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GENERALIZED QUATERNIONS AND THEIR ALGEBRAIC PROPERTIES

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Abstract. The aim of this paper is to study the generalized quaternions, $H_{\alpha\beta}$, and their basic properties. $H_{\alpha\beta}$ has a generalized inner product that allows us to identify it with four-dimensional space $E^4_{\alpha\beta}$. Also, it is shown that the set of all unit generalized quaternions with the group operation of quaternion multiplication is a Lie group of 3-dimension and its Lie algebra is found.

1. Introduction

Quaternion algebra, customarily denoted by H (in honor of William R. Hamilton [7], who enunciated this algebra for a first) recently has played a significant role in several areas of science; namely, in differential geometry, in analysis, synthesis of mechanism and machines, simulation of particle motion in molecular physics and quaternionic formulation of equation of motion in theory of relativity [1, 2]. After his discovery of quaternions, split quaternions, H', were initially introduced by James Cackle in 1849, which are also called coquaternion or para-quaternion [3]. Manifolds endowed with coquaternion structures are studied in differential geometry and superstring theory. Quaternion and split quaternion algebras both are associative and non-commutative *4*-dimensional Clifford algebras. A brief introduction of the generalized quaternions is provided in [20]. Also, this subject have investigated in algebra [22, 23]. It was pointed out that the group G of all unit quaternions with the group operation of quaternion multiplication is a Lie group of 3-dimension and its Lie algebra were worked out in [14]. Subsequently, Inoguchi [6] showed that the set of all unit split quaternions is a Lie group and found its Lie algebra. Here, we study the generalized quaternions, $H_{\alpha\beta}$, and give some of their algebraic properties. Ultimately, we aim to show that the set of all unit generalized quaternions is a Lie group. Its Lie algebra and properties of the bracket multiplication are investigated. Also, we point out that every generalized quaternion has an exponential representation and find De-Moivre's formula for it.

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2. Preliminaries

In this section, we define a new inner product and give a brief summary of real and split quaternions.

Definition 2.1. A real quaternion is defined *as*

$$q = a_0 + a_1 i + a_2 j + a_3 k$$

where a_0, a_1, a_2 and a_3 are real numbers and $1, i, j,\ k$ of q may be interpreted as the four basic vectors of Cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$\begin{aligned} i^2 &= j^2 = k^2 = ijk = -1 \\ ij &= k = -ji, \quad jk = i = -kj \end{aligned}$$

and

$$ki = j = -ik.$$

A quaternion may be defined as a pair (S_q, V_q), where $S_q = a_0 \in \mathbb{R}$ is scalar part and $V_q = a_1 i + a_2 j + a_3 k \in \mathbb{R}^3$ is the vector part of q. The quaternion product of two quaternions p and q is defined as

$$pq = S_p S_q - \langle V_p, V_q \rangle + S_p V_q + S_q V_p + V_p \wedge V_q$$

where"$\langle , \rangle$"and "$\wedge$" are the inner and vector products in $\mathbb{R}^3$, respectively. The norm of a quaternion is given by the sum of the squares of its components: $N_q = a_\circ^2 + a_1^2 + a_2^2 + a_3^2$, $N_q \in \mathbb{R}$. It can also be obtained by multiplying the quaternion by its conjugate, in either order since a quaternion and conjugated commute: $N_q = \overline{q}q = q\overline{q}$. Every non-zero quaternion has a multiplicative inverse given by its conjugate divided by its norm: $q^{-1} = \frac{\overline{q}}{N_q}$. The quaternion algebra H is a normed division algebra, meaning that for any two quaternions p and q, $N_{pq} = N_p N_q$, and the norm of every non-zero quaternion is non-zero (and positive) and therefore the multiplicative inverse exists for any non-zero quaternion. Of course, as is well known, multiplication of quaternions is not commutative, so that in general for any two quaternions p and q, $pq \neq qp$. Also, the algebra H' of split quaternions is defined as the four-dimensional vector space over $\mathbb{R}$ having a basis $\{1, i, j, k\}$ with the following properties;

$$\begin{aligned} i^2 &= -1, \quad j^2 = k^2 = +1 \\ ij &= k = -ji, \quad jk = -i = -kj \end{aligned}$$

and

$$ki = j = -ik.$$

The quaternion product of two split quaternions p and q is defined as

$$pq = S_pS_q + \langle V_p, V_q\rangle_l + S_pV_q + S_qV_p + V_p \wedge_l V_q$$

where"$\langle , \rangle_l$" and "$\wedge_l$" are Lorentzian inner and vector products, respectively. It is clear that H and H' are associative and non-commutative algebras and 1 is the identity element [13, 15, 24].

Definition 2.2. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be in $\mathbb{R}^3$. If $\alpha, \beta \in \mathbb{R}^+$, the generalized inner product of u and v is defined by

$$g(u, v) = \alpha u_1v_1 + \beta u_2v_2 + \alpha\beta u_3v_3. \qquad (1)$$

It could be written

$$g(u, v) = u^t \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix} v = u^tGv.$$

If $\alpha = \beta = 1$, then $\mathbf{E}^3_{\alpha\beta}$ is an Euclidean 3-space $\mathbf{E}^3$.

Also, if $\alpha > 0, \beta < 0$, $g(u, v)$ is called the generalized Lorentzian inner product. The vector space on $\mathbb{R}^3$ equipped with the generalized inner product, is called 3-dimensional generalized space, and is denoted by $\mathbf{E}^3_{\alpha\beta}$. The vector product in $\mathbf{E}^3_{\alpha\beta}$ is defined by

$$\begin{aligned} u \wedge v &= \begin{vmatrix} \beta i & \alpha j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \beta(u_2v_3 - u_3v_2)i + \alpha(u_3v_1 - u_1v_3)j + (u_1v_2 - u_2v_1)k, \end{aligned}$$

where $i \wedge j = k$, $j \wedge k = \beta i$ and $k \wedge i = \alpha j$ [8].

Proposition 2.1. *For $\alpha, \beta \in \mathbb{R}^+$, the inner and the vector product satisfy the following properties;*

1. $u \wedge v = -v \wedge u$,
2. $g(u \wedge v, w) = g(v \wedge w, u) = g(w \wedge u, v) = det(u, v, w)$,
3. $g(u, v \wedge w) = -g(v, u \wedge w)$,
4. $u \wedge (v \wedge w) = g(u, w)v - g(u, v)w$.

3. GENERALIZED QUATERNIONS

Definition 3.1. A generalized quaternion q is an expression of the form

$$q = a_0 + a_1 i + a_2 j + a_3 k$$

where a_0, a_1, a_2 and a_3 are real numbers and i, j, k are quaternionic units which satisfy the equalities

$$\begin{aligned} i^2 &= -\alpha, \quad j^2 = -\beta, \quad k^2 = -\alpha\beta \\ ij &= k = -ji \ , \quad jk = \beta i = -kj \end{aligned}$$

and

$$ki = \alpha j = -ik, \quad \alpha, \beta \in \mathbb{R}.$$

The set of all generalized quaternions are denoted by $H_{\alpha\beta}$. A generalized quaternion q is a sum of a scalar and a vector, called scalar part, $S_q = a_0$, and vector part $V_q = a_1 i + +a_2 j + a_3 k \in \mathbb{R}^3_{\alpha\beta}$. Therefore, $H_{\alpha\beta}$ forms a *4*-dimensional real space which contains the real axis $\mathbb{R}$ and a *3*-dimensional real linear space $E^3_{\alpha\beta}$, so that, $H_{\alpha\beta} = \mathbb{R} \oplus E^3_{\alpha\beta}$.

Special cases:

1) If $\alpha = \beta = 1$ is considered, then $H_{\alpha\beta}$ is the algebra of real quaternions H.

2) If $\alpha = 1, \beta = -1$ is considered, then $H_{\alpha\beta}$ is the algebra of split quaternions H'.
3) If $\alpha = 1, \beta = 0$ is considered, then $H_{\alpha\beta}$ is the algebra of semi quaternions H° [17].
4) If $\alpha = -1, \beta = 0$ is considered, then $H_{\alpha\beta}$ is the algebra of split semiquaternions H'°.
5) If $\alpha = 0, \beta = 0$ is considered, then $H_{\alpha\beta}$ is the algebra of $\frac{1}{4}$quaternions $H^{\circ\circ}$ (see[7, 21]).

The addition rule for generalized quaternions, $H_{\alpha\beta}$, is:

$$p + q = (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k,$$

for $p = a_0 + a_1 i + a_2 j + a_3 k$ and $q = b_0 + b_1 i + b_2 j + b_3 k$.

This rule preserves the associativity and commutativity properties of addition, and provides a consistent behavior for the subset of quaternions corresponding to real numbers, *i.e.*,

$$S_{p+q} = S_p + S_q = a_0 + b_0.$$

The product of a scalar and a generalized quaternion is defined in a straightforward manner. If c is a scalar and $q \in H_{\alpha\beta}$,

$$cq = cS_q + cV_q = (ca_0)1 + (ca_1)i + (ca_2)j + (ca_3)k.$$

The multiplication rule for generalized quaternions is defined as

$$pq \quad = \quad S_pS_q - g(V_p, V_q) + S_pV_q + S_qV_p + V_p \wedge V_q,$$

which could also be expressed as

$$pq = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Obviously, quaternion multiplication is an associative and distributive with respect to addition and subtraction, but the commutative law does not hold in general.

Corollary 3.1. *$H_{\alpha\beta}$ with addition and multiplication has all the properties of a number field expect commutativity of the multiplication. It is therefore called the skew field of quaternions.*

4. Some Properties of Generalized Quaternions

1) The Hamilton conjugate of $q = a_\circ + a_1 i + a_2 j + a_3 k = S_q + V_q$ is

$$\overline{q} = a_0 - (a_1 i + a_2 j + a_3 k) = S_q - V_q.$$

It is clear that the scalar and vector part of q denoted by $S_q = \frac{q+\overline{q}}{2}$ and $V_q = \frac{q-\overline{q}}{2}$.

2) The norm of q is defined as $N_q = |q\overline{q}| = |\overline{q}q| = |a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2|$.

Proposition 4.1. *Let $p, q \in H_{\alpha\beta}$ and $\lambda, \delta \in \mathbb{R}$. The conjugate and norm of generalized quaternions satisfies the following properties;*

i) $\overline{\overline{q}} = q$, *ii)* $\overline{pq} = \overline{q}\ \overline{p}$, *iii)* $\overline{\lambda p + \delta q} = \lambda\overline{p} + \delta\overline{q}$,

iv) $N_{pq} = N_p N_q$, *v)* $N_{\lambda q} = \lambda^2 N_q$, *vi)* $N_{\frac{p}{q}} = \frac{N_p}{N_q}$.

If $N_q = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 = 1$, then q is called a unit generalized quaternion.

3) The inverse of q is defined as $q^{-1} = \frac{\overline{q}}{N_q}$, $N_q \neq 0$, with the following properties;
i) $(pq)^{-1} = q^{-1}p^{-1}$, *ii)* $(\lambda q)^{-1} = \frac{1}{\lambda}q^{-1}$, *iii)* $N_{q^{-1}} = \frac{1}{N_q}$.

4) For $\alpha, \beta > 0$, division of a generalized quaternion p by the generalized quaternion $q(\neq 0)$, one simply has to resolve the equation

$$xq = p \quad or \quad qy = p,$$

with the respective solutions

$$\begin{aligned} x &= pq^{-1} = p\frac{\overline{q}}{N_q}, \\ y &= q^{-1}p = \frac{\overline{q}}{N_q}p, \end{aligned}$$

and the relation $N_x = N_y = \frac{N_p}{N_q}$.

If $S_q = 0$, then q is called pure generalized quaternion, or generalized vector. We also note that since

$$qp - pq = V_q \wedge V_p - V_p \wedge V_q,$$

and if p is a quaternion which commutes with every other quaternion then $V_p = 0$ and p is a real number.

Theorem 4.1. *Let p and q are two generalized quaternions, then we have the following properties;*

i) $S_{pq} = S_{qp}$, *ii)* $S_{p(qr)} = S_{(pq)r}$.

5) The scalar product of two generalized quaternions $p = S_p + V_p$ and $q = S_q + V_q$ is defined as

$$\begin{aligned} \langle p, q\rangle &= S_p S_q + g(V_p, V_q) \\ &= S_{p\overline{q}} \end{aligned}$$

The above expression defines a metric in $\mathbf{E}^4_{\alpha\beta}$. In the case $\alpha, \beta > 0$, using the scalar product we can define an angle λ between two quaternions p, q to be such;

$$\cos\lambda = \frac{S_{p\overline{q}}}{\sqrt{N_p}\sqrt{N_q}}.$$

Theorem 4.2. *The scalar product has a properties;*

1) $\langle pq_1, pq_2\rangle = N_p\langle q_1, q_2\rangle$
2) $\langle q_1 p, q_2 p\rangle = N_p\langle q_1, q_2\rangle$
3) $\langle pq_1, q_2\rangle = \langle q_1, \overline{p}q_2\rangle$
4) $\langle pq_1, q_2\rangle = \langle p, q_2\overline{q_1}\rangle$.

Proof. We proof identities *(1)* and *(3)*.

$$\begin{aligned}\langle pq_1, pq_2\rangle &= S_{(pq_1,\overline{pq_2})} = S_{(pq_1,\overline{q_2 p})}\\ &= S_{(\overline{q_2 p},pq_1)} = N_p S_{(\overline{q_2},q_1)}\\ &= N_p S_{(q_1,\overline{q_2})} = N_p\langle q_1, q_2\rangle\end{aligned}$$

and

$$\begin{aligned}\langle pq_1, q_2\rangle &= S_{(pq_1,\overline{q_2})} = S_{(q_1,\overline{q}_2 p)}\\ &= S_{(q_1,\overline{\overline{p}q_2})} = \langle q_1, \overline{p}q_2\rangle.\end{aligned}$$

□

6) The cross product of two generalized quaternion p, q is a sum of a real number and a pure generalized vectors, we defined as

$$p \times q = V_p \times V_q = -g(V_p, V_q) + V_p \wedge V_q.$$

here $p = V_p = a_1 i + a_2 j + a_3 k$ and $q = V_q = b_1 i + b_2 j + b_3 k$. This is clearly a general quaternion expect in two special cases; if $V_p \parallel V_q$, the product is a real part of generalized quaternion equal to $-g(V_p, V_q)$ and if $V_p \perp V_q$ the product is a generalized vector equal to $V_p \wedge V_q$.

7) We call generalized quaternions p and q are parallel if their vector parts $V_p = \frac{p-\overline{p}}{2}$ and $V_q = \frac{q-\overline{q}}{2}$ are parallel; *i.e.*, if $(S - \overline{S}) = 0$, where $S = V_p \wedge V_q$. Similarly, we call they are perpendicular if V_p and V_q are perpendicular; *i.e.*, if $(S + \overline{S}) = 0$.

8) Polar form: Let $\alpha, \beta > 0$, then every generalized quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ can be written in the form

$$q = r(\cos\theta + \overrightarrow{u}\sin\theta)\ , \qquad 0 \le \theta \le 2\pi$$

with

$$r = \sqrt{N_q} = \sqrt{a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2},$$

$\cos\theta = \frac{a_0}{r}$ and

$$\sin\theta = \frac{\sqrt{\alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2}}{r}.$$

The unit vector $\overrightarrow{u}$ is given by

$$\overrightarrow{u} = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{\alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2}},$$

with $\alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 \neq 0$. We can view θ as the angle between the vector $q \in H_{\alpha\beta}$ and the real axis and $\overrightarrow{u}\sin\theta$ as the projection of q onto the subspace $\mathbb{R}^3_{\alpha\beta}$ of pure quaternions. Since $\overrightarrow{u}^2 = -1$ for any $u \in S^2_{\alpha\beta}$, we have a natural generalization of Euler's formula for generalized quaternions with $\alpha, \beta > 0$,

$$\begin{aligned} e^{\overrightarrow{u}\theta} &= 1 + \overrightarrow{u}\theta - \frac{\theta^2}{2!} - \overrightarrow{u}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - ... \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ... + \overrightarrow{u}(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...) \\ &= \cos\theta + \overrightarrow{u}\sin\theta, \end{aligned}$$

for any real θ.

Theorem 4.3. (De-Moivre's formula) *Let $q = e^{\overrightarrow{u}\theta} = \cos\theta + \overrightarrow{u}\sin\theta$ be a unit generalized quaternion with positive alfa and beta, we have*

$$q^n = e^{n\overrightarrow{u}\theta} = \cos n\theta + \overrightarrow{u}\sin n\theta,$$

for every integer n.

The formula holds for all integer n since

$$\begin{aligned} q^{-1} &= \cos\theta - \overrightarrow{u}\sin\theta, \\ q^{-n} &= \cos(-n\theta) + \overrightarrow{u}\sin(-n\theta) \\ &= \cos n\theta - \overrightarrow{u}\sin n\theta. \end{aligned}$$

Example 4.1. $q_1 = \frac{1}{2} + \frac{1}{2}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}, \frac{1}{\sqrt{\alpha\beta}}) = \cos\frac{\pi}{3} + \frac{1}{\sqrt{3}}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}} \frac{1}{\sqrt{\alpha\beta}})\sin\frac{\pi}{3}$ is of order 6 and $q_2 = \frac{-1}{2} + \frac{1}{2}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}, \frac{1}{\sqrt{\alpha\beta}}) = \cos\frac{2\pi}{3} + \frac{1}{\sqrt{3}}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}} \frac{1}{\sqrt{\alpha\beta}})\sin\frac{2\pi}{3}$ is of order 3.

Note that theorem 4.3 holds for $\alpha\beta < 0$ (see [16]).

Special case: If $\alpha = \beta = 1$ is considered, then q becomes a unit real quaternion and its De-Moivre form reads [4].

Corollary 4.1. *There are uncountably many unit generalized quaternions satisfying $q^n = 1$ for every integer $n \geq 3$.*

Proof. For every $\overrightarrow{u} \in S^2_{\alpha\beta}$, the quaternion $q = \cos 2\pi/n + \overrightarrow{u}\, sin 2\pi/n$ is of order n. For $n = 1$ or $n = 2$, the generalized quaternion q is independent of $\overrightarrow{u}$. $\square$

5. Lie Group and Lie Algebra of $H_{\alpha\beta}$

Theorem 5.1. *Let α, β be positive numbers. The set G containing all of the unit generalized quaternions is a Lie group of dimension 3.*

Proof. G with multiplication action is a group. let us consider the differentiable function

$$\begin{aligned} f &: H_{\alpha\beta} \to \mathbb{R}, \\ f(q) &= a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2. \end{aligned}$$

$G = f^{-1}(1)$ is a submanifold of $H_{\alpha\beta}$, since 1 is a regular value of function f . Also, the following maps $\mu : G \times G \to G$ sending (q,p) to qp and $\zeta : G \to G$ sending q to q^{-1} are both differentiable. □

So, we put Lie group structure on unit ellipse

$$S^3_{\alpha\beta} = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0^2 + \alpha x_1^2 + \beta x_2^2 + \alpha\beta x_3^2 = 1,\ \alpha, \beta > 0\right\}$$

in four-dimensional space $\mathbf{E}^4_{\alpha\beta}$.

Theorem 5.2. *The Lie algebra $\Im$ of G is the imaginary part of $H_{\alpha\beta}$, i.e.*

$$\Im = ImH_{\alpha\beta} = \{a_1 i + a_2 j + a_3 k : a_1, a_2, a_3 \in \mathbb{R}\}.$$

Proof. Let $g(s) = a_0(s) + a_1(s)i + a_2(s)j + a_3(s)k$ be a curve on G, and let $g(0) = 1$, *i.e.,* $a_0(0) = 1$, $a_m(0) = 0$ for $m = 1, 2, 3$. By differentiation the equation

$$a_0^2(s) + \alpha a_1^2(s) + \beta a_2^2(s) + \alpha\beta a_3^2(s) = 1,$$

yields the equation

$$2a_0(s)a_0'(s) + 2\alpha a_1(s)a_1'(s) + 2\beta a_2(s)a_2'(s) + 2\alpha\beta a_3(s)a_3'(s) = 0.$$

Substituting $s = 0$, we obtain $a_0'(0) = 0$. The Lie algebra $\Im$ is constituted by vector of the form $\xi = \xi^m(\frac{\partial}{\partial a_m}) \mid_{g=1}$ where $m = 1, 2, 3$. The vector ξ is formally written in the form $\xi = \xi^1 i + \xi^2 j + \xi^3 k$. Thus $\Im = ImH_{\alpha\beta} \simeq T_G(e)$. □

Let us find the left invariant vector field X on G for which $X_{g=1} = \xi$. Let $\beta(s)$ be a curve on G such that $\beta(0) = 1$, $\beta'(0) = \xi$. Then $L_g(\beta(s)) = g\beta(s)$ is the left translation of the curve $\beta(s)$ by the unit generalized quaternion $g \in G$. Its tangent vector is $g\beta'(0) = g\xi$. In particular, denote by X_m those left invariant vector field on G for which

$$X_m \mid_{g=1} = (\frac{\partial}{\partial a_m}) \mid_{g=1},$$

where $m = 1, 2, 3$. These three vector fields are represented at the point $g = 1$, in quaternion notation, by the quaternions i, j and k.

For the components of these vector fields at the point $g = a_0 + a_1 i + a_2 j + a_3 k$, we have $(X_1)_g = g \times i$, $(X_2)_g = g \times j$, $(X_3)_g = g \times k$. The computations yield

$$\begin{aligned} X_1 &= -\alpha a_1 \frac{\partial}{\partial a_0} + a_0 \frac{\partial}{\partial a_1} + \alpha a_3 \frac{\partial}{\partial a_2} - a_2 \frac{\partial}{\partial a_3}, \\ X_2 &= -\beta a_2 \frac{\partial}{\partial a_0} - \beta a_3 \frac{\partial}{\partial a_1} + a_0 \frac{\partial}{\partial a_2} + a_1 \frac{\partial}{\partial a_3}, \\ X_3 &= -\alpha\beta a_3 \frac{\partial}{\partial a_0} + \beta a_2 \frac{\partial}{\partial a_1} - \alpha a_1 \frac{\partial}{\partial a_2} + a_0 \frac{\partial}{\partial a_3}, \end{aligned}$$

where all the partial derivatives are at the point g. Further, we obtain

$$[X_1, X_2] = 2X_3, \;\; [X_2, X_3] = 2\beta X_1, \;\; [X_3, X_1] = 2\alpha X_2.$$

If we limit ourselves to the values at the point $e = 1$, we obtain, in quaternion notation,

$$[i, j] = 2k, \;\; [j, k] = 2\beta i, \;\; [k, i] = 2\alpha j.$$

Special case:

1) If $\alpha = \beta = 1$ is considered, then Lie bracket of $\Im$ is given for real quaternions [14].

2) If $\alpha = 1, \beta = -1$ is considered, then Lie bracket of $\Im$ is given for split quaternions [6].

Definition 5.1. Let $\Im$ be a Lie algebra. For $X \in \Im$, we denote $Ad_X : \Im \to \Im$, $Y \to [X, Y]$ for all $Y \in \Im$. Let us define $K(X, Y) = Tr(Ad_X, Ad_Y)$ for all $X, Y \in \Im$. The form $K(X, Y)$ is called the *Killing bilinear* form on $\Im$ [14].

Theorem 5.3. *For every $X = x_1 i + x_2 j + x_3 k \in \Im$, the corresponding matrix Ad_X is*

$$Ad_X = \begin{bmatrix} 0 & -2\beta x_3 & 2\beta x_2 \\ 2\alpha x_3 & 0 & -2\alpha x_1 \\ -2x_2 & 2x_1 & 0 \end{bmatrix}$$

and $K(X, Y) = -8g(X, Y)$.

Proof. The above expression of Ad_X, we have

$$\begin{aligned} Ad_X(i) &= [x_1 i + x_2 j + x_3 k, i] = x_1[i, i] + x_2[j, i] + x_3[k, i] = 0 + x_2(-2k) + x_3(2\alpha j) \\ &= 0i + 2\alpha x_3 j - 2x_2 k \\ Ad_X(j) &= [x_1 i + x_2 j + x_3 k, j] = x_1[i, j] + x_2[j, j] + x_3[k, j] = x_1(2k) + 0 + x_3(-2\beta i) \\ &= -2\beta x_3 i + 0j + 2x_1 k \\ Ad_X(k) &= [x_1 i + x_2 j + x_3 k, k] = x_1[i, k] + x_2[j, k] + x_3[k, k] = x_1(-2\alpha j) + x_2(2\beta i) + 0 \\ &= 2\beta x_2 i - 2\alpha x_1 j + 0k. \end{aligned}$$

Thus, we find the matrix representation of the linear operator Ad_X as follows:

$$Ad_X = \begin{bmatrix} 0 & -2\beta x_3 & 2\beta x_2 \\ 2\alpha x_3 & 0 & -2\alpha x_1 \\ -2x_2 & 2x_1 & 0 \end{bmatrix}$$

So

$$Tr(Ad_X, Ad_Y) = -8(\alpha x_1 y_1 + \beta x_2 y_2 + \alpha\beta x_3 y_3) = -8g(X, Y).$$

□

Theorem 5.4. *The matrix corresponding to the* Killing bilinear *form for the Lie group G is $K = -8\breve{I}$,*

where $\breve{I} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}$.

Proof. By Theorem 5.3, Killing form is defined as

$$\begin{aligned} K &: T_G(e) \times T_G(e) \to T_G(e) \\ (X, Y) &\to K(X, Y) = -8g(X, Y), \end{aligned}$$

also, $T_G(e) \simeq sp\{i, j, k\}$ then we have

$$\begin{aligned} K &= \begin{bmatrix} K(i,i) & K(i,j) & K(i,k) \\ K(j,i) & K(j,j) & K(j,k) \\ K(k,i) & K(k,j) & K(k,k) \end{bmatrix} \\ &= -8\breve{I}. \end{aligned}$$

□

Theorem 5.5. *For $\alpha, \beta > 0$, the set of all unit generalized quaternions G is a compact Lie group.*

Proof. For $\alpha, \beta > 0$, we have $K(X, Y) < 0$, thus G is a compact Lie group. □

In the next work, we will introduce the quaternion rotation operator in 3-space $\mathbf{E}^3_{\alpha\beta}$ and giving the algebraic properties of Hamilton operators of generalized quaternion. In [10] we considered the homothetic motions associated with these operators in four-dimensional space $\mathbf{E}^4_{\alpha\beta}$. Dual generalized quaternions and screw motion in spatial kinematics are also under study by authors [11].

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[0]Başlık: Genelleştirilmiş Kuaterniyonlar ve Cebirsel Özellikleri
Anahtar Kelimeler: De-Moivre formülü, genellştirilmiş kuaterniyon, Lie grubu