)-(2.6), (2.8), (2.10), (2.12), (2.14) and (2.16) we obtain the asserted formula. □

References

[1] A. Alaca, Ş. Alaca, K.S. Williams, *Evaluation of the convolution sums* $\sum_{l+12m=n}\sigma(l)\sigma(m)$ *and* $\sum_{3l+4m=n}\sigma(l)\sigma(m)$, Adv. Theoretical Appl. Math. **1** (1) (2006), 27-48.

[2] A. Alaca, Ş. Alaca, K.S. Williams, *Evaluation of the convolution sums* $\sum_{l+24m=n}\sigma(l)\sigma(m)$ *and* $\sum_{3l+8m=n}\sigma(l)\sigma(m)$, Math. J. Okayama Univ. **49** (2007), 93-111.

[3] Ş. Alaca, K.S. Williams, *Evaluation of the convolution sums* $\sum_{l+6m=n}\sigma(l)\sigma(m)$ *and* $\sum_{2l+3m=n}\sigma(l)\sigma(m)$, J. Number Theory **124** (2007), 491-510.

[4] Ş. Alaca, K.S. Williams, *The number of representation of a positive integer by certain octonary quadratic forms*, Funct. Approx. **43** (1) (2010), 45-54.

[5] A. Alaca, Ş. Alaca, K.S. Williams, *Fourteen octonary quadratic forms*, Int. J. Number Theory **6** (2010), 37-50.

[6] A. Alaca, Ş. Alaca, K.S. Williams, *Seven octonary quadratic forms*, Acta Arith. **135** (2008), 339-350.

[7] A. Alaca, Ş. Alaca, M. F. Lemire, K.S. Williams, *Nineteen quaternary quadratic forms*, Acta Arith. **130** (2007), 277-310.

[8] J.W.L. Glaisher, *On the square of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **14** (1885), 156-163.

[9] J.G. Huard, Z.M. Ou, B.K. Spearman, K.S. Williams, *Elementary Theory for the Millenium II*, edited by M.A. Bennet, B. C. Berndt, N. Boston, H. G. Diamond, A. J. H. Hildebrand, and W. Philipp, A. K. Peters, Natick, Massachusetts, 2002, pp. 229-274.

[10] C.G.J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, 1829, in Gesammelte Werke, Erster Band (Chelsea Publishing Co., New York, 1969), pp. 49–239.

[11] B. Köklüce, *The representation numbers of three octonary quadratic forms*, Int. J. Number Theory **9** (2013), 505-516.

[12] B. Köklüce, *On the number of representation of positive integers by some octonary quadratic forms*, 2nd International Eurasian Conference on Mathematical Sciences and Applications (IECMSA), Sarajevo, Bosnia and Herzegovina, August, 2013.

[13] G. Melfi, *On some modular identities*, Number Theory (K. Györy, A.Pethö, and V. Sos, eds), de Gruyter, Berlin, 1998, pp. 371-382.

[14] G. Melfi, *Some Problems in Elementary Number Theory and Modular Forms*, Ph. D. thesis, University of Pisa, 1998.

[15] K.S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), 159-184.
[16] K.S. Williams, *A cubic transformation formula for* ${}_2F(\frac{1}{2}, \frac{2}{3}; 1, z)$ *and some arithmetic convolution formulae*, Math. Proc. Cambridge Philos. Soc. **137** (2004), 519-539.
[17] K.S. Williams, *An arithmetic proof of Jacobi's eight squares theorem*, Far East J. Math. Sci. **3** (2001), 1001-1005. , Pacific J. Math. **228** (2006), 387-396.
[18] K. S. Williams, The convolution sum $\sum_{m<\frac{n}{8}} \sigma(m)\sigma(n-8m)$, Pacific J. Math, 228 (2006) 387–396.

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