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STABILITY AND SUPER STABILITY OF FUZZY APPROXIMATELY *-HOMOMORPHISMS

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Abstract. In this paper we introduce the concept of fuzzy Banach *-algebra. Then we study the stability and super stability of approximately *-homomorphisms in the fuzzy sense.

1. Introduction

It seems that the stability problem of functional equations had been first raised by Ulam [12]. In 1941, Hyers [3] showed that if $\delta > 0$ and if $f : E_1 \to E_2$ is a mapping between Banach spaces E_1 and E_2 with $||f(x+y)-f(x)-f(y)|| \leq \delta$ for all $x, y \in E_1$, then there exists a unique $T : E_1 \to E_2$ such that $T(x+y) = T(x)+T(y)$ with $||f(x) - T(x)|| \leq \delta$ for all $x, y \in E_1$. In 1978, a generalized solution to Ulam's problem for approximately linear mappings was given by Th. M. Rassias [10]. Suppose E_1 and E_2 are two real Banach spaces and $f : E_1 \to E_2$ is a mapping. If there exist $\delta \geq 0$ and $0 \leq p < 1$ such that $||f(x+y)-f(x)-f(y)|| \leq \delta(||x||^p+||y||^p)$ for all $x, y \in E_1$, then there is a unique additive mapping $T : E_1 \to E_2$ such that $||f(x) - T(x)|| \leq 2\delta||x||^p/|2 - 2^p|$ for every $x \in E_1$. In 1991, Gajda [1] gave a solution to this question for $p > 1$. For the case $p = 1$, Th. M. Rassias and Šemrl [11] showed that there exists a continuous real-valued function $f : \mathbb{R} \to \mathbb{R}$ such that f can not be approximated with an additive map.

Găvruta [2] generalized Rassias's result: Let G be an abelian group and X a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to X$ is a mapping satisfying

$$||f(x + y) - f(x) - f(y)|| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to X$ such that

$$||f(x) - T(x)|| \leq 1/2\tilde{\varphi}(x, x)$$

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for all $x \in G$. Recently, Park [9] applied Găvruta's result to linear functional equations in Banach modules over a C*-algebra.

B. E. Johnson [4] also investigated almost algebra *-homomorphisms between Banach *-algebras.

Fuzzy notion introduced firstly by Zadeh [13] that has been widely involved in different subjects of mathematics. Zadeh's definition of a fuzzy set characterized by a function from a nonempty set X to $[0, 1]$.

Later, in 1984 Katsaras [7] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Defining the class of approximately solutions of a given functional equation one can ask whether every mapping from this class can be somehow approximated by an exact solution of the considered equation in the fuzzy Banach *-algebra. To answer this question, we use here the definition of fuzzy normed spaces given in [7] to exhibit some reasonable notions of fuzzy approximately *-homomorphism in fuzzy normed algebras and we will prove that if A is a Banach *-algebra, then under some suitable conditions a fuzzy approximately *-homomorphism $f : A \to A$ can be approximated in a fuzzy sense by a *-homomorphism $H : A \to A$. This is applied to show that for a fuzzy approximately map $f : A \to A$ on a C*-algebra A, there exists a unique *-homomorphism $H : A \to A$ such that $f = H$.

2. Preliminaries

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

Definition 2.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,

(N1) $N(x, c) = 0$ for $c \leq 0$;
(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$;
(N5) $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty} N(x, t) = 1$;
(N6) for $x \neq 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space.

Example 2.2. Let $(X, ||.||)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} 0, & t \leq 0; \\ \frac{t}{||x||}, & 0 < t \leq ||x||; \\ 1, & t > ||x||. \end{cases}$$

is a fuzzy norm on X.

Definition 2.3. Let (X, N) be a fuzzy normed linear space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$lim_{n\to\infty}N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - lim_{n\to\infty}x_n = x$.

Definition 2.4. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a complete fuzzy normed space.

Let X be an algebra and (X, N) be complete fuzzy normed space. The pair (X, N) is said to be a fuzzy Banach algebra if for every $x, y \in X$ and $s, t \in \mathbb{R}$ we have $N(xy, st) \geq min\{N(x, s), N(y, t)\}$.

Definition 2.5. Let X be a linear space and $\varphi : X \times X \to [0, \infty)$. We say that φ is control function if we have

$\tilde{\varphi}(x, y) = \sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty,$

for all $x, y \in X$.

We give the following results proved in [8].

Theorem 2.6. *Let X be a linear space and (Y, N) be a fuzzy Banach space. Suppose that $\varphi : X \times X \to [0, \infty)$ is a control function and $f : X \to Y$ is a uniformly approximately additive function with respect to φ in the sense that*

$lim_{t\to\infty}N(f(x + y) - f(x) - f(y), t\varphi(x, y)) = 1$

uniformly on $X \times X$. Then $T(x) = N - lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ for all $x \in X$ exists and defines an additive mapping $T : X \to Y$ such that if for some $\delta > 0$, $\alpha > 0$

$N(f(x + y) - f(x) - f(y), \delta\varphi(x, y)) > \alpha,$

for all $x, y \in X$, then

$N(T(x) - f(x), \delta/2\tilde{\varphi}(x, x)) > \alpha,$

for every $x \in X$.

Corollary 2.7. *Let X be a linear space and (Y, N) be a fuzzy Banach space. Let $\varphi : X \times X \to [0, \infty)$ be a control function and $f : X \to Y$ be a uniformly approximately additive function with respect to φ in the sense that*

$lim_{t\to\infty}N(f(x + y) - f(x) - f(y), t\varphi(x, y)) = 1$

uniformly on $X \times X$. Then there is a unique additive mapping $T : X \to Y$ such that

$lim_{t\to\infty}N(T(x) - f(x), t\tilde{\varphi}(x, x)) = 1,$

uniformly on X.

Theorem 2.8. *Let X be a linear space and let (Z, N') be a fuzzy normed space. Let $\psi : X \times X \to Z$ be a function such that for some $0 < \alpha < 2$,*

$N'(\psi(2x, 2y), t) \geq N'(\alpha\psi(x, y), t)$

for all $x, y \in X$ and $t > 0$. Let (Y, N) be a fuzzy Banach space and let $f : X \to Y$ be a mapping in the sense that

$N(f(x+y)-f(x)-f(y),t) \geq N'(\psi(x,y),t)$
for each $t>0$ and $x,y \in X$. Then there exists a unique additive mapping $T: X \to Y$ such that
$N(f(x)-T(x),t) \geq N'(\frac{2\psi(x,x)}{2-\alpha},t)$,
where $x \in X$ and $t>0$.

3. Stability and super stability of fuzzy approximately *-homomorphisms on a fuzzy Banach *-algebra in uniform version

We start our work with definition of fuzzy Banach *-algebra.

Definition 3.1. A fuzzy Banach *-algebra A is a *-algebra A with a fuzzy complete N- norm N such that $N(a,t) = N(a^*,t)$ for all $a \in A$.

Throughout this paper, let A_{sa} be the set of self-adjoint elements of A and $U(A)$ the set of unitary elements in A.

Lemma 3.2. *Let X be a fuzzy normed *-algebra and $N - lim_{n\to\infty} x_n = x$. Then $N - lim_{n\to\infty} x_n^* = x^*$.*

Proof. By Definition 2.3 we have $lim_{t\to\infty} N(x_n - x, t) = 1$. So $lim_{t\to\infty} N(x_n^* - x^*, t) = lim_{t\to\infty} N((x_n - x)^*, t) = 1$. It means that $N - lim_{n\to\infty} x_n^* = x^*$. □

Theorem 3.3. *Let A be a fuzzy Banach *-algebra and let $\varphi : A \times A \to [0,\infty)$ be a control function and suppose that $f : A \to A$ is a function such that*

$$lim_{t\to\infty} N(f(\mu x + \mu y) - \mu f(x) - \mu f(y), t\varphi(x,y)) = 1, \tag{3.1}$$

uniformly on $A \times A$,

$$lim_{t\to\infty} N(f(x^*) - f(x)^*, t\varphi(x,x)) = 1, \tag{3.2}$$

uniformly on A, and

$$lim_{t\to\infty} N(f(zw) - f(z)f(w), t\varphi(z,w)) = 1, \tag{3.3}$$

*uniformly on $A \times A$ for all $\mu \in T^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, all $z,w \in A_{sa}$, and all $x,y \in A$. Then there exists a unique algebra *-homomorphism $H : A \to A$ such that*

$$lim_{t\to\infty} N(H(x) - f(x), t\tilde{\varphi}(x,x)) = 1 \tag{3.4}$$

uniformly on A.

Proof. Put $\mu = 1 \in T^1$. It follows from Theorem 2.6 and Corollary 2.7 that, there exists a unique additive mapping $H : A \to A$ such that the equality (3.4) holds. The additive mapping $H : A \to A$ is given by $H(x) = N - lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ for all $x \in A$.

By the assumption we have,
$lim_{t\to\infty} N(f(2^n \mu x) - 2\mu f(2^{n-1}x), t\varphi(2^{n-1}x, 2^{n-1}x)) = 1,$

for all $\mu \in T^1$ and all $x \in A$. We have

$$\begin{aligned} & N(\mu f(2^n x) - 2\mu f(2^{n-1}x), t\varphi(2^{n-1}x, 2^{n-1}x)) \\ = \quad & N(f(2^n x) - 2f(2^{n-1}x), |\mu|^{-1} t\varphi(2^{n-1}x, 2^{n-1}x)) \\ = \quad & N(f(2^n x) - 2f(2^{n-1}x), t\varphi(2^{n-1}x, 2^{n-1}x)), \end{aligned}$$

for all $\mu \in T^1$ and all $x \in A$. On the other hand

$$\begin{aligned} & N(f(2^n \mu x) - \mu f(2^n x), t\varphi(2^{n-1}x, 2^{n-1}x)) \\ \geq \quad & min\{N(f(2^n \mu x) - 2\mu f(2^{n-1}x), t/2\varphi(2^{n-1}x, 2^{n-1}x)), \\ & N(2\mu f(2^{n-1}x) - \mu f(2^n x), t/2\varphi(2^{n-1}x, 2^{n-1}x))\}, \end{aligned}$$

for all $\mu \in T^1$ and $x \in A$. Thus

$lim_{t\to\infty} N(f(2^n \mu x) - \mu f(2^n x),\ t\varphi(2^{n-1}x, 2^{n-1}x)) = 1.$

So

$lim_{t\to\infty} N(2^{-n} f(2^n \mu x) - 2^{-n}\mu f(2^n x),\ 2^{-n} t\varphi(2^{n-1}x, 2^{n-1}x)) = 1.$

Since $lim_{n\to\infty} 2^{-n} t\varphi(2^{n-1}x, 2^{n-1}x) = 0$, there is some $n_0 > 0$ such that $2^{-n} t\varphi(2^{n-1}x, 2^{n-1}x) < t$,

for all $n \geq n_0$ and $t > 0$. Hence

$N(2^{-n} f(2^n \mu x) - 2^{-n}\mu f(2^n x), t) \geq N(2^{-n} f(2^n \mu x) - 2^{-n}\mu f(2^n x),\ 2^{-n} t\varphi(2^{n-1}x, 2^{n-1}x)).$

Given $\varepsilon > 0$ we can find some $t_0 > 0$ such that

$N(2^{-n} f(2^n \mu x) - 2^{-n}\mu f(2^n x), 2^{-n} t\varphi(2^{n-1}x, 2^{n-1}x)) \geq 1 - \varepsilon,$

for all $x \in A$ and all $t \geq t_0$. So $N(2^{-n} f(2^n \mu x) - 2^{-n}\mu f(2^n x), t) = 1$ for all $t > 0$. Hence by items (N5) and (N2) of definition 2.1 we have

$N - lim_{n\to\infty} 2^{-n} f(2^n \mu x) = N - lim_{n\to\infty} 2^{-n}\mu f(2^n x),$

for all $\mu \in T^1$ and all $x \in A$. Hence

$H(\mu x) = N - lim_{n\to\infty} \frac{f(2^n \mu x)}{2^n} = N - lim_{n\to\infty} \frac{\mu f(2^n x)}{2^n} = \mu H(x),$

for all $\mu \in T^1$ and all $x \in A$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and let M be an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < 1/4 < 1/3$. By ([5], Theorem 1), there exist three elements $\mu_1, \mu_2, \mu_3 \in T^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. We have $H(x) = H(3.1/3x) = 3H(1/3x)$ for all $x \in A$. So $H(1/3x) = 1/3H(x)$ for all $x \in A$. Thus

$H(\lambda x) = H(\frac{M}{3} 3.\frac{\lambda}{M} x) = MH(1/3.3\frac{\lambda}{M} x) = M/3H(\mu_1 x + \mu_2 x + \mu_3 x)$

$= M/3(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) = M/3(\mu_1 + \mu_2 + \mu_3)H(x) = \frac{M}{3} 3\frac{\lambda}{M} H(x) = \lambda H(x),$

for all $x \in A$. Hence

$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y),$

for all $\zeta, \eta \in \mathbb{C}$ ($\zeta, \eta \neq 0$) and all $x, y \in A$, and $H(0x) = 0 = 0H(x)$ for all $x \in A$. So the unique additive mapping $H : A \to A$ is a $\mathbb{C}$-linear mapping.

By using (3.2) we have

$lim_{t\to\infty} N(2^{-n} f(2^n x^*) - 2^{-n} f(2^n x)^*, 2^{-n} t\varphi(x, x)) = 1.$

Since $lim_{n\to\infty} 2^{-n} t\varphi(x, x) = 0$, there is some $n_0 > 0$ such that $2^{-n} t\varphi(x, x) < t$ for all $n \geq n_0$ and $t > 0$. Hence

$N(2^{-n} f(2^n x^*) - 2^{-n} f(2^n x)^*, t) \geq N(2^{-n} f(2^n x^*) - 2^{-n} f(2^n x)^*, 2^{-n} t\varphi(x, x)).$

Given $\varepsilon > 0$ we can find some $t_0 > 0$ such that
$N(2^{-n}f(2^n x^*) - 2^{-n}f(2^n x)^*, 2^{-n}t\varphi(x,x)) \geq 1-\varepsilon$,
for all $x \in A$ and all $t \geq t_0$. So $N(2^{-n}f(2^n x^*) - 2^{-n}f(2^n x)^*, t) = 1$ for all $t > 0$. Hence by items (N5) and (N2) of Definition 2.1 we have

$$N - lim_{n\to\infty}(2^{-n}f(2^n x^*)) = N - lim_{n\to\infty}2^{-n}f(2^n x)^*. \tag{3.5}$$

By (3.5) and Lemma 3.2, we get
$H(x^*) = N-lim_{n\to\infty}\frac{f(2^n x^*)}{2^n} = N-lim_{n\to\infty}\frac{(f(2^n x))^*}{2^n} = (N-lim_{n\to\infty}\frac{f(2^n x)}{2^n})^* = H(x)^*$,
for all $x \in A$.

Now it follows from (3.3) that
$lim_{t\to\infty}N(4^{-n}f(2^{-n}z2^{-n}w) - 4^{-n}f(2^{-n}z)f(2^{-n}w), 4^{-n}t\varphi(2^{-n}z, 2^{-n}w)) = 1$.
Since $lim_{n\to\infty}4^{-n}t\varphi(2^{-n}z, 2^{-n}w) = 0$, there is some $n_0 > 0$ such that
$4^{-n}t\varphi(2^{-n}z, 2^{-n}w) < t$,
for all $n \geq n_0$ and $t > 0$. Hence

$$\begin{aligned} & N(4^{-n}f(2^{-n}z2^{-n}w) - 4^{-n}f(2^{-n}z)f(2^{-n}w), t) \\ \geq \;& N(4^{-n}f(2^{-n}z2^{-n}w) - 4^{-n}f(2^{-n}z)f(2^{-n}w), 4^{-n}t\varphi(2^{-n}z, 2^{-n}w)). \end{aligned}$$

Given $\varepsilon > 0$ we can find some $t_0 > 0$ such that
$N(4^{-n}f(2^{-n}z2^{-n}w) - 4^{-n}f(2^{-n}z)f(2^{-n}w), 4^{-n}t\varphi(2^{-n}z, 2^{-n}w)) \geq 1-\varepsilon$,
for all $x \in A$ and all $t \geq t_0$. So $N(4^{-n}f(2^{-n}z2^{-n}w) - 4^{-n}f(2^{-n}z)f(2^{-n}w), t) = 1$ for all $t > 0$. Hence by items (N5) and (N2) of definition 2.1 we have
$N - lim_{n\to\infty}4^{-n}f(2^{-n}z2^{-n}w) = N - lim_{n\to\infty}4^{-n}f(2^{-n}z)f(2^{-n}w)$,
for all $z, w \in A_{sa}$; but $\sum_{j=0}^{\infty} 4^{-j}\varphi(2^j z, 2^j w) \leq \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j z, 2^j w)$ for all $z, w \in A_{sa}$. So
$H(zw) = N-lim_{n\to\infty}\frac{f(4^n zw)}{4^n} = N-lim_{n\to\infty}\frac{f(2^n z)f(2^n w)}{2^n 2^n} = N-lim_{n\to\infty}\frac{f(2^n z)}{2^n}.N-lim_{n\to\infty}\frac{f(2^n w)}{2^n} = H(z)H(w)$,
for all $z, w \in A_{sa}$.

For elements $x, y \in A$, $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i}$ and $y = \frac{y+y^*}{2} + i\frac{y-y^*}{2i}$, where $x_1 = \frac{x+x^*}{2}$, $x_2 = \frac{x-x^*}{2i}$, $y_1 = \frac{y+y^*}{2}$ and $y_2 = \frac{y-y^*}{2i}$ are self-adjoint. Since H is $\mathbb{C}$-linear,
$H(xy) = H(x_1y_1 - x_2y_2 + i(x_1y_2 + x_2y_1)) = H(x_1y_1) - H(x_2y_2) + iH(x_1y_2) + iH(x_2y_1)$
$= H(x_1)H(y_1) - H(x_2)H(y_2) + iH(x_1)H(y_2) + iH(x_2)H(y_1)$
$= (H(x_1) + iH(x_2))(H(y_1) + iH(y_2))$
$= H(x_1 + ix_2)H(y_1 + iy_2) = H(x)H(y)$,
for all $x, y \in A$. Hence the additive mapping H is an algebra *-homomorphism satisfying the inequality (3.4), as desired.

The proof of the uniqueness property of H is similar to the proof of Corollary 2.7. □

Corollary 3.4. *Let A be a fuzzy Banach *-algebra, $\theta \geq 0$ and $q > 0$,$q \neq 1$. Suppose that $f : A \to A$ is a function such that*

$$lim_{t\to\infty} N(f(\mu x + \mu y) - \mu f(x) - \mu f(y), t\theta(||x||^q + ||y||^q)) = 1, \tag{3.6}$$

uniformly on $A \times A$,

$$lim_{t\to\infty} N(f(x^*) - f(x)^*, 2t\theta||x||^q) = 1, \tag{3.7}$$

uniformly on A, and

$$lim_{t\to\infty} N(f(zw) - f(z)f(w), t\theta(||z||^q + ||w||^q)) = 1, \tag{3.8}$$

*uniformly on $A \times A$ for all $\mu \in T^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, all $z, w \in A_{sa}$, and all $x, y \in A$. Then there exists a unique algebra *-homomorphism $H : A \to A$ such that*

$$lim_{t\to\infty} N(H(x) - f(x), \frac{2\theta t||x||^q}{|1 - 2^{q-1}|}) = 1, \tag{3.9}$$

uniformly on A.

Proof. Considering the control function $\varphi(x, y) = \theta(||x||^q + ||y||^q)$ for some $\theta > 0$, we obtain this corollary. □

In the following example we will show that Corollary 3.4 does not necessarily hold for $q = 1$.

Example 3.5. Let X be a Banach *-algebra, $x_0 \in X$ and α, β are real numbers such that $|\alpha| \geq 1 - (||x|| + ||y||)$ and $|\beta| \leq ||x|| + ||y||$ for every $x, y \in X$. Put
$f(x) = \alpha x + \beta x_0||x||,\ (x \in X)$.
Moreover for each fuzzy norm N on X, we have
$N(f(x + y) - f(x) - f(y), t(||x|| + ||y||))$
$= N(\beta x_0(||x + y|| - ||x|| - ||y||), t(||x|| + ||y||))$
$= N(\beta x_0, \frac{t(||x||+||y||)}{||x+y||-|x||-||y||}) \geq N(\beta x_0, t)\ (x, y \in X,\ t \in \mathbb{R})$.
Therefore by the item (N5) of the Definition 2.1, we get
$lim_{t\to\infty} N(f(x + y) - f(x) - f(y), t(||x|| + ||y||)) = 1$,
uniformly on $X \times X$.

Also

$$\begin{aligned} & N(f(xy)-f(x)f(y), t(||x||+||y||)) \\ = \;& N(\alpha xy+\beta x_0||xy||-(\alpha x+\beta x_0||x||)(\alpha y+\beta x_0||y||), t(||x||+||y||)) \\ = \;& N(\alpha xy+\beta x_0||xy||-\alpha^2 xy-\alpha\beta x x_0||y||-\alpha\beta x_0 y||x||-\beta^2 x_0^2||x||||y||, t(||x||+||y||)) \\ \geq \;& min\{N((1-\alpha)\alpha xy, \frac{t(||x||+||y||)}{5}), N(||xy||\beta x_0, \frac{t(||x||+||y||)}{5}), \\ & N(\beta^2 x_0^2||x||||y||, \frac{t(||x||+||y||)}{5}), N(\alpha\beta x x_0||y||, \frac{t(||x||+||y||)}{5}), \\ & N(\alpha\beta x_0 y||x||, \frac{t(||x||+||y||)}{5})\} \end{aligned}$$

where $x \in X$ and $t \in \mathbb{R}$.

Taking into account the following inequalities

$$N((1-\alpha)\alpha xy, \frac{t(||x||+||y||)}{5}) = N(\alpha xy, \frac{t(||x||+||y||)}{5|1-\alpha|}) \geq N(\alpha xy, t/5), \quad (3.10)$$

$$N(||xy||\beta x_0, \frac{t(||x||+||y||)}{5}) = N(||xy||x_0, \frac{t(||x||+||y||)}{5|\beta|}) \geq N(||xy||x_0, t/5), \quad (3.11)$$

$$N(\beta^2 x_0^2||x||||y||, \frac{t(||x||+||y||)}{5}) = N(\beta||x||||y||x_0^2, \frac{t}{5|\beta|}) \geq N(\beta||x||||y||x_0^2, \frac{t}{5}), \quad (3.12)$$

$$N(\alpha\beta x x_0||y||, \frac{t(||x||+||y||)}{5}) = N(\alpha x x_0||y||, \frac{t(||x||+||y||)}{5|\beta|}) \geq N(\alpha x x_0||y||, t/5), \quad (3.13)$$

$$N(\alpha\beta x_0 y||x||, \frac{t(||x||+||y||)}{5}) = N(\alpha x_0 y||x||, \frac{t(||x||+||y||)}{5|\beta|}) \geq N(\alpha x_0 y||x||, t/5), \quad (3.14)$$

it can be easily seen that $lim_{t\to\infty} N(f(xy)-f(x)f(y), t(||x||+||y||)) = 1$ uniformly on $X \times X$.

Also we have

$$\begin{aligned} & N(f(x^*)-f(x)^*, 2t||x||) \\ = \;& N(\alpha x^*-\alpha x^*+\beta x_0||x^*||-\beta x_0^*||x||, 2t||x||) \\ \geq \;& min\{N(\beta x_0, \frac{2t||x||}{||x^*||}), N(\beta x_0^*, \frac{2t||x||}{||x||})\}. \end{aligned}$$

So $lim_{t\to\infty} N(f(x^*)-f(x)^*, 2t||x||) = 1$ uniformly on X and therefore the conditions of Corollary 3.4 are fulfilled.

Now we suppose that there exists a unique *-homomorphism H satisfying the conditions of Corollary 3.4. By the equation

$$lim_{t\to\infty} N(f(x+y)-f(x)-f(y), t(||x||+||y||)) = 1, \tag{3.15}$$

for given $\varepsilon > 0$, we can find some $t_0 > 0$ such that
$N(f(x+y)-f(x)-f(y), t(||x||+||y||)) \geq 1-\varepsilon$,
for all $x, y \in X$ and all $t \geq t_0$. By using the simple induction on n, we shall show that

$$N(f(2^n x) - 2^n f(x), tn2^n||x||) \geq 1-\varepsilon. \tag{3.16}$$

Putting $y = x$ in (3.15), we get (3.16) for $n = 1$. Let (3.16) holds for some positive integer n. Then

$$\begin{aligned} & N(f(2^{n+1}x) - 2^{n+1}f(x), t(n+1)2^{n+1}||x||) \\ \geq\ & min\{N(f(2^{n+1}x) - 2f(2^n x), t(||2^n x|| + ||2^n x||)), \\ & N(2f(2^n x) - 2^{n+1}f(x), 2tn(||2^{n-1}x|| + ||2^{n-1}x||)) \\ \geq\ & 1-\varepsilon. \end{aligned}$$

This completes the induction argument. We observe that
$lim_{n\to\infty} N(H(x)-f(x), nt||x||) \geq 1-\varepsilon$.
Hence

$$lim_{n\to\infty} N(H(x)-f(x), nt||x||) = 1. \tag{3.17}$$

One may regard $N(x,t)$ as the truth value of the statement 'the norm of x is less than or equal to the real number t. So (3.17) is a contradiction with the non-fuzzy sense. This means that there is no such the H.

Theorem 3.6. *Let A be a C*-algebra and let $f : A \to A$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ for which there exists function $\varphi : A \times A \to [0.\infty)$ satisfying (3.1) and (3.3) such that*

$$lim_{t\to\infty} N(f(u^*) - f(u)^*, t\varphi(u,u)) = 1, \tag{3.18}$$

*for all $u \in U(A)$. Assume that $N - lim_{n\to\infty} \frac{f(2^n e)}{2^n}$ is invertible, where e is the identity of A. Then the bijective mapping f is a bijective *-homomorphism.*

Proof. By the same reasoning as in the proof of Theorem 3.3 there exists a unique $\mathbb{C}$-linear mapping $H : A \to A$ such that

$$lim_{t\to\infty} N(H(x)-f(x), t\tilde{\varphi}(x,x)) = 1, \tag{3.19}$$

for all $x \in A$. The $\mathbb{C}$-linear mapping $H : A \to A$ is given by
$H(x) = N - lim_{n\to\infty} \frac{f(2^n x)}{2^n}$,

for all $x \in A$.

By using (3.18) we have

$lim_{t\to\infty} N(2^{-n}f(2^n u^*) - 2^{-n}f(2^n u)^*, 2^{-n}t\varphi(u,u)) = 1$.

Since $lim_{n\to\infty} 2^{-n}t\varphi(u,u) = 0$, there is some $n_0 > 0$ such that $2^{-n}t\varphi(u,u) < t$ for all $n \geq n_0$ and $t > 0$. Hence

$N(2^{-n}f(2^n u^*) - 2^{-n}f(2^n u)^*, t) \geq N(2^{-n}f(2^n u^*) - 2^{-n}f(2^n u)^*, 2^{-n}t\varphi(u,u))$.

Given $\varepsilon > 0$ we can find some $t_0 > 0$ such that

$N(2^{-n}f(2^n u^*) - 2^{-n}f(2^n u)^*, 2^{-n}t\varphi(u,u)) \geq 1 - \varepsilon$,

for all $x \in A$ and all $t \geq t_0$. So $N(2^{-n}f(2^n u^*) - 2^{-n}f(2^n u)^*, t) = 1$ for all $t > 0$. Hence by items (N5) and (N2) of definition 2.1 we have

$$N - lim_{n\to\infty}(2^{-n}f(2^n u^*) = N - lim_{n\to\infty} 2^{-n}f(2^n u)^*. \tag{3.20}$$

By (3.20) and Lemma 3.2, we get

$H(u^*) = N - lim_{n\to\infty} \frac{f(2^n u^*)}{2^n} = N - lim_{n\to\infty} \frac{(f(2^n u))^*}{2^n} = (N - lim_{n\to\infty} \frac{f(2^n u)}{2^n})^* = H(u)^*$,

for all $u \in U(A)$.

Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements [6],

$H(x^*) = H(\sum_{j=1}^m \bar{\lambda_j} u_j^*) = \sum_{j=1}^m \bar{\lambda_j} H(u_j^*) = \sum_{j=1}^m \bar{\lambda_j} H(u_j)^* = (\sum_{j=1}^m \lambda_j H(u_j))^* = H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^*$,

for all $x \in A$.

Since $f(xy) = f(x)f(y)$ for all $x, y \in A$,

$$H(xy) = N - lim_{n\to\infty} \frac{f(2^n xy)}{2^n} = N - lim_{n\to\infty} \frac{f(2^n x)f(y)}{2^n} = H(x)f(y) \tag{3.21}$$

for all $x, y \in A$. By the additivity of H and (3.21),

$2^n H(xy) = H(2^n xy) = H(x(2^n y)) = H(x)f(2^n y)$,

for all $x, y \in A$. Hence

$$H(xy) = \frac{H(x)f(2^n y)}{2^n} = H(x)\frac{f(2^n y)}{2^n}, \tag{3.22}$$

for all $x, y \in A$. Taking the N-limit in (3.22) as $n \to \infty$, we obtain

$H(xy) = H(x)H(y)$,

for all $x, y \in A$. By (3.21) we have,

$$H(x) = H(ex) = H(e)f(x), \tag{3.23}$$

for all $x \in A$. Since $H(e) = N - lim_{n\to\infty} \frac{2^n e}{2^n}$ is invertible and the mapping f is bijective, the $\mathbb{C}$-linear mapping H is a bijective *-homomorphism.

Now we have,

$H(e)H(x) = H(ex) = H(x) = H(e)f(x)$,

for all $x \in A$. Since $H(e)$ is invertible, $H(x) = f(x)$ for all $x \in A$. Hence the bijective mapping f is a bijective *-homomorphism. □

4. Non-uniform type of Stability and super stability of fuzzy approximately *-homomorphisms

We are in a position to give non-uniform type of Theorems 3.3 and 3.6.

Theorem 4.1. *Let (B, N') be a fuzzy normed algebra, A a fuzzy Banach *-algebra and let $\varphi : A \times A \to B$ be a function such that for some $0 < \alpha < 2$,*
$N'(\varphi(2x, 2y), t) \geq N'(\varphi(x, y), t)$
for all $x, y \in A$ and $t > 0$. Let $f : A \to A$ be a function such that
$N(f(\mu x + \mu y) - \mu f(x) - \mu f(y), t) \geq N'(\varphi(x, y), t),$
for all $x, y \in A$,

$$N(f(x^*) - f(x)^*, t) \geq N'(\varphi(x, x), t), \tag{4.1}$$

for all $x \in A$ and

$$N(f(zw) - f(z)f(w), t) \geq N'(\varphi(z, w), t), \tag{4.2}$$

*for all $t > 0$, all $\mu \in T^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and all $z, w \in A_{sa}$. Then there exists a unique algebra *-homomorphism $H : A \to A$ such that*
$N(H(x) - f(x), t) \geq N'(\frac{2\varphi(x,x)}{2-\alpha}, t)$
for all $x \in A$ and all $t > 0$.

Proof. Theorem 2.8 shows that there exists an additive function $H : A \to A$ such that
$N(f(x) - T(x), t) \geq N'(\frac{2\varphi(x,x)}{2-\alpha}, t),$
where $x \in A$ and $t > 0$.

Put $\mu = 1 \in T^1$. The additive mapping $H : A \to A$ is given by $H(x) = N - lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ for all $x \in A$.

By assumption for each $\mu \in T^1$,
$N(f(2^n \mu x) - 2\mu f(2^{n-1}x), t) \geq N'^{n-1}x, 2^{n-1}x), t),$
for all $x \in A$. We have
$N(\mu f(2^n x) - 2\mu f(2^{n-1}x), t) = N(f(2^n x) - 2f(2^{n-1}x), |\mu|^{-1}t) = N(f(2^n x) - 2f(2^{n-1}x), t) \geq N'^{n-1}x, 2^{n-1}x), t),$
for all $\mu \in T^1$ and all $x \in A$. So

$$N(f(2^n \mu x) - \mu f(2^n x), t) \geq min\{N(f(2^n \mu x) - 2\mu f(2^{n-1}x), t/2), \tag{4.3}$$

$N(2\mu f(2^{n-1}x) - \mu f(2^n x), t/2)\} \geq N'^{n-1}x, 2^{n-1}x), t/2),$
for all $\mu \in T^1$ and all $x \in A$. Taking n to infinity in (4.3) and using the items (N2) and (N5) of Definition 2.1, we see that
$N - lim_{n\to\infty} 2^{-n} f(2^n \mu x) = N - lim_{n\to\infty} 2^{-n} \mu f(2^n x),$
for all $\mu \in T^1$ and all $x \in A$.

Now by using the similar proof of the Theorem 3.3 the unique additive mapping $H : A \to A$ is a $\mathbb{C}$-linear mapping.

By using (4.1) we have

$$N(2^{-n}f(2^n x^*) - 2^{-n}f(2^n x)^*, t) \geq N'^n x, 2^n x), 2^n t), \tag{4.4}$$

for all $x \in A$. Taking n to infinity in (4.4) and using the items (N2) and (N5) of Definition 2.1, we see that

$N - lim_{n\to\infty} 2^{-n}f(2^n x^*) = N - lim_{n\to\infty} 2^{-n}f(2^n x)^*$.

Again by using the similar proof of the Theorem 3.3 we have $H(x^*) = H(x)^*$. Now it follows from (4.2) that

$$N(4^{-n}f(2^{-n}z2^{-n}w) - 4^{-n}f(2^{-n}z)f(2^{-n}w), t) \geq N'^n z, 2^n w), 4^n t). \tag{4.5}$$

for all $z, w \in A_{sa}$. Taking n to infinity in (4.5) and using the items (N2) and (N5) of Definition 2.1, we see that

$N - lim_{n\to\infty} 4^{-n}f(2^{-n}z2^{-n}w) = N - lim_{n\to\infty} 4^{-n}f(2^{-n}z)f(2^{-n}w)$,

for all $z, w \in A_{sa}$. By the proof of Theorem 3.3, H is a *-homomorphism as desired.

To prove the uniqueness property of H, assume that H^* is another *-homomorphism satisfying $N(f(x) - H^*(x), t) \geq N'(\frac{2\varphi(x,x)}{2-\alpha}, t)$. Since both H and H^* are additive we deduce that

$N(H(a) - H^*(a), t) \geq min\{N(H(a) - n^{-1}f(na), t/2), N(n^{-1}f(na) - H^*(a), t/2)\} \geq N'(\frac{2\varphi(na,na)}{2-\alpha}, nt/2)$

for all $a \in A$ and all $t > 0$. Letting n tend to infinity we get that $H(a) = H^*(a)$ for all $a \in A$. □

Theorem 4.2. *Let A be a C^*-algebra, (B, N') a fuzzy normed algebra and let $\varphi : A \times A \to B$ be a function such that for some $0 < \alpha < 2$,*

$N'(\varphi(2x, 2y), t) \geq N'(\varphi(x, y), t)$

for all $x, y \in A$ and $t > 0$. Let $f : A \to A$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ such that

$N(f(\mu x + \mu y) - \mu f(x) - \mu f(y), t) \geq N'(\varphi(x, y), t)$,

and

$lim_{t\to\infty} N(f(u^*) - f(u)^*, t\varphi(u, u)) = 1$,

*for all $x, y \in A$ and $u \in U(A)$. Assume that $N - lim_{n\to\infty} \frac{f(2^n e)}{2^n}$ is invertible, where e is the identity of A. Then the bijective mapping f is a bijective *-homomorphism.*

Proof. As same as the proof of the Theorems 3.6 and 4.1, we can prove this Theorem. □

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[0] Başlık: Bulanık yaklaşık *-homomorfizmin kararlılığı ve süper kararlılığı
Anahtar Kelimeler: Bulanık normlu uzay, yaklaşık *-homomorfizm, kararlılık