CONVERGENCE OF CERTAIN NONLINEAR COUNTERPART OF THE BERNSTEIN OPERATORS

HARUN KARSLI AND H. ERHAN ALTIN

ABSTRACT. The present paper concerns with the nonlinear Bernstein operators NB_nf of the form

$$(NB_nf)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \ , \ 0 \le x \le 1 \ , \quad n \in \mathbb{N},$$

acting on bounded functions on an interval $[0,1]$, where $P_{n,k}$ satisfy some suitable assumptions. As a continuation of the very recent paper of the authors [13], we establish some pointwise convergence results for these type operators on the interval $[0,1]$.

1. INTRODUCTION

We consider the problem of approximating a given real-valued function f, defined on $[0,1]$, by means of a sequence of nonlinear Bernstein operators (NB_nf). Operators like positive linear, convolution, moment and sampling operators play an important role in several branches of Mathematics, for instance in reconstruction of signals and images, in Fourier analysis, operator theory, probability theory and approximation theory.

In this paper, we deal with a certain nonlinear counterpart of the Bernstein operators, considered in [13].

Let f be a function defined on the interval $[0,1]$ and let $\mathbb{N} := \{1, 2, ...\}$. The classical Bernstein operators B_nf applied to f are defined as

$$(B_nf)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \ , \ 0 \le x \le 1 \ , \quad n \in \mathbb{N}, \tag{1}$$

---

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis. These polynomials were introduced by Bernstein [7] in 1912 to give the first constructive proof of the Weierstrass approximation theorem. Some properties of the polynomials (1) can be found in Lorentz [14].

We now state a brief and technical explanation of the relation between approximation by linear and nonlinear operators. Approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in [15] and widely developed in [5] (and the references contained therein). In [15], the assumption of linearity of the singular integral operators was replaced by an assumption of a Lipschitz condition for the kernel function $K_\lambda(t, u)$ with respect to the second variable. Especially, nonlinear integral operators of type

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t - x, f(t))\, dt, \quad x \in (a, b),$$

and its special cases were studied by Bardaro-Karsli and Vinti [2], [3] and Karsli [9], [10] in some Lebesgue spaces.

For further reading, we also refer the reader to [1], [6], [11] and the very recent paper of the authors [13] as well as the monographs [5] and [8] where other kind of convergence results of linear and nonlinear operators in the Lebesgue spaces, Musielak-Orlicz spaces, BV-spaces and BV_φ-spaces have been considered.

Very recently, by using the techniques due to Musielak [15], Karsli-Tiryaki and Altin [13] introduced the following type nonlinear counterpart of the well-known Bernstein operators;

$$(NB_n f)(x) = \sum_{k=0}^n P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) , \quad 0 \le x \le 1 , \quad n \in \mathbb{N}, \tag{2}$$

acting on bounded functions f on an interval $[0, 1]$, where $P_{n,k}$ satisfy some suitable assumptions. They proved some existence and approximation theorems for the nonlinear Bernstein operators. In particular, they obtain some pointwise convergence for the nonlinear sequence of Bernstein operators (2) to some point x of f, as $n \to \infty$.

As a continuation of the very recent paper of the authors [13], we estimate the rate of pointwise convergence for the nonlinear sequence of Bernstein operators (2) to the point x, at the Lebesgue points of f, as $n \to \infty$.

An outline of the paper is as follows: The next section contains basic definitions and notations. In Section 3, the main approximation results of this study are given.

In Section 4, we give some certain results which are necessary to prove the main result. The final section, that is Section 5, concerns with the proof of the main results presented in Section 3.

2. Preliminaries

In this section, we recall the following structural assumptions according to [13], which will be fundamental in proving our convergence theorems.

Let X be the set of all bounded Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the function ψ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$.

We now introduce a sequence of functions. Let $\{P_{n,k}\}_{n \in \mathbb{N}}$ be a sequence functions $P_{n,k} : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$P_{n,k}(t, u) = p_{n,k}(t) H_n(u) \tag{3}$$

for every $t \in [0,1]$, $u \in \mathbb{R}$, where $H_n : \mathbb{R} \to \mathbb{R}$ is such that $H_n(0) = 0$ and $p_{n,k}(t)$ is the Bernstein basis.

Throughout the paper we assume that $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing and continuous function such that $\lim_{n \to \infty} \mu(n) = \infty$.

First of all we assume that the following conditions hold:

$a)$ $H_n : \mathbb{R} \to \mathbb{R}$ is such that

$$|H_n(u) - H_n(v)| \leq \psi\left(|u - v|\right), \quad \psi \in \Psi,$$

holds for every $u, v \in \mathbb{R}$, for every $n \in \mathbb{N}$. That is, H_n satisfies a $(L - \psi)$ Lipschitz condition.

$b)$ We now set

$$K_n(x, u) := \begin{cases} \sum_{k \leq nu} p_{n,k}(x) & , & 0 < u \leq 1 \\ \\ 0 & , & u = 0 \end{cases} \tag{4}$$

and

$$B_n(x) := \int_{x - x/n^{\gamma/\beta}}^{x + (1-x)/n^{\gamma/\beta}} d_t\left(K_n(x, t)\right) \quad \text{for any fixed } x \in (0, 1)$$

where $\beta > 0$, $\gamma \geq 1$ and

$$\lambda_n(x, t) := \int_0^t d_u K_n(x, u). \tag{5}$$

Similar approach and some particular examples can be found in [6], [11], [12], [13] and [16].

c) Denoting by $r_n(u) := H_n(u) - u$, $u \in \mathbb{R}$ and $n \in \mathbb{N}$. Assume that for n sufficiently large

$$\sup_u |r_n(u)| \leq \frac{1}{\mu(n)},$$

holds.

The symbol $[a]$ will denote the greatest integer not greater than a.

3. CONVERGENCE RESULTS

We will consider the following type nonlinear Bernstein operators,

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right)$$

defined for every $f \in X$ for which $NB_n f$ is well-defined, where

$$P_{n,k}(x,u) = p_{n,k}(x) H_n(u)$$

for every $x \in [0,1], u \in \mathbb{R}$.

We are now ready to establish the main results of this study:

Definition 1. A point $x_0 \in \mathbb{R}$ is called a Lebesgue point of the function f, if

$$\lim_{h \to 0+} \frac{1}{h} \int_0^h |f(x_0 + t) - f(x_0)|\, dt = 0, \tag{6}$$

holds.

Theorem 1. Let $\psi \in \Psi$ and $f \in L_1([0,1])$ be such that $\psi \circ |f| \in BV([0,1])$. Suppose that $P_{n,k}(x,u)$ satisfies condition (a), (b) and (c). Then at each point $x \in (0,1)$ for which (6) holds we have for each $\epsilon > 0$ and for sufficiently large $n \in \mathbb{N}$,

$$
\begin{aligned}
|(NB_n f)(x) - f(x)| \quad \leq \quad & \epsilon B_n^*(x)\left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1} \\
& + \frac{B_n^*(x)}{n^{\frac{\gamma}{\beta}}}\left[\bigvee_0^1 \psi(|f_x|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{x - x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|)\right] \\
& + \frac{1}{\mu(n)}
\end{aligned}
$$

where $B_n^*(x) = B_n(x)\max\left\{x^{-\beta}, (1-x)^{-\beta}\right\}$, $(\beta > 0)$.

Theorem 2. Let $\psi \in \Psi$ and $f \in L_1([0,1])$ be such that $\psi \circ |f| \in BV([0,1])$. Suppose that $P_{n,k}(x,u)$ satisfies condition (a), (b) and (c). Then at each point $x \in (0,1)$ for which (6) holds we have

$$\lim_{n \to \infty} |(NB_n f)(x) - f(x)| = 0.$$

Proof . From Theorem 1 and the definition of ψ function we reach the result, by the arbitrariness of $\epsilon > 0$.

Corollary 1 . Let $\psi \in \Psi$ and $f \in L_1([0,1])$ be such that $\psi \circ |f| \in BV([0,1])$. Suppose that $P_{n,k}(x,u)$ satisfies condition (a), (b) and (c). Then

$$\lim_{n \to \infty} |(NB_n f)(x) - f(x)| = 0$$

holds almost everywhere in $(0,1)$.

Since almost all $x \in (0,1)$ are Lebesgue points of the function f, then the assertion follows by Theorem 2.

4. Auxiliary Result

In this section we give certain results, which are necessary to prove our theorems.

Lemma 1. ([13], Lemma 2). *For all $x \in (0,1)$ and for each $n \in \mathbb{N}$, let*

$$NB_n((t-x)^\beta; x) := \int_0^1 |u-x|^\beta d_u(K_n(x,u)) \leq \frac{B_n(x)}{n^{\gamma/\beta}}, \quad (\beta > 0) \qquad (7)$$

holds, where $B_n(x)$ is as defined in Section 2. Then one has

$$\lambda_n(x,t) =: \int_0^t d_u(K_n(x,u)) \leq \frac{B_n(x)}{(x-t)^\beta n^{\gamma/\beta}}, \quad 0 \leq t < x, \qquad (8)$$

and

$$1 - \lambda_n(x,t) = \int_t^1 d_u(K_n(x,u)) \leq \frac{B_n(x)}{(t-x)^\beta n^{\gamma/\beta}}, \quad x < t < 1. \qquad (9)$$

The following lemma is the slight modification of the Lemma 1 in [4].

Lemma 2 . Let $\psi \in \Psi$. Then, if $x_0 \in \mathbb{R}$ is a Lebesgue point of the function f, we have

$$\left| \int_0^h \psi(|f(x_0+t) - f(x_0)|) dt \right| = o(|h|) \ as \ h \to 0. \qquad (10)$$

Proof . In order to prove our lemma we will show the following two statements:

$$\left| \int_0^h \psi(|f(x_0+t) - f(x_0)|) dt \right| = o(h) \ as \ h \to 0^+,$$

$$\left| \int_h^0 \psi(|f(x_0+t) - f(x_0)|) dt \right| = o(-h) \ as \ h \to 0^-.$$

Since ψ is concave, one has for $h < 0$ and $h > 0$, respectively,

$$\frac{1}{-h} \int_{h}^{0} \psi\left(\left|f\left(x_0 + t\right) - f\left(x_0\right)\right|\right) dt \leq \psi\left(\frac{1}{-h} \int_{h}^{0} \left|f\left(x_0 + t\right) - f\left(x_0\right)\right| dt\right)$$

and

$$\frac{1}{h} \int_{0}^{h} \psi\left(\left|f\left(x_0 + t\right) - f\left(x_0\right)\right|\right) dt \leq \psi\left(\frac{1}{h} \int_{0}^{h} \left|f\left(x_0 + t\right) - f\left(x_0\right)\right| dt\right).$$

Hence, by continuity of ψ and $\psi\left(0\right) = 0$, we reach the desired result.

5. Proof of the Theorems

Proof of Theorem 1. Suppose that

$$x + \delta < 1 \ , \ x - \delta > 0 \ , \tag{11}$$

for any $0 < \delta$.

Let

$$\begin{aligned}
\left|I_n\left(x\right)\right| &= \left|\left(NB_n f\right)\left(x\right) - f\left(x\right)\right| \\
&= \left|\sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f\left(x\right)\right|.
\end{aligned}$$

From (2) and using triangle inequality, we can rewrite $\left|I_n\left(x\right)\right|$ as follows:

$$\begin{aligned}
\left|I_n\left(x\right)\right| &\leq \left|\sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - \sum_{k=0}^{n} P_{n,k}\left(x, f\left(x\right)\right)\right| \\
&\quad + \left|\sum_{k=0}^{n} P_{n,k}\left(x, f\left(x\right)\right) - f\left(x\right)\right| \\
&= I_{n,1}\left(x\right) + I_{n,2}\left(x\right)
\end{aligned}$$

From (c) it is easy to see that the second term of the right-hand-side of the above inequality is less than or equal to $\frac{1}{\mu(n)}$. Indeed;

$$
\begin{aligned}
I_{n,2}(x) &= \left| \sum_{k=0}^{n} P_{n,k}(x, f(x)) - f(x) \right| \\
&= \left| \sum_{k=0}^{n} p_{n,k}(x) H_n(f(x)) - \sum_{k=0}^{n} p_{n,k}(x) f(x) \right| \\
&= |H_n(f(x)) - f(x)| \sum_{k=0}^{n} p_{k,n}(x) \\
&\leq \frac{1}{\mu(n)}
\end{aligned}
$$

holds for n sufficiently large.

As to the first term, by (a) and using Lebesgue-Stieltjes integral representation of Bernstein polynomial, we have the following inequality,

$$
\begin{aligned}
I_{n,1}(x) &\leq \sum_{k=0}^{n} \psi\left(\left| f\left(\frac{k}{n}\right) - f(x) \right| \right) p_{n,k}(x) \\
&= \int_{0}^{1} \psi(|f(t) - f(x)|) \, d_t(K_n(x,t)).
\end{aligned}
$$

According to (b), we can split the last integral in three terms as follows:

$$
\begin{aligned}
I_{n,1}(x) &\leq \left(\int_{0}^{x - x/n^{\frac{\gamma}{\beta}}} + \int_{x - x/n^{\frac{\gamma}{\beta}}}^{x + (1-x)/n^{\frac{\gamma}{\beta}}} + \int_{x + (1-x)/n^{\frac{\gamma}{\beta}}}^{1} \right) \psi(|f(t) - f(x)|) \, d_t(K_n(x,t)) \\
&= I_1(n,x) + I_2(n,x) + I_3(n,x).
\end{aligned}
$$

First, we estimate $I_2(n, x)$. We have for $t \in \left[x - x/n^{\frac{\gamma}{\beta}}, x + (1 - x)/n^{\frac{\gamma}{\beta}} \right]$

$$
\begin{aligned}
|I_2(n, x)| &= \int_{x - x/n^{\frac{\gamma}{\beta}}}^{x + (1-x)/n^{\frac{\gamma}{\beta}}} \psi\left(|f(t) - f(x)|\right) d_t\left(K_n(x, t)\right) \\
&\leq \int_{x - x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f(t) - f(x)|\right) d_t\left(K_n(x, t)\right) \\
&\quad + \int_{x}^{x + (1-x)/n^{\frac{\gamma}{\beta}}} \psi\left(|f(t) - f(x)|\right) d_t\left(K_n(x, t)\right) \\
&= I_{2,1}(n, x) + I_{2,2}(n, x).
\end{aligned}
$$

Setting

$$
F(t) := \int_{t}^{x} \psi\left(|f(y) - f(x)|\right) dy,
$$

then, according to Lemma 2, for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$
F(t) \leq \epsilon(x - t) \tag{12}
$$

for all $0 < x - t \leq \delta$.

We now fix this δ and estimate $I_{2,1}(n, x)$ and $I_{2,2}(n, x)$ respectively.

Now, we recall the Lebesgue-Stieltjes integral representation and using (5), we can write $I_{2,1}(n, x)$ as

$$
\begin{aligned}
I_{2,1}(n, x) &= \int_{x - x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f(t) - f(x)|\right) d_t K_n(x, t) \\
&= \int_{x - x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f(t) - f(x)|\right) \frac{\partial}{\partial t} \lambda_n(x, t) \, dt \\
&= \int_{x - x/n^{\frac{\gamma}{\beta}}}^{x} \frac{\partial}{\partial t} \lambda_n(x, t) \, d\left(-F(t)\right). \tag{13}
\end{aligned}
$$

Applying partial Lebesgue-Stieltjes integration (13) and using (12), we obtain,

$$
\begin{aligned}
I_{2,1}\left(n,x\right) &= F\left(x - x/n^{\frac{\gamma}{\beta}}\right) \frac{\partial}{\partial t}\left(\lambda_n\left(x, x - x/n^{\frac{\gamma}{\beta}}\right)\right) \\
&\quad + \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} F\left(t\right) \frac{\partial^2}{\partial t^2}\left(\lambda_n\left(x,t\right)\right) dt \\
&\leq \epsilon\, x/n^{\frac{\gamma}{\beta}} \frac{\partial}{\partial t}\left(\lambda_n\left(x, x - x/n^{\frac{\gamma}{\beta}}\right)\right) \\
&\quad + \epsilon \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \left(x - t\right) \frac{\partial^2}{\partial t^2}\left(\lambda_n\left(x,t\right)\right) dt.
\end{aligned}
$$

Integration by parts again gives

$$
\begin{aligned}
I_{2,1}\left(n,x\right) &= \epsilon\, x/n^{\frac{\gamma}{\beta}} \frac{\partial}{\partial t}\left(\lambda_n\left(x, x - x/n^{\frac{\gamma}{\beta}}\right)\right) \\
&\quad + \epsilon \left\{ -x/n^{\frac{\gamma}{\beta}} \frac{\partial}{\partial t}\left(\lambda_n\left(x, x - x/n^{\frac{\gamma}{\beta}}\right)\right) + \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \frac{\partial}{\partial t}\left(\lambda_n\left(x,t\right)\right) dt \right\} \\
&= \epsilon \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \frac{\partial}{\partial t}\left(\lambda_n\left(x,t\right)\right) dt \\
&= \epsilon \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} d_t\left(K_n\left(x,t\right)\right) \\
&\leq \epsilon\, B_n\left(x\right) x^{-\beta} \left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1}.
\end{aligned}
$$

We can use a similar method for $I_{2,2}\left(n,x\right)$. Then, we find the following inequality,

$$
\begin{aligned}
I_{2,2}\left(n,x\right) &\leq \epsilon \int_{x}^{x+(1-x)/n^{\frac{\gamma}{\beta}}} d_t\left(K_n\left(x,t\right)\right) \\
&\leq \epsilon\, B_n\left(x\right) \left(1 - x\right)^{-\beta} \left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1}.
\end{aligned}
$$

Next, we estimate $I_1(n, x)$. Using partial Lebesgue-Stieltjes integration, we obtain

$$
\begin{aligned}
|I_1(n, x)| &= \int_0^{x - x/n^{\frac{\gamma}{\beta}}} \psi(|f(t) - f(x)|) \, d_t(K_n(x, t)) \\
&= \int_0^{x - x/n^{\frac{\gamma}{\beta}}} \psi(|f_x(t)|) \frac{\partial}{\partial t}(\lambda_n(x, t)) \, dt \\
&= \psi\left(\left|f_x\left(x - \frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right|\right) \lambda_n\left(x, x - \frac{x}{n^{\frac{\gamma}{\beta}}}\right) \\
&\quad - \int_0^{x - x/n^{\frac{\gamma}{\beta}}} \lambda_n(x, t) \, d_t(\psi(|f_x(t)|)).
\end{aligned}
$$

Let $y = x - x/n^{\frac{\gamma}{\beta}}$. By Lemma 1, it is clear that

$$
\lambda_n(x, y) \leq B_n(x)(x - y)^{-\beta}\left(n^{\frac{\gamma}{\beta}}\right)^{\beta - 1}. \tag{14}
$$

Here we note that

$$
\begin{aligned}
\psi\left(\left|f_x\left(x - \frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right|\right) &= \left|\psi\left(\left|f_x\left(x - \frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right|\right) - \psi(|f_x(x)|)\right| \\
&\leq \bigvee_{x - x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f_x|).
\end{aligned}
$$

Using partial integration and applying (14), we obtain

$$
\begin{aligned}
|I_1(n, x)| &\leq \bigvee_{x - x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f_x|) \left|\lambda_n\left(x, x - \frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right| \\
&\quad + \int_0^{x - x/n^{\frac{\gamma}{\beta}}} \lambda_n(x, t) \, d_t\left(-\bigvee_t^{x} \psi(|f_x|)\right) \\
&\leq \bigvee_{x - x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f_x|) B_n(x) x^{-\beta}\left(n^{\frac{\gamma}{\beta}}\right)^{\beta - 1} \\
&\quad + \frac{B_n(x)}{n^{\frac{\gamma}{\beta}}} \int_0^{x - x/n^{\frac{\gamma}{\beta}}} (x - t)^{-\beta} \, d_t\left(-\bigvee_t^{x} \psi(|f_x|)\right)
\end{aligned}
$$

$$= \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f_x|\right) B_n\left(x\right) x^{-\beta} \left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1}$$

$$+ \frac{B_n\left(x\right)}{n^{\frac{\gamma}{\beta}}} \left[-x^{-\beta} n^{\gamma} \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f_x|\right) + x^{-\beta} \bigvee_{0}^{x} \psi\left(|f_x|\right) \right.$$

$$\left. + \int_{0}^{x-x/n^{\frac{\gamma}{\beta}}} \bigvee_{t}^{x} \psi\left(|f_x|\right) \frac{\beta}{(x-t)^{\beta+1}} dt \right]$$

$$= \frac{B_n\left(x\right)}{n^{\frac{\gamma}{\beta}}} \left[x^{-\beta} \bigvee_{0}^{x} \psi\left(|f_x|\right) + \int_{0}^{x-x/n^{\frac{\gamma}{\beta}}} \bigvee_{t}^{x} \psi\left(|f_x|\right) \frac{\beta}{(x-t)^{\beta+1}} dt \right].$$

Changing the variable t by $x - x/u^{1/\beta}$ in the last integral, we have

$$\int_{0}^{x-x/n^{\frac{\gamma}{\beta}}} \bigvee_{t}^{x} \psi\left(|f_x|\right) \frac{\beta}{(x-t)^{\beta+1}} dt = \frac{1}{x^{\beta}} \int_{1}^{n^{\gamma}} \bigvee_{x-x/u^{1/\beta}}^{x} \psi\left(|f_x|\right) du$$

$$\leq \frac{1}{x^{\beta}} \sum_{k=1}^{[n^{\gamma}]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi\left(|f_x|\right).$$

Consequently, we obtain

$$|I_1\left(n,x\right)| \leq \frac{B_n\left(x\right)}{n^{\frac{\gamma}{\beta}}} x^{-\beta} \left[\bigvee_{0}^{x} \psi\left(|f_x|\right) + \sum_{k=1}^{[n^{\gamma}]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi\left(|f_x|\right) \right].$$

Using a similar method, we can find

$$|I_3\left(n,x\right)| \leq \frac{B_n\left(x\right)}{n^{\frac{\gamma}{\beta}}} \left(1-x\right)^{-\beta} \left[\bigvee_{x}^{1} \psi\left(|f_x|\right) + \sum_{k=1}^{[n^{\gamma}]} \bigvee_{x}^{x+(1-x)/k^{1/\beta}} \psi\left(|f_x|\right) \right].$$

Collecting the above estimates we get the required result.

REFERENCES

[1] L. Angeloni and G. Vinti, Convergence and rate of approximation for linear integral operators in BV_{φ}-spaces in multidimensional setting, J. Math. Anal.and Appl., Vol. 349, (2009), 317-334.

[2] C. Bardaro, H. Karsli and G. Vinti, Nonlinear integral operators with homogeneous kernels: pointwise approximation theorems, Appl. Anal., Vol. 90, Nos. 3–4, March–April (2011), 463–474.

[3] C. Bardaro, H. Karsli and G. Vinti, On pointwise convergence of linear integral operators with homogeneous kernels , Integral Transforms and Special Functions, 19(6), (2008), 429-439.

[4] C. Bardaro, I. Mantellini, Pointwise convergence theorems for nonlinear Mellin convolution operators, Int. J. Pure Appl. Math. 27(4) (2006), 431-447.

[5] C. Bardaro, J. Musielak and G. Vinti , Nonlinear integral operators and applications, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 9, xii + 201 pp., 2003.

[6] C. Bardaro, S. Sciamannini, G. Vinti, Convergence in BV_φ by nonlinear Mellin-type convolution operators, Func. et Approx., 29, (2001), 17-28.

[7] S. N. Bernstein, *Demonstration du Thěoreme de Weierstrass fonděe sur le calcul des probabilitěs*, Comm. Soc. Math. Kharkow **13**, (1912/13), 1-2.

[8] P.L. Butzer and R.J. Nessel, Fourier Analysis and Approximation, V.1, Academic Press, New York, London, 1971.

[9] H. Karsli, Convergence and rate of convergence by nonlinear singular integral operators depending on two parameters, Appl. Anal. 85(6,7), (2006), 781-791.

[10] H. Karsli, On approximation properties of a class of convolution type nonlinear singular integral operators , Georgian Math. Jour., Vol. 15, No. 1, (2008), 77–86.

[11] H. Karsli, Some convergence results for nonlinear singular integral operators, Demonstratio. Math., Vol. XLVI No 4, 729-740 (2013).

[12] H. Karsli and V. Gupta, Rate of convergence by nonlinear integral operators for functions of bounded variation, Calcolo, Vol. 45, 2, (2008), 87-99.

[13] H. Karsli, I. U. Tiryaki, H. E. Altin, Some approximation properties of a certain nonlinear Bernstein operators, Filomat, 28(2014), 1295-1305.

[14] G.G. Lorentz, Bernstein Polynomials, University of Toronto Press,Toronto (1953).

[15] J. Musielak, On some approximation problems in modular spaces, In Constructive Function Theory 1981, (Proc. Int. Conf., Varna, June 1-5, 1981), pp. 455-461, Publ. House Bulgarian Acad. Sci., Sofia 1983.

[16] S.Y. Shaw, W.C. Liaw and Y.L. Lin, Rates for approximation of functions in $BV[a,b]$ and $DBV[a,b]$ by positive linear operators, Chinese J. Math. Vol 21., No: 2, (1993), 171-193.

Address: Abant Izzet Baysal University Faculty of Science and Arts Department of Mathematics 14280 Golkoy Bolu

E-mail: `karsli_h@ibu.edu.tr`

E-mail: `erhanaltin@ibu.edu.tr`