Commun.Fac.Sci.Univ.Ank.Series A1
Volume 64, Number 2, Pages 1–34 (2015)
DOI: 10.1501/Commua1_0000000730
ISSN 1303–5991

ON THE GROWTH ANALYSIS OF WRONSKIANS IN THE LIGHT OF SOME GENERALIZED GROWTH INDICATORS

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Abstract. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using m-th generalized ${}_pL^*$-order with rate p, m-th generalised ${}_pL^*$- type with rate p and m-th generalised ${}_pL^*$-weak type with rate p and wronskians generated by one of the factors where m and p are any two positive integers.

1. Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M(r,f)$ corresponding to f is defined on $|z|=r$ as follows:

$$M(r,f)=\max_{|z|=r}|f(z)| \ .$$

When f is meromorphic, $M(r,f)$ cannot be defined as f is not analytic throughout the complex plane. In this situation, one may introduce another function $T(r,f)$ known as Nevanlinna's characteristic function of f, playing the same role as maximum modulus function in the following manner:

$$T(r,f)=N(r,f)+m(r,f)\,,$$

where

$$N(r,f)=\int_0^r\frac{n(t,f)-n(0,f)}{t}dt+n(0,f)\log r$$

is the pole-counting contribution, where $n(r,f)$ is the number of poles of f, including multiplicities, for $|z|\le r$. On the other hand, the function $m(r,f)$ known as the

Received by the editors: April 09, 2015; Accepted: July 12, 2015 .
2010 *Mathematics Subject Classification.* 30D35, 30D30.
Key words and phrases. Transcendental entire function, transcendental meromorphic function, composition, growth, m-th generalised ${}_pL^*$- order with rate p, m-th generalised ${}_pL^*$- type with rate p and m-th generalised ${}_pL^*$-weak type with rate p, wronskian, slowly changing function.

proximity function is defined as

$$m(r,f)=\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}\left|f\left(re^{i\theta}\right)\right|d\theta,$$

$$\text{where } \log^{+}x=\max\left(\log x,0\right) \text{ for all } x\geqslant 0\ .$$

In addition, we denote the order and lower order of growth of f by ρ_f and λ_f respectively and they are defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}\text{ and }\lambda_f=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}\ .$$

When f is entire, one can easily verify that

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}\text{ and }\lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}$$

where $\log^{[k]}x=\log\left(\log^{[k-1]}x\right)$ for $k=1,2,3,....$ and $\log^{[0]}x=x$.

Somasundaram and Thamizharasi [12] introduced the notions of L-order and L-type for entire function where $L\equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar)\sim L(r)$ as $r\to\infty$ for every positive constant 'a'. The more generalized concept for L-order and L-type for entire and meromorphic functions are L^{*}-order and L^{*}-type. Their definitions are as follows:

Definition 1. [12] *The L^{*}-order $\rho_f^{L^{*}}$ and the L^{*}-lower order $\lambda_f^{L^{*}}$ of an entire function f are defined as*

$$\rho_f^{L^{*}}=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log\left[re^{L(r)}\right]}\text{ and }\lambda_f^{L^{*}}=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log\left[re^{L(r)}\right]},$$

When f is meromorphic, one can easily verify that

$$\rho_f^{L^{*}}=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}\text{ and }\lambda_f^{L^{*}}=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}\ .$$

Definition 2. [12] *The L^{*}-type $\sigma_f^{L^{*}}$ of an entire function f is defined as*

$$\sigma_f^{L^{*}}=\limsup_{r\to\infty}\frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^{*}}}}\ 0<\ \rho_f^{L^{*}}\ <\infty\ .$$

For meromorphic f,

$$\sigma_f^{L^{*}}=\limsup_{r\to\infty}\frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^{*}}}}\ 0<\ \rho_f^{L^{*}}\ <\infty\ .$$

In the line of Somasundaram and Thamizharasi [12] , for any two positive integers m and p, Datta and Biswas [3] introduced the following definition:

Definition 3. [3] *The m-th generalized ${}_{p}L^{*}$-order with rate p denoted by ${}_{(p)}^{(m)}\rho_{f}^{L^{*}}$ and the m-th generalized ${}_{p}L^{*}$-lower order with rate p denoted as ${}_{(p)}^{(m)}\lambda_{f}^{L^{*}}$ of an entire function f are defined in the following way:*

$$ {}_{(p)}^{(m)}\rho_{f}^{L^{*}} = \limsup_{r\to\infty}\frac{\log^{[m+1]}M(r,f)}{\log\left[r\exp^{[p]}L(r)\right]} \text{ and } {}_{(p)}^{(m)}\lambda_{f}^{L^{*}} = \liminf_{r\to\infty}\frac{\log^{[m+1]}M(r,f)}{\log\left[r\exp^{[p]}L(r)\right]}, $$

where both m and p are positive integers.

When f is meromorphic, it can be easily verified that

$$ {}_{(p)}^{(m)}\rho_{f}^{L^{*}} = \limsup_{r\to\infty}\frac{\log^{[m]}T(r,f)}{\log\left[r\exp^{[p]}L(r)\right]} \text{ and } {}_{(p)}^{(m)}\lambda_{f}^{L^{*}} = \liminf_{r\to\infty}\frac{\log^{[m]}T(r,f)}{\log\left[r\exp^{[p]}L(r)\right]}, $$

where both m and p are positive integers.

These definitions extend the generalized L^{*}-order $\rho_{f}^{[m]L^{*}}$ (respectively generalized L^{*}-lower order $\lambda_{f}^{[m]L^{*}}$) of an entire or meromorphic function f for each integer $m\geq 2$ since these correspond to the particular case $\rho_{f}^{[m]L^{*}} = {}_{(1)}^{(m)}\rho_{f}^{L^{*}}$ (respectively $\lambda_{f}^{[m]L^{*}} = {}_{(1)}^{(m)}\lambda_{f}^{L^{*}}$). Clearly ${}_{(p)}^{(1)}\rho_{f}^{L^{*}} = {}_{(p)}\rho_{f}^{L^{*}}$ (respectively ${}_{(p)}^{(1)}\lambda_{f}^{L^{*}} = {}_{(p)}\lambda_{f}^{L^{*}}$) and ${}_{(1)}^{(1)}\rho_{f}^{L^{*}} = \rho_{f}^{L^{*}}$ (respectively ${}_{(1)}^{(1)}\lambda_{f}^{L^{*}} = \lambda_{f}^{L^{*}}$).

In order to compare the relative growth of two entire or meromorphic functions having same non zero finite generalized ${}_{p}L^{*}$-order with rate p, one may introduce the definitions of generalised ${}_{p}L^{*}$-type with rate p and generalised ${}_{p}L^{*}$-lower type with rate p of entire and meromorphic functions having finite positive generalised ${}_{p}L^{*}$-order with rate p in the following manner:

Definition 4. *The m-th generalised ${}_{p}L^{*}$-type with rate p denoted by ${}_{(p)}^{(m)}\sigma_{f}^{L^{*}}$ and m-th generalised ${}_{p}L^{*}$-lower type with rate p of an entire function f denoted by ${}_{(p)}^{(m)}\overline{\sigma}_{f}^{L^{*}}$ are respectively defined as follows:*

$$ \begin{aligned} {}_{(p)}^{(m)}\sigma_{f}^{L^{*}} &= \limsup_{r\to\infty}\frac{\log^{[m]}M(r,f)}{\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}^{(m)}\rho_{f}^{L^{*}}}} \text{ and} \\ {}_{(p)}^{(m)}\overline{\sigma}_{f}^{L^{*}} &= \liminf_{r\to\infty}\frac{\log^{[m]}M(r,f)}{\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}^{(m)}\rho_{f}^{L^{*}}}},\ 0<{}_{(p)}^{(m)}\rho_{f}^{L^{*}}<\infty, \end{aligned} $$

where m and p are any two positive integers.
For meromorphic f,

$$ \begin{aligned} {}_{(p)}^{(m)}\sigma_{f}^{L^{*}} &= \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f)}{\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}^{(m)}\rho_{f}^{L^{*}}}} \text{ and} \\ {}_{(p)}^{(m)}\overline{\sigma}_{f}^{L^{*}} &= \liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f)}{\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}^{(m)}\rho_{f}^{L^{*}}}},\ 0<{}_{(p)}^{(m)}\rho_{f}^{L^{*}}<\infty, \end{aligned} $$

where both m and p are positive integers.

If $m = p = 1$, then Definition 4 becomes the classical one given in Definition 2. If $p = 1$ and m is any positive integer, we get the definition of generalized L^*-type $\sigma_f^{[m]L^*}$ (respectively generalized L^*-lower type $\overline{\sigma}_f^{[m]L^*}$) and if $m = 1$ and p is any positive integer, then ${}_{(p)}^{(1)}\sigma_f^{L^*} = {}_{(p)}\sigma_f^{L^*}$ and ${}_{(p)}^{(1)}\overline{\sigma}_f^{L^*} = {}_{(p)}\overline{\sigma}_f^{L^*}$ are respectively called as ${}_pL^*$-type with rate p and ${}_pL^*$-lower type with rate p of an entire or meromorphic function f.

Analogusly in order to determine the relative growth of two entire or meromorphic functions having same non zero finite generalized ${}_pL^*$-lower order with rate p one may introduce the definition of generalised ${}_pL^*$-weak type with rate p of entire and meromorphic functions having finite positivegeneralized ${}_pL^*$-lower order with rate p in the following way:

Definition 5. *The m-th generalised ${}_pL^*$-weak type with rate p denoted by ${}_{(p)}^{(m)}\tau_f^{L^*}$ of an entire function f is defined as follows:*

$$ {}_{(p)}^{(m)}\tau_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\lambda_f^{L^*}}} \ , \ 0 < {}_{(p)}^{(m)}\lambda_f^{L^*} < \infty, $$

where both m and p are positive integers.

Also one may define the growth indicator ${}_{(p)}^{(m)}\overline{\tau}_f^{L^*}$ of an entire function f in the following manner :

$$ {}_{(p)}^{(m)}\overline{\tau}_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\lambda_f^{L^*}}} \ , \ 0 < {}_{(p)}^{(m)}\lambda_f^{L^*} < \infty, $$

where m and p are any two positive integers.
For meromorphic f,

$$ {}_{(p)}^{(m)}\overline{\tau}_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r,f)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\lambda_f^{L^*}}} \ \text{and} $$

$$ {}_{(p)}^{(m)}\tau_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[m-1]} T(r,f)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\lambda_f^{L^*}}}, \ 0 < {}_{(p)}^{(m)}\lambda_f^{L^*} < \infty, $$

where both m and p are positive integers.

Particularly, when $p = 1$ and m is any positive integer, then ${}_{(1)}^{(m)}\tau_f^{L^*} = \tau_f^{[m]L^*}$ (respectively ${}_{(1)}^{(m)}\overline{\tau}_f^{L^*} = \overline{\tau}_f^{[m]L^*}$) and $m = 1$ and p is any positive integer, then ${}_{(p)}^{(1)}\tau_f^{L^*} = {}_{(p)}\tau_f^{L^*}$ (respectively ${}_{(p)}^{(1)}\overline{\tau}_f^{L^*} = {}_{(p)}\overline{\tau}_f^{L^*}$). Clearly ${}_{(1)}^{(1)}\tau_f^{L^*} = \tau_f^{L^*}$ (respectively ${}_{(1)}^{(1)}\overline{\tau}_f^{L^*} = \overline{\tau}_f^{L^*}$).

The following definitions are also well known:

Definition 6. *A meromorphic function $a \equiv a(z)$ is called small with respect to f if $T(r,a) = S(r,f)$ where $S(r,f) = o\{T(r,f)\}$ i.e., $\frac{S(r,f)}{T(r,f)} \to 0$ as $r \to \infty$.*

Definition 7. *Let $a_1, a_2, ...a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ...a_k; f)$, the Wronskian determinant of $a_1, a_2, ..., a_k,\ f$ i.e.,*

$$L(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1' & a_2' & . & . & . & a_k' & f' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix} .$$

Definition 8. *If $a \in \mathbb{C} \cup \{\infty\}$,the quantity*

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

is called the Nevanlinna deficiency of the value 'a'.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf.[[7], p.43]). If in particular, $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

Lakshminarasimhan [8] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [10] worked on the entire functions of L-bounded index and of non uniform L-bounded index. Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using generalised ${}_pL^*$-order with rate p, generalised ${}_pL^*$- type with rate p and generalised ${}_pL^*$-weak type with rate p and wronskians generated by one of the factors. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [7] and [13].

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [1]*If f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T(r, f\circ g) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f) .$$

Lemma 2. [2]*Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T\left(r, f \circ g\right) \geq T\left(\exp\left(r^{\mu}\right), f\right) \ .$$

Lemma 3. [9] *Let g be an entire function with $\lambda_g < \infty$ and $a_i (i = 1,\ 2,\ 3, \cdots, n;$ $n \leq \infty)$ are entire functions satisfying $T\left(r, a_i\right) = o\left\{T\left(r, g\right)\right\}$. If $\sum_{i=1}^{n} \delta\left(a_i, g\right) = 1$ then $\lim_{r\to\infty} \frac{T(r,g)}{\log M(r,g)} = \frac{1}{\pi}$.*

Lemma 4. [11] *Let f be a transcendental meromorphic function having the maximum deficiency sum . Then*

$$\lim_{r\to\infty} \frac{T\left(r, L\left(f\right)\right)}{T\left(r, f\right)} = 1 + k - k\delta\left(\infty; f\right).$$

Lemma 5. *Let f be a transcendental meromorphic function having the maximum deficiency sum and m and p are any two positive integers.Then the m-th generalized ${}_pL^*$-order with rate p (the m-th generalized ${}_pL^*$-lower order with rate p) of $L\left(f\right)$ and that of f are same.*

Proof. By Lemma 4, $\lim_{r\to\infty} \frac{\log^{[m]} T(r,L(f))}{\log^{[m]} T(r,f)}$ exists and is equal to 1 for $m \geq 1$. Now

$$\begin{aligned} {}^{(m)}_{(p)}\rho^{L^*}_{L(f)} &= \limsup_{r\to\infty} \frac{\log^{[m]} T\left(r, L(f)\right)}{\log\left[r \exp^{[p]} L\left(r\right)\right]} \\ &= \lim_{r\to\infty} \frac{\log^{[m]} T\left(r, L(f)\right)}{\log^{[m]} T\left(r, f\right)} \cdot \limsup_{r\to\infty} \frac{\log^{[m]} T\left(r, f\right)}{\log\left[r \exp^{[p]} L\left(r\right)\right]} \\ &= {}^{(m)}_{(p)}\rho^{L^*}_{f} \ . \end{aligned}$$

In a similar manner, ${}^{(m)}_{(p)}\lambda^{L^*}_{L(f)} = {}^{(m)}_{(p)}\lambda^{L^*}_{f}$.
This proves the lemma. □

Lemma 6. *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$(i)\ \ {}^{(m)}_{(p)}\sigma^{L^*}_{L(f)} = \begin{cases} \left\{1 + k - k\delta\left(\infty; f\right)\right\} \cdot \ {}^{(m)}_{(p)}\sigma^{L^*}_{f} & \text{for } \ m = 1 \\ {}^{(m)}_{(p)}\sigma^{L^*}_{f} & \text{for } m > 1 \end{cases}$$

and

$$(ii)\ \ {}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{L(f)} = \begin{cases} \left\{1 + k - k\delta\left(\infty; f\right)\right\} \cdot \ {}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f} & \text{for } \ m = 1 \\ {}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f} & \text{for } m > 1 \ . \end{cases}$$

Proof. By Lemma 4 and Lemma 5 we get that

$$\begin{aligned} {}_{(p)}\sigma_{L(f)}^{L^*} &= \limsup_{r\to\infty} \frac{T(r, L(f))}{\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_{L(f)}^{L^*}}} \\ &= \lim_{r\to\infty} \frac{T(r, L(f))}{T(r,f)} \cdot \limsup_{r\to\infty} \frac{T(r,f)}{\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_{f}^{L^*}}} \\ &= \{1 + k - k\delta(\infty; f)\} \cdot {}_{(p)}\sigma_{f}^{L^*} . \end{aligned}$$

Also for $m > 1$, $\lim\limits_{r\to\infty} \frac{\log^{[m-1]} T(r,L(f))}{\log^{[m-1]} T(r,f)}$ exists and is equal to 1 and therefore in view of Lemma 5 we obtain that

$$\begin{aligned} {}_{(p)}^{(m)}\sigma_{L(f)}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, L(f))}{\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_{f}^{L^*}}} \\ &= \lim_{r\to\infty} \frac{\log^{[m-1]} T(r, L(f))}{\log^{[m-1]} T(r,f)} \cdot \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r,f)}{\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_{f}^{L^*}}} \\ &= {}_{(p)}^{(m)}\sigma_{f}^{L^*} . \end{aligned}$$

In a similar manner,

$$\begin{aligned} {}_{(p)}^{(m)}\overline{\sigma}_{L(f)}^{L^*} &= \{1 + k - k\delta(\infty; f)\} \cdot {}_{(p)}^{(m)}\overline{\sigma}_{f}^{L^*} \text{ for } m = 1 \\ \text{and } {}_{(p)}^{(m)}\overline{\sigma}_{L(f)}^{L^*} &= {}_{(p)}^{(m)}\overline{\sigma}_{f}^{L^*} \text{ otherwise.} \end{aligned}$$

Thus the lemma follows. □

Lemma 7. *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$(i)\ {}_{(p)}^{(m)}\tau_{L(f)}^{L^*} = \begin{cases} \{1 + k - k\delta(\infty; f)\} \cdot {}_{(p)}^{(m)}\tau_{f}^{L^*} & \textit{for } m = 1 \\ {}_{(p)}^{(m)}\tau_{f}^{L^*} & \textit{for } m > 1 \end{cases}$$

and

$$(ii)\ {}_{(p)}^{(m)}\overline{\tau}_{L(f)}^{L^*} = \begin{cases} \{1 + k - k\delta(\infty; f)\} \cdot {}_{(p)}^{(m)}\overline{\tau}_{f}^{L^*} & \textit{for } m = 1 \\ {}_{(p)}^{(m)}\overline{\tau}_{f}^{L^*} & \textit{for } m > 1 . \end{cases}$$

We omit the proof of Lemma 7 because it can be carried out in the line of Lemma 6.

3. Theorems.

In this section we present the main results of the paper.

Theorem 1. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be entire such that $0<{}^{(m)}_{(p)}\lambda_f^{L^*}<\infty$ and ${}_{(p)}\sigma_g^{L^*}<\infty$ where m and p are any two positive integers. If $\exp^{[p-1]}L(M(r,g))=o\left(\left[r\exp^{[p]}L(r)\right]^{\alpha}\right)$ as $r\to\infty$ and for some positive $\alpha<{}_{(p)}\rho_g^{L^*}$, then*

$$\liminf_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{\log^{[m]}T\left(\exp\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\rho_g^{L^*}},L(f)\right)}\leq {}_{(p)}\sigma_g^{L^*}.$$

Proof. Since $T(r,g)\leq\log^{+}M(r,g)$ and by Lemma 1, we get for a sequence of values of r tending to infinity that

$$\log T(r,f\circ g)\leq\log\{1+o(1)\}+\log T(M(r,g),f)$$
$$i.e.,\ \log^{[m]}T(r,f\circ g)\leq O(1)+\log^{[m]}T(M(r,g),f)$$

$$i.e.,\ \log^{[m]}T(r,f\circ g)\leq O(1)+\left({}^{(m)}_{(p)}\lambda_f^{L^*}+\varepsilon\right)\cdot\left[\log M(r,g)+\exp^{[p-1]}L(M(r,g))\right]$$

$$i.e.,\ \log^{[m]}T(r,f\circ g)\leq O(1)+\left({}^{(m)}_{(p)}\lambda_f^{L^*}+\varepsilon\right)\cdot\left[\left({}_{(p)}\sigma_g^{L^*}+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\rho_g^{L^*}}+\exp^{[p-1]}L(M(r,g))\right].\tag{1}$$

Further in view of Lemma 5, we obtain for all sufficiently large values of r that

$$\log^{[m]}T\left(\exp\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\rho_g^{L^*}},L(f)\right)\geq$$
$$\left({}^{(m)}_{(p)}\lambda_{L(f)}^{L^*}-\varepsilon\right)\left[\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\rho_g^{L^*}}+\exp^{[p-1]}\left[L\left(\exp\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\rho_g^{L^*}}\right)\right]\right]$$

$$i.e.,\ \log^{[m]}T\left(\exp\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\rho_g^{L^*}},L(f)\right)\geq\left({}^{(m)}_{(p)}\lambda_f^{L^*}-\varepsilon\right)\cdot\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\rho_g^{L^*}}.$$

Now from (1) and above it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}, L(f)\right)} \leq \frac{O(1) + \left({}_{(p)}^{(m)}\lambda_f^{L^*} + \varepsilon\right)\cdot\left[\left({}_{(p)}\sigma_g^{L^*} + \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}} + \exp^{[p-1]} L(M(r,g))\right]}{\left({}_{(p)}^{(m)}\lambda_f^{L^*} - \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}}$$

i.e., $$\frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}, L(f)\right)} \leq \frac{O(1)}{\left({}_{(p)}^{(m)}\lambda_f^{L^*} - \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}} +$$

$$\frac{\left({}_{(p)}^{(m)}\lambda_f^{L^*} + \varepsilon\right)\cdot\left[\left({}_{(p)}\sigma_g^{L^*} + \varepsilon\right) + \frac{\exp^{[p-1]} L(M(r,g))}{\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}}\right]}{\left({}_{(p)}^{(m)}\lambda_f^{L^*} - \varepsilon\right)} . \tag{2}$$

As $\alpha < {}_{(p)}\rho_g^{L^*}$ and $\exp^{[p-1]} L(M(r,g)) = o\left(\left[r\exp^{[p]} L(r)\right]^{\alpha}\right)$ as $r\to\infty$, we obtain that

$$\lim_{r\to\infty}\frac{\exp^{[p-1]} L(M(r,g))}{\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}} = 0 . \tag{3}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (2) and (3) that

$$\liminf_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}, L(f)\right)} \leq {}_{(p)}\sigma_g^{L^*} .$$

Thus the theorem is established. □

In the line of Theorem 1, the following two theorems can be carried out and therefore their proofs are omitted:

Theorem 2. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be entire with $0 < {}_{(p)}^{(m)}\rho_f^{L^*} < \infty$ and ${}_{(p)}\sigma_g^{L^*} < \infty$ where m and p are any two positive integers. If $\exp^{[p-1]} L(M(r,g)) = o\left(\left[r\exp^{[p]} L(r)\right]^{\alpha}\right)$ as $r\to\infty$ and for some positive $\alpha < {}_{(p)}\rho_g^{L^*}$, then*

$$\liminf_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}, L(f)\right)} \leq {}_{(p)}\sigma_g^{L^*} .$$

Theorem 3. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be entire such that $0<{}_{(p)}^{(m)}\lambda_f^{L^*}\leq{}_{(p)}^{(m)}\rho_f^{L^*}<\infty$ and ${}_{(p)}\sigma_g^{L^*}<\infty$ where m and p are any two positive integers. If*

$$\exp^{[p-1]}L(M(r,g))=o\left(\left[r\exp^{[p]}L(r)\right]^{\alpha}\right)\ \text{as } r\to\infty$$

and for some positive $\alpha<{}_{(p)}\rho_g^{L^*}$, then

$$\limsup_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{\log^{[m]}T\left(\exp\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\rho_g^{L^*}},L(f)\right)}\leq\frac{{}_{(p)}^{(m)}\rho_f^{L^*}\cdot{}_{(p)}\sigma_g^{L^*}}{{}_{(p)}^{(m)}\lambda_f^{L^*}}.$$

Remark 1. *For $p=1$, Theorem 3 reduces to Theorem* 14 *of* [5].

Using the notion of ${}_pL^*$ -lower type with rate p (p is any positive integer) we may state the following theorem without its proof because it can be proved in the line of Theorem 3:

Theorem 4. *Let f be a transcendental meromorphic function having maximum deficiency sum and g be entire with $0<{}_{(p)}^{(m)}\lambda_f^{L^*}\leq{}_{(p)}^{(m)}\rho_f^{L^*}<\infty$ and ${}_{(p)}\overline{\sigma}_g^{L^*}<\infty$ where m and p are any two positive integers. If $\exp^{[p-1]}L(M(r,g))=o\left(\left[r\exp^{[p]}L(r)\right]^{\alpha}\right)$ as $r\to\infty$ and for some positive $\alpha<{}_{(p)}\rho_g^{L^*}$, then*

$$\liminf_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{\log^{[m]}T\left(\exp\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\rho_g^{L^*}},L(f)\right)}\leq\frac{{}_{(p)}^{(m)}\rho_f^{L^*}\cdot{}_{(p)}\overline{\sigma}_g^{L^*}}{{}_{(p)}^{(m)}\lambda_f^{L^*}}.$$

Now we state the following three theorems without their proofs as those can be carried out in the line of Theorem 1, Theorem 2 and Theorem 3 repectively.

Theorem 5. *Let f be meromorphic and g be transcendental entire with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, ${}_{(p)}\lambda_g^{L^*}>0$, ${}_{(p)}^{(m)}\lambda_f^{L^*}<\infty$ and ${}_{(p)}\sigma_g^{L^*}<\infty$ where m and p are any two positive integers. If $\exp^{[p-1]}L(M(r,g))=o\left(\left[r\exp^{[p]}L(r)\right]^{\alpha}\right)$ as $r\to\infty$ and for some positive $\alpha<{}_{(p)}\rho_g^{L^*}$, then*

$$\liminf_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{\log^{[m]}T\left(\exp\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\rho_g^{L^*}},L(g)\right)}\leq\frac{{}_{(p)}^{(m)}\lambda_f^{L^*}\cdot{}_{(p)}\sigma_g^{L^*}}{{}_{(p)}\lambda_g^{L^*}}.$$

Theorem 6. *Let f be meromorphic and g be transcendental entire having the maximum deficiency sum such that ${}_{(p)}\rho_g^{L^*}>0$, ${}_{(p)}^{(m)}\rho_f^{L^*}<\infty$ and ${}_{(p)}\sigma_g^{L^*}<\infty$ where m and p are any two positive integers. If $\exp^{[p-1]}L(M(r,g))=o\left(\left[r\exp^{[p]}L(r)\right]^{\alpha}\right)$*

as $r \to \infty$ *and for some positive* $\alpha < {}_{(p)}\rho_g^{L^*}$, *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}, L(g)\right)} \leq \frac{{}_{(p)}^{(m)}\rho_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{{}_{(p)}\rho_g^{L^*}} .$$

Theorem 7. *Let* f *be meromorphic and* g *be transcendental entire with* $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$, ${}_{(p)}\lambda_g^{L^*} > 0$, ${}_{(p)}^{(m)}\rho_f^{L^*} < \infty$ *and* ${}_{(p)}\sigma_g^{L^*} < \infty$ *where* m *and* p *are any two positive integers. If* $\exp^{[p-1]} L(M(r,g)) = o\left(\left[r\exp^{[p]} L(r)\right]^{\alpha}\right)$ *as* $r \to \infty$ *and for some positive* $\alpha < {}_{(p)}\rho_g^{L^*}$, *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}, L(g)\right)} \leq \frac{{}_{(p)}^{(m)}\rho_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{{}_{(p)}\lambda_g^{L^*}} .$$

Remark 2. *Theorem 7 improves Theorem* 15 *of Datta et. al.* { *cf.* [5]}.

Theorem 8. *Let* f *be meromorphic and* g *be transcendental entire having the maximum deficiency sum such that* ${}_{(p)}\lambda_g^{L^*} > 0$, ${}_{(p)}^{(m)}\rho_f^{L^*} < \infty$ *and* ${}_{(p)}\overline{\sigma}_g^{L^*} < \infty$ *where* m *and* p *are any two positive integers. If* $\exp^{[p-1]} L(M(r,g)) = o\left(\left[r\exp^{[p]} L(r)\right]^{\alpha}\right)$ *as* $r \to \infty$ *and for some positive* $\alpha < {}_{(p)}\rho_g^{L^*}$, *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}, L(g)\right)} \leq \frac{{}_{(p)}^{(m)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\sigma}_g^{L^*}}{{}_{(p)}\lambda_g^{L^*}} .$$

We omit the proof of Theorem 8 as it can easily be established in the line of Theorem 4.

Further we state the following two theorems which are based on ${}_{p}L^*$ -weak type with rate p (p is any positive integer):

Theorem 9. *Let* f *be a transcendental meromorphic function having the maximum deficiency sum and* g *be entire with* $0 < {}_{(p)}^{(m)}\lambda_f^{L^*} \leq {}_{(p)}^{(m)}\rho_f^{L^*} < \infty$ *and* ${}_{(p)}\tau_g^{L^*} < \infty$ *where* m *and* p *are any two positive integers. If*

$$\exp^{[p-1]} L(M(r,g)) = o\left(\left[r\exp^{[p]} L(r)\right]^{\alpha}\right) \text{ as } r \to \infty$$

and for some positive $\alpha < {}_{(p)}\lambda_g^{L^*}$, *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\lambda_g^{L^*}}, L(f)\right)} \leq \frac{{}_{(p)}^{(m)}\rho_f^{L^*} \cdot {}_{(p)}\tau_g^{L^*}}{{}_{(p)}^{(m)}\lambda_f^{L^*}} .$$

Theorem 10. *Let* f *be meromorphic and* g *be transcendental entire having the maximum deficiency sum such that* ${}_{(p)}\lambda_g^{L^*} > 0$, ${}_{(p)}^{(m)}\rho_f^{L^*} < \infty$ *and* ${}_{[p]}\tau_g^{L^*} < \infty$ *where* m

and p are any two positive integers. If $\exp^{[p-1]} L(M(r,g)) = o\left(\left[r\exp^{[p]} L(r)\right]^{\alpha}\right)$ *as $r \to \infty$ and for some positive* $\alpha < {}_{(p)}\lambda_g^{L^*}$, *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\lambda_g^{L^*}}, L(g)\right)} \leq \frac{{}_{(p)}^{(m)}\rho_f^{L^*} \cdot {}_{(p)}\tau_g^{L^*}}{{}_{(p)}\lambda_g^{L^*}} .$$

The proofs of the above two theorems can be carried out in the line of Theorem 4 and Theorem 8 respectively and therefore their proofs are omitted.

Using the concept of the growth indicator ${}_{[p]}\overline{\tau}_g^{L^*}$ (where p is any positive integer) of an entire function g, we may state the subsequent six theorems without their proofs since those can be carried out in the line of Theorem 1, Theorem 2, Theorem 3, Theorem 5, Theorem 6 and Theorem 7 respectively.

Theorem 11. *Let f be a transcendental meromorphic function with* $\sum_{a\neq\infty}\delta(a;f) + \delta(\infty;f) = 2$ *and g be entire such that* $0 < {}_{(p)}^{(m)}\lambda_f^{L^*} < \infty$ *and* ${}_{(p)}\overline{\tau}_g^{L^*} < \infty$ *where m and p are any two positive integers. If* $\exp^{[p-1]} L(M(r,g)) = o\left(\left[r\exp^{[p]} L(r)\right]^{\alpha}\right)$ *as $r \to \infty$ and for some positive* $\alpha < {}_{(p)}\lambda_g^{L^*}$, *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\lambda_g^{L^*}}, L(f)\right)} \leq {}_{(p)}\overline{\tau}_g^{L^*} .$$

Theorem 12. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be entire with* $0 < {}_{(p)}^{(m)}\rho_f^{L^*} < \infty$ *and* ${}_{(p)}\overline{\tau}_g^{L^*} < \infty$ *where m and p are any two positive integers. If* $\exp^{[p-1]} L(M(r,g)) = o\left(\left[r\exp^{[p]} L(r)\right]^{\alpha}\right)$ *as $r \to \infty$ and for some positive* $\alpha < {}_{(p)}\lambda_g^{L^*}$, *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\lambda_g^{L^*}}, L(f)\right)} \leq {}_{(p)}\overline{\tau}_g^{L^*} .$$

Theorem 13. *Let f be a transcendental meromorphic function with* $\sum_{a\neq\infty}\delta(a;f) + \delta(\infty;f) = 2$ *and g be entire such that* $0 < {}_{(p)}^{(m)}\lambda_f^{L^*} \leq {}_{(p)}^{(m)}\rho_f^{L^*} < \infty$ *and* ${}_{(p)}\overline{\tau}_g^{L^*} < \infty$ *where m and p are any two positive integers. If*

$$\exp^{[p-1]} L(M(r,g)) = o\left(\left[r\exp^{[p]} L(r)\right]^{\alpha}\right) \text{ as } r\to\infty$$

and for some positive $\alpha < {}_{(p)}\lambda_g^{L^*}$, *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m]} T\left(\exp\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\lambda_g^{L^*}}, L(f)\right)} \leq \frac{{}_{(p)}^{(m)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{{}_{(p)}^{(m)}\lambda_f^{L^*}} .$$

Theorem 14. *Let f be meromorphic and g be transcendental entire with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, ${}_{(p)}\lambda_g^{L^*}>0$, ${}_{(p)}^{(m)}\lambda_f^{L^*}<\infty$ and ${}_{(p)}\overline{\tau}_g^{L^*}<\infty$ where m and p are any two positive integers. If $\exp^{[p-1]}L(M(r,g))=o\left(\left[r\exp^{[p]}L(r)\right]^{\alpha}\right)$ as $r\to\infty$ and for some positive $\alpha<{}_{(p)}\lambda_g^{L^*}$, then*

$$\liminf_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{\log^{[m]}T\left(\exp\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\lambda_g^{L^*}},L(g)\right)}\leq\frac{{}_{(p)}^{(m)}\lambda_f^{L^*}\cdot{}_{(p)}\overline{\tau}_g^{L^*}}{{}_{(p)}\lambda_g^{L^*}}\ .$$

Theorem 15. *Let f be meromorphic and g be transcendental entire having the maximum deficiency sum such that ${}_{(p)}\rho_g^{L^*}>0$, ${}_{(p)}^{(m)}\rho_f^{L^*}<\infty$ and ${}_{(p)}\overline{\tau}_g^{L^*}<\infty$ where m and p are any two positive integers. If $\exp^{[p-1]}L(M(r,g))=o\left(\left[r\exp^{[p]}L(r)\right]^{\alpha}\right)$ as $r\to\infty$ and for some positive $\alpha<{}_{(p)}\lambda_g^{L^*}$, then*

$$\liminf_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{\log^{[m]}T\left(\exp\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\lambda_g^{L^*}},L(g)\right)}\leq\frac{{}_{(p)}^{(m)}\rho_f^{L^*}\cdot{}_{(p)}\overline{\tau}_g^{L^*}}{{}_{(p)}\rho_g^{L^*}}\ .$$

Theorem 16. *Let f be meromorphic and g be transcendental entire with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, ${}_{(p)}\lambda_g^{L^*}>0$, ${}_{(p)}^{(m)}\rho_f^{L^*}<\infty$ and ${}_{(p)}\overline{\tau}_g^{L^*}<\infty$ where m and p are any two positive integers. If $\exp^{[p-1]}L(M(r,g))=o\left(\left[r\exp^{[p]}L(r)\right]^{\alpha}\right)$ as $r\to\infty$ and for some positive $\alpha<{}_{(p)}\lambda_g^{L^*}$, then*

$$\limsup_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{\log^{[m]}T\left(\exp\left[r\exp^{[p]}L(r)\right]^{{}_{(p)}\lambda_g^{L^*}},L(g)\right)}\leq\frac{{}_{(p)}^{(m)}\rho_f^{L^*}\cdot{}_{(p)}\overline{\tau}_g^{L^*}}{{}_{(p)}\lambda_g^{L^*}}\ .$$

Theorem 17. *Let f be transcendental meromorphic function having the maximum deficiency sum such that $0<{}_{(p)}^{(m)}\rho_f^{L^*}<\rho_g$ and ${}_{(p)}^{(m)}\sigma_f^{L^*}>0$ where m and p are any two positive integers and g be an entire function. If $\exp^{[p-1]}L\left(\exp\left(r\exp^{[p]}L(r)\right)^{\alpha}\right)=o\left(\left[r\exp^{[p]}L(r)\right]^{\alpha}\right)$ as $r\to\infty$ for any $\alpha>0$, then*

$$\limsup_{r\to\infty}\frac{\log^{[m]}T\left(r\exp^{[p]}L(r),f\circ g\right)}{\log^{[m-1]}T(r,L(f))}\geq\begin{cases}\frac{{}_{(p)}\lambda_f^{L^*}}{\{1+k-k\delta(\infty;f)\}\cdot{}_{(p)}\sigma_f^{L^*}} & \text{for } m=1\\ \frac{{}_{(p)}^{(m)}\lambda_f^{L^*}}{{}_{(p)}^{(m)}\sigma_f^{L^*}} & \text{for } m>1\ .\end{cases}$$

Proof. From Definition 4 and any arbitrary $\varepsilon\,(>0)$, we obtain for all sufficiently large values of r that

$$\log^{[m-1]} T\left(r, L(f)\right) \leq \left({}_{(p)}^{(m)}\sigma_{L(f)}^{L^*} + \varepsilon\right)\left[r\exp^{[p]} L\left(r\right)\right]^{{}_{(p)}^{(m)}\rho_{L(f)}^{L^*}} . \tag{4}$$

Now in view of Lemma 5 and Lemma 6, it follows from (4) for all sufficiently large values of r that

$$\begin{aligned}&\log^{[m-1]} T\left(r, L(f)\right)\\ \leq\ & \begin{cases}\left(\{1+k-k\delta\left(\infty;f\right)\}\cdot\ {}_{(p)}^{(m)}\sigma_f^{L^*}+\varepsilon\right)\left[r\exp^{[p]} L\left(r\right)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}} & \text{for } m=1\\ \left({}_{(p)}^{(m)}\sigma_f^{L^*}+\varepsilon\right)\left[r\exp^{[p]} L\left(r\right)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}} & \text{for } m>1\ .\end{cases}\end{aligned} \tag{5}$$

As $0 < {}_{(p)}^{(m)}\rho_f^{L^*} < \rho_g$, we obtain in view of Lemma 2 for a sequence of values of r tending to infinity that

$$\log^{[m]} T\left(r\exp^{[p]} L\left(r\right), f\circ g\right) \geq \log^{[m]} T\left(\exp\left(r\exp^{[p]} L\left(r\right)\right)^{{}_{(p)}^{(m)}\rho_f^{L^*}}, f\right)$$

$$i.e.,\ \log^{[m]} T\left(r\exp^{[p]} L\left(r\right), f\circ g\right) \geq$$
$$\left({}_{(p)}^{(m)}\lambda_f^{L^*}-\varepsilon\right)\left[\left[r\exp^{[p]} L\left(r\right)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}} + \exp^{[p-1]} L\left(\exp\left(r\exp^{[p]} L\left(r\right)\right)^{{}_{(p)}^{(m)}\rho_f^{L^*}}\right)\right] .$$

Therefore from (5) and above, it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[m]} T\left(r\exp^{[p]} L\left(r\right), f\circ g\right)}{\log^{[m-1]} T\left(r, L(f)\right)} \geq$$
$$\begin{cases}\dfrac{\left({}_{(p)}^{(m)}\lambda_f^{L^*}-\varepsilon\right)\left[\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}}+\exp^{[p-1]} L\left(\exp\left(r\exp^{[p]} L(r)\right)^{{}_{(p)}^{(m)}\rho_f^{L^*}}\right)\right]}{\left(\{1+k-k\delta(\infty;f)\}\cdot\ {}_{(p)}^{(m)}\sigma_f^{L^*}+\varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}}} & \text{for } m=1\\ \dfrac{\left({}_{(p)}^{(m)}\lambda_f^{L^*}-\varepsilon\right)\left[\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}}+\exp^{[p-1]} L\left(\exp\left(r\exp^{[p]} L(r)\right)^{{}_{(p)}^{(m)}\rho_f^{L^*}}\right)\right]}{\left({}_{(p)}^{(m)}\sigma_f^{L^*}+\varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}}} & \text{for } m>1.\end{cases}$$

Since $\lim_{r\to\infty} \frac{\exp^{[p-1]} L\left(\exp\left(r \exp^{[p]} L(r)\right)^{{}_{(p)}^{(m)}\rho_f^{L^*}}\right)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}}} = 0$ as $\exp^{[p-1]} L\left(\exp\left(r \exp^{[p]} L(r)\right)^{\alpha}\right) = o\left(\left[r \exp^{[p]} L(r)\right]^{\alpha}\right)$ $(r \to \infty)$ for any $\alpha > 0$, we obtain from above that

$$\limsup_{r\to\infty} \frac{\log^{[m]} T\left(r \exp^{[p]} L(r), f \circ g\right)}{\log^{[m-1]} T(r, L(f))} \geq \begin{cases} \frac{{}_{(p)}\lambda_f^{L^*}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\sigma_f^{L^*}} & \text{for } m = 1 \\ \frac{{}_{(p)}^{(m)}\lambda_f^{L^*}}{{}_{(p)}^{(m)}\sigma_f^{L^*}} & \text{for } m > 1 . \end{cases}$$

Thus the theorem follows. $\square$

Now using the concept of the growth indicator ${}_{(p)}^{(m)}\overline{\tau}_f^{L^*}$ (m and p are any two positive integers) of a meromorphic function f, we may state thefollowing theorem without its proof since it can be carried out in the line of Theorem 17.

Theorem 18. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, $0 < {}_{(p)}^{(m)}\rho_f^{L^*} < \rho_g$ and ${}_{(p)}^{(m)}\overline{\tau}_f^{L^*} > 0$ where m and p are any two positive integers and g be an entire function. If $\exp^{[p-1]} L\left(\exp\left(r \exp^{[p]} L(r)\right)^{\alpha}\right) = o\left(\left[r \exp^{[p]} L(r)\right]^{\alpha}\right)$ as $r \to \infty$ for any $\alpha > 0$, then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T\left(r \exp^{[p]} L(r), f \circ g\right)}{\log^{[m-1]} T(r, L(f))} \geq \begin{cases} \frac{{}_{(p)}\lambda_f^{L^*}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\overline{\tau}_f^{L^*}} & \text{for } m = 1 \\ \frac{{}_{(p)}^{(m)}\lambda_f^{L^*}}{{}_{(p)}^{(m)}\overline{\tau}_f^{L^*}} & \text{for } m > 1 . \end{cases}$$

Theorem 19. *Let f be meromorphic function and g be entire function having the maximum deficiency sum such that (i) ${}_{(p)}^{(m)}\rho_f^{L^*} < \infty$ and (ii) $0 < {}_{(p)}\overline{\sigma}_g^{L^*} \leq {}_{(p)}\sigma_g^{L^*} < \infty$ where m and p are any two positive integers. Then*
(a) If $\exp^{[p-1]} L(M(r, g)) = o\{T(r, L(g))\}$ then

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f \circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r, g))} \leq \frac{{}_{(p)}^{(m)}\rho_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{(1 + k - k\delta(\infty; g)) \cdot {}_{(p)}\overline{\sigma}_g^{L^*}}$$

and (b) if $T(r, L(g)) = o\left\{\exp^{[p-1]} L(M(r, g))\right\}$ then

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f \circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r, g))} \leq {}_{(p)}^{(m)}\rho_f^{L^*} .$$

Proof. Since $T(r,g) \leq \log^{+} M(r,g)$ in view of Lemma 1, we obtain for all sufficiently large values of r that

$$\log T(r, f\circ g) \leq \log\{1+o(1)\} + \log T(M(r,g), f)$$
$$i.e.,\ \log^{[m]} T(r, f\circ g) \leq o(1) + \log^{[m]} T(M(r,g), f)$$
$$i.e.,\ \log^{[m]} T(r, f\circ g) \leq o(1) + \left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)\left\{\log M(r,g) + \exp^{[p-1]} L(M(r,g))\right\}. \tag{6}$$

Using the definition of ${}_{(p)}L^*$-type, we obtain from (6) for all sufficiently large values of r that

$$\log^{[m]} T(r, f\circ g) \leq o(1) + \left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)\left({}_{(p)}\sigma_g^{L^*} + \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}} + \left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)\exp^{[p-1]} L(M(r,g)). \tag{7}$$

Again from the definition of ${}_{(p)}L^*$-lower type and in view of Lemma 5 and Lemma 6, we get for all sufficiently large values of r that

$$T(r, L(g)) \geq \left({}_{(p)}\overline{\sigma}_{L(g)}^{L^*} - \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_{L(g)}^{L^*}}$$
$$i.e.,\ T(r, L(g)) \geq \left\{(1+k-k\delta(\infty;g))\cdot{}_{(p)}\overline{\sigma}_g^{L^*} - \varepsilon\right\}\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}}$$
$$i.e.,\ \left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho_g^{L^*}} \leq \frac{T(r, L(g))}{\left((1+k-k\delta(\infty;g))\cdot{}_{(p)}\overline{\sigma}_g^{L^*} - \varepsilon\right)}. \tag{8}$$

Now from (7) and (8), it follows for all sufficiently large values of r that

$$\log^{[m]} T(r, f\circ g) \leq o(1) + \left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)\exp^{[p-1]} L(M(r,g))$$
$$+\left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)\left({}_{(p)}\sigma_g^{L^*} + \varepsilon\right)\frac{T(r, L(g))}{\left((1+k-k\delta(\infty;g))\cdot{}_{(p)}\overline{\sigma}_g^{L^*} - \varepsilon\right)}$$
$$ie.,\ \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq \frac{o(1)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))}$$
$$+\frac{\frac{\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)\left({}_{(p)}\sigma_g^{L^*}+\varepsilon\right)}{\left((1+k-k\delta(\infty;g))\cdot{}_{(p)}\overline{\sigma}_g^{L^*}-\varepsilon\right)}}{1+\frac{\exp^{[p-1]} L(M(r,g))}{T(r,L(g))}} + \frac{\left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)}{1+\frac{T(r,L(g))}{\exp^{[p-1]} L(M(r,g))}}. \tag{9}$$

If $\exp^{[p-1]} L(M(r,g)) = o\{T(r, L(g))\}$ then from (9) we get that

$$\limsup_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq \frac{{}^{(m)}_{(p)}\rho_f^{L^*}\cdot{}_{(p)}\sigma_g^{L^*}}{(1+k-k\delta(\infty;g))\cdot{}_{(p)}\overline{\sigma}_g^{L^*}}.$$

Thus the first part of theorem follows.
Again if $T\left(r,L(g)\right)=o\left\{\exp^{[p-1]}L\left(M\left(r,g\right)\right)\right\}$ then from (9) it follows that

$$\limsup_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{T\left(r,L(g)\right)+\exp^{[p-1]}L\left(M\left(r,g\right)\right)}\leq {}_{(p)}^{(m)}\rho_f^{L^*}$$

which is the second part of the theorem. □

Theorem 20. *Let f be a meromorphic function and g be an entire function having the maximum deficiency sum with (i) ${}_{(p)}^{(m)}\lambda_f^{L^*}<\infty$ and (ii) $0<{}_{(p)}\overline{\sigma}_g^{L^*}\leq {}_{(p)}\sigma_g^{L^*}<\infty$ where m and p are any two positive integers. Then*
(a) If $\exp^{[p-1]}L\left(M\left(r,g\right)\right)=o\left\{T\left(r,L(g)\right)\right\}$ then

$$\liminf_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{T\left(r,L(g)\right)+\exp^{[p-1]}L\left(M\left(r,g\right)\right)}\leq\frac{{}_{(p)}^{(m)}\lambda_f^{L^*}\cdot {}_{(p)}\sigma_g^{L^*}}{\left(1+k-k\delta\left(\infty;g\right)\right)\cdot {}_{(p)}\overline{\sigma}_g^{L^*}}$$

and (b) if $T\left(r,L(g)\right)=o\left\{\exp^{[p-1]}L\left(M\left(r,g\right)\right)\right\}$ then

$$\liminf_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{T\left(r,L(g)\right)+\exp^{[p-1]}L\left(M\left(r,g\right)\right)}\leq {}_{(p)}^{(m)}\lambda_f^{L^*}\ .$$

We omit the proof of the above theorem as it can be carried out in the line of Theorem 19.

Using the concept of the growth indicator ${}_{(p)}\tau_g^{L^*}$ and ${}_{(p)}\overline{\tau}_g^{L^*}$ (p is any positive integer) of an entire function g, we may state the subsequent two theorems without their proofs since those can be carried out in the line of Theorem 19 and Theorem 20 respectively.

Theorem 21. *Let f be a meromorphic function and g be an entire function having the maximum deficiency sum such that (i) ${}_{(p)}^{(m)}\rho_f^{L^*}<\infty$ and (ii) $0<{}_{(p)}\tau_g^{L^*}\leq {}_{(p)}\overline{\tau}_g^{L^*}<\infty$ where m and p are any two positive integers. Then*
(a) If $\exp^{[p-1]}L\left(M\left(r,g\right)\right)=o\left\{T\left(r,L(g)\right)\right\}$ then

$$\limsup_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{T\left(r,L(g)\right)+\exp^{[p-1]}L\left(M\left(r,g\right)\right)}\leq\frac{{}_{(p)}^{(m)}\rho_f^{L^*}\cdot {}_{(p)}\overline{\tau}_g^{L^*}}{\left(1+k-k\delta\left(\infty;g\right)\right)\cdot {}_{(p)}\tau_g^{L^*}}$$

and (b) if $T\left(r,L(g)\right)=o\left\{\exp^{[p-1]}L\left(M\left(r,g\right)\right)\right\}$ then

$$\limsup_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{T\left(r,L(g)\right)+\exp^{[p-1]}L\left(M\left(r,g\right)\right)}\leq {}_{(p)}^{(m)}\rho_f^{L^*}\ .$$

Theorem 22. *Let f be a meromorphic function and g be an entire function having the maximum deficiency sum with (i) ${}_{(p)}^{(m)}\lambda_f^{L^*}<\infty$ and (ii) $0<{}_{(p)}\tau_g^{L^*}\leq {}_{(p)}\overline{\tau}_g^{L^*}<\infty$ where m and p are any two positive integers. Then*
(a) If $\exp^{[p-1]}L\left(M\left(r,g\right)\right)=o\left\{T\left(r,L(g)\right)\right\}$ then

$$\liminf_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{T\left(r,L(g)\right)+\exp^{[p-1]}L\left(M\left(r,g\right)\right)}\leq\frac{{}_{(p)}^{(m)}\lambda_f^{L^*}\cdot {}_{(p)}\overline{\tau}_g^{L^*}}{\left(1+k-k\delta\left(\infty;g\right)\right)\cdot {}_{(p)}\tau_g^{L^*}}$$

and (b) if $T(r, L(g)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq {}^{(m)}_{(p)}\lambda_f^{L^*} .$$

Now we state the following four theorems under some different conditions which can also be carried out using the same technique of Theorem 19 and therefore their proofs are omitted.

Theorem 23. *Let* f *be a meromorphic function and* g *be an entire function having the maximum deficiency sum such that (i)* ${}^{(m)}_{(p)}\rho_f^{L^*} < \infty$, *(ii)* ${}_{(p)}\sigma_g^{L^*} < \infty$ *and (iii)* ${}_{(p)}\tau_g^{L^*} > 0$ *where* m *and* p *are any two positive integers. Then*
(a) If $\exp^{[p-1]} L(M(r,g)) = o\{T(r, L(g))\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq \frac{{}^{(m)}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{(1 + k - k\delta(\infty; g)) \cdot {}_{(p)}\tau_g^{L^*}}$$

and (b) if $T(r, L(g)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq {}^{(m)}_{(p)}\rho_f^{L^*} .$$

Theorem 24. *Let* f *be a meromorphic function and* g *be an entire function having the maximum deficiency sum with (i)* ${}^{(m)}_{(p)}\lambda_f^{L^*} < \infty$, *(ii)* ${}_{(p)}\sigma_g^{L^*} < \infty$ *and (iii)* ${}_{(p)}\tau_g^{L^*} > 0$ *where* m *and* p *are any two positive integers. Then*
(a) If $\exp^{[p-1]} L(M(r,g)) = o\{T(r, L(g))\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq \frac{{}^{(m)}_{(p)}\lambda_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{(1 + k - k\delta(\infty; g)) \cdot {}_{(p)}\tau_g^{L^*}}$$

and (b) if $T(r, L(g)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq {}^{(m)}_{(p)}\lambda_f^{L^*} .$$

Theorem 25. *Let* f *be a meromorphic function and* g *be an entire function having the maximum deficiency sum such that (i)* ${}^{(m)}_{(p)}\rho_f^{L^*} < \infty$, *(ii)* ${}_{(p)}\overline{\sigma}_g^{L^*} > 0$ *and (iii)* ${}_{(p)}\overline{\tau}_g^{L^*} < \infty$ *where* m *and* p *are any two positive integers. Then*
(a) If $\exp^{[p-1]} L(M(r,g)) = o\{T(r, L(g))\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq \frac{{}^{(m)}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{(1 + k - k\delta(\infty; g)) \cdot {}_{(p)}\overline{\sigma}_g^{L^*}}$$

and (b) if $T(r, L(g)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq {}^{(m)}_{(p)}\rho_f^{L^*} .$$

Theorem 26. *Let f be a meromorphic function and g be an entire function having the maximum deficiency sum with (i) ${}_{(p)}^{(m)}\lambda_f^{L^*} < \infty$, (ii) ${}_{(p)}\overline{\sigma}_g^{L^*} > 0$ and (iii) ${}_{(p)}\overline{\tau}_g^{L^*} < \infty$ where m and p are any two positive integers. Then*
(a) If $\exp^{[p-1]} L(M(r,g)) = o\{T(r, L(g))\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq \frac{{}_{(p)}^{(m)}\lambda_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{(1+k-k\delta(\infty; g)) \cdot {}_{(p)}\overline{\sigma}_g^{L^*}}$$

and (b) if $T(r, L(g)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq {}_{(p)}^{(m)}\lambda_f^{L^*} .$$

The following theorem can also be carried out in the line of Theorem 19 and therefore its proof is omitted.

Theorem 27. *Let f be a meromorphic function with ${}_{(p)}^{(m)}\rho_f^{L^*} < \infty$ where m and p are any two positive integers. Also let g be an entire function with $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$ and also satisfy any one of the following conditiona:*
(i) $0 < {}_{(p)}\overline{\sigma}_g^{L^*} < \infty$, (ii) $0 < {}_{(p)}\sigma_g^{L^*} < \infty$, (iii) $0 < {}_{(p)}\overline{\tau}_g^{L^*} < \infty$, or (iv) $0 < {}_{(p)}\tau_g^{L^*} < \infty$. Then
(a) If $\exp^{[p-1]} L(M(r,g)) = o\{T(r, L(g))\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq \frac{{}_{(p)}^{(m)}\rho_f^{L^*}}{(1+k-k\delta(\infty; g))}$$

and (b) if $T(r, L(g)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \leq {}_{(p)}^{(m)}\rho_f^{L^*} .$$

Remark 3. *Theorem 27 extends Theorem 26 of Datta et. al. { cf. [5]}.*

Theorem 28. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ and g be entire such that (i) ${}_{(p)}^{(m)}\rho_f^{L^*} = {}_{(p)}\rho_g^{L^*}$, (ii) $0 < {}_{(p)}\sigma_g^{L^*} < \infty$ and (iii) ${}_{(p)}^{(m)}\overline{\sigma}_f^{L^*} > 0$ where m and p are any two positive integers. Then*

(a) If $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq \begin{cases} \frac{{}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\overline{\sigma}_f^{L^*}} & \text{for } m=1 \\ \frac{{}_{(p)}^{(m)}\rho_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{{}_{(p)}^{(m)}\overline{\sigma}_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) if $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq {}_{(p)}^{(m)}\rho_f^{L^*} .$$

Proof. In view of condition (ii) we obtain from (7) for all sufficiently large values of r that

$$\log^{[m]} T(r, f\circ g) \leq o(1) + \left({}_{(p)}^{(m)}\rho_f^{L^*} + \varepsilon\right)\left({}_{(p)}\sigma_g^{L^*} + \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}} + \left({}_{(p)}^{(m)}\rho_f^{L^*} + \varepsilon\right)\exp^{[p-1]} L(M(r,g)) . \quad (10)$$

Again from the definition of ${}_{(p)}L^*$-lower type and in view of Lemma 5, we get for all sufficiently large values of r that

$$\log^{[m-1]} T(r, L(f)) \geq \left({}_{(p)}^{(m)}\overline{\sigma}_{L(f)}^{L^*} - \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_{L(f)}^{L^*}}$$

$$i.e.,\ \log^{[m-1]} T(r, L(f)) \geq \left({}_{(p)}^{(m)}\overline{\sigma}_{L(f)}^{L^*} - \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}}$$

$$i.e.,\ \left[r\exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}} \leq \frac{\log^{[m-1]} T(r, L(f))}{\left({}_{(p)}^{(m)}\overline{\sigma}_{L(f)}^{L^*} - \varepsilon\right)} . \quad (11)$$

Now from (10) and (11), it follows for all sufficiently large values of r that

$$\log^{[m]} T(r, f\circ g) \leq o(1) + \left({}_{(p)}^{(m)}\rho_f^{L^*} + \varepsilon\right)\left({}_{(p)}\sigma_g^{L^*} + \varepsilon\right)\frac{\log^{[m-1]} T(r, L(f))}{\left({}_{(p)}^{(m)}\overline{\sigma}_{L(f)}^{L^*} - \varepsilon\right)} + \left({}_{(p)}^{(m)}\rho_f^{L^*} + \varepsilon\right)\exp^{[p-1]} L(M(r,g))$$

$$ie.,\ \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq \frac{o(1)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))}$$

$$+\frac{\frac{\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)\left({}_{(p)}\sigma_g^{L^*}+\varepsilon\right)}{\left({}^{(m)}_{(p)}\overline{\sigma}_{L(f)}^{L^*}-\varepsilon\right)}}{1+\frac{\exp^{[p-1]}L(M(r,g))}{\log^{[m-1]}T(r,L(f))}}+\frac{\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)}{1+\frac{\log^{[m-1]}T(r,L(f))}{\exp^{[p-1]}L(M(r,g))}}\ . \tag{12}$$

If $\exp^{[p-1]} L\left(M\left(r,g\right)\right) = o\left\{\log^{[m-1]} T\left(r, L(f)\right)\right\}$ then from (12) we get that

$$\limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m-1]} T\left(r, L(f)\right)+\exp^{[p-1]} L\left(M\left(r,g\right)\right)}\le\frac{\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)\left({}_{(p)}\sigma_g^{L^*}+\varepsilon\right)}{\left({}^{(m)}_{(p)}\overline{\sigma}_{L(f)}^{L^*}-\varepsilon\right)}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m-1]} T\left(r, L(f)\right)+\exp^{[p-1]} L\left(M\left(r,g\right)\right)}\le\frac{{}^{(m)}_{(p)}\rho_f^{L^*}\cdot\ {}_{(p)}\sigma_g^{L^*}}{{}^{(m)}_{(p)}\overline{\sigma}_{L(f)}^{L^*}}\ .$$

Now in view of Lemma 6, we get from above that

$$\limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m-1]} T\left(r, L(f)\right)+\exp^{[p-1]} L\left(M\left(r,g\right)\right)}$$

$$\le\begin{cases}\dfrac{{}_{(p)}\rho_f^{L^*}\cdot\ {}_{(p)}\sigma_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot\ {}_{(p)}\overline{\sigma}_f^{L^*}} & \text{for}\quad m=1\\[2ex] \dfrac{{}^{(m)}_{(p)}\rho_f^{L^*}\cdot\ {}_{(p)}\sigma_g^{L^*}}{{}^{(m)}_{(p)}\overline{\sigma}_f^{L^*}} & \text{for } m>1\ .\end{cases}$$

Thus the first part of the theorem follows.
Again if $\log^{[m-1]} T\left(r, L(f)\right) = o\left\{\exp^{[p-1]} L\left(M\left(r,g\right)\right)\right\}$ then from (12) it follows that

$$\limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m-1]} T\left(r, L(f)\right)+\exp^{[p-1]} L\left(M\left(r,g\right)\right)}\le\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)\ .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m-1]} T\left(r, L(f)\right)+\exp^{[p-1]} L\left(M\left(r,g\right)\right)}\le\ {}^{(m)}_{(p)}\rho_f^{L^*}\ .$$

Thus the second part of the theorem is established. □

Theorem 29. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be entire with (i) ${}^{(m)}_{(p)}\lambda_f^{L^*}<\infty$, (ii) ${}^{(m)}_{(p)}\rho_f^{L^*}={}_{(p)}\rho_g^{L^*}$, (iii) ${}_{(p)}\sigma_g^{L^*}<\infty$ and (iv) ${}^{(m)}_{(p)}\overline{\sigma}_f^{L^*}>0$ where m and p are any two positive integers. Then*

(a) If $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq \begin{cases} \frac{{}_{(p)}\lambda_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\overline{\sigma}_f^{L^*}} & \text{for } m=1 \\ \frac{{}_{(p)}^{(m)}\lambda_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{{}_{(p)}^{(m)}\overline{\sigma}_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) if $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq {}_{(p)}^{(m)}\lambda_f^{L^*}.$$

Theorem 30. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ and g be entire such that (i) ${}_{(p)}^{(m)}\rho_f^{L^*} = {}_{(p)}\rho_g^{L^*}$, (ii) $0 < {}_{(p)}\sigma_g^{L^*} < \infty$ and (iii) ${}_{(p)}^{(m)}\sigma_f^{L^*} > 0$ where m and p are any two positive integers. Then*
(a) If $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq \begin{cases} \frac{{}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\sigma_f^{L^*}} & \text{for } m=1 \\ \frac{{}_{(p)}^{(m)}\rho_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{{}_{(p)}^{(m)}\sigma_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) if $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq {}_{(p)}^{(m)}\rho_f^{L^*}.$$

Theorem 31. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ and g be entire such that (i) ${}_{(p)}^{(m)}\rho_f^{L^*} = {}_{(p)}\rho_g^{L^*}$, (ii) $0 < {}_{(p)}\overline{\sigma}_g^{L^*} < \infty$ and (iii) ${}_{(p)}^{(m)}\overline{\sigma}_f^{L^*} > 0$ where m and p are any two positive integers. Then*

(a) If $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \le \begin{cases} \frac{{}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\sigma}_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\overline{\sigma}_f^{L^*}} & \text{for } m=1 \\ \frac{{}^{(m)}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\sigma}_g^{L^*}}{{}^{(m)}_{(p)}\overline{\sigma}_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) if $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \le {}^{(m)}_{(p)}\rho_f^{L^*}.$$

We omit the proof of the above three theorems as those can be carried out in the line of Theorem 28.

Remark 4. *For* $p=1$*, Theorem 30 reduces to Theorem* 27 *of* [5].

Similarly using the concept of the growth indicator ${}^{(m)}_{(p)}\tau_f^{L^*}$ and ${}_{(p)}\overline{\tau}_g^{L^*}$ we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 28, Theorem 29, Theorem 30 and Theorem 31 respectively.

Theorem 32. *Let* f *be a transcendental meromorphic function with* $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ *and* g *be entire such that (i)* ${}^{(m)}_{(p)}\rho_f^{L^*} < \infty$*, (ii)* ${}^{(m)}_{(p)}\lambda_f^{L^*} = {}_{(p)}\lambda_g^{L^*}$*, (iii)* ${}_{(p)}\overline{\tau}_g^{L^*} < \infty$ *and (iv)* ${}^{(m)}_{(p)}\tau_f^{L^*} > 0$ *where* m *and* p *are any two positive integers. Then*
(a) If $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \le \begin{cases} \frac{{}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\tau_f^{L^*}} & \text{for } m=1 \\ \frac{{}^{(m)}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{{}^{(m)}_{(p)}\tau_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) if $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \le {}^{(m)}_{(p)}\rho_f^{L^*}.$$

Theorem 33. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be entire with (i) ${}^{(m)}_{(p)}\lambda_f^{L^*} = {}_{(p)}\lambda_g^{L^*}$, (ii) $0 < {}_{(p)}\overline{\tau}_g^{L^*} < \infty$ and (iii) ${}^{(m)}_{(p)}\tau_f^{L^*} > 0$ where m and p are any two positive integers. Then*
(a) *If* $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq \begin{cases} \frac{{}_{(p)}\lambda_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\tau_f^{L^*}} & \text{for } m=1 \\ \frac{{}^{(m)}_{(p)}\lambda_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{{}^{(m)}_{(p)}\tau_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) *if* $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq {}^{(m)}_{(p)}\lambda_f^{L^*}.$$

Theorem 34. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f) + \delta(\infty;f) = 2$ and g be entire such that (i) ${}^{(m)}_{(p)}\rho_f^{L^*} < \infty$, (ii) ${}^{(m)}_{(p)}\lambda_f^{L^*} = {}_{(p)}\lambda_g^{L^*}$, (iii) ${}_{(p)}\overline{\tau}_g^{L^*} < \infty$ and (iv) ${}^{(m)}_{(p)}\overline{\tau}_f^{L^*} > 0$ where m and p are any two positive integers. Then*
(a) *If* $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq \begin{cases} \frac{{}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\overline{\tau}_f^{L^*}} & \text{for } m=1 \\ \frac{{}^{(m)}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{{}^{(m)}_{(p)}\overline{\tau}_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) *if* $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq {}^{(m)}_{(p)}\rho_f^{L^*}.$$

Theorem 35. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f) + \delta(\infty;f) = 2$ and g be entire such that (i) ${}^{(m)}_{(p)}\rho_f^{L^*} < \infty$, (ii) ${}^{(m)}_{(p)}\lambda_f^{L^*} = {}_{(p)}\lambda_g^{L^*}$, (iii) ${}_{(p)}\tau_g^{L^*} < \infty$ and (iv) ${}^{(m)}_{(p)}\tau_f^{L^*} > 0$ where m and p are any two positive integers. Then*

(a) If $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \le \begin{cases} \frac{{}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\tau_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\tau_f^{L^*}} & \text{for } m=1 \\ \frac{{}^{(m)}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\tau_g^{L^*}}{{}^{(m)}_{(p)}\tau_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) if $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \le {}^{(m)}_{(p)}\rho_f^{L^*} .$$

Remark 5. *Theorem 35 extends Theorem* 1 *of Datta et. al. { cf.* [4]*}.*

Analogously we state the following four theorems under some different conditions which can also be carried out using the same technique of Theorem 28 and therefore their proofs are omitted.

Theorem 36. *Let* f *be a transcendental meromorphic function with* $\sum_{a\neq\infty}\delta(a;f) + \delta(\infty;f) = 2$ *and* g *be entire such that (i)* ${}^{(m)}_{(p)}\rho_f^{L^*} < \infty$, *(ii)* ${}^{(m)}_{(p)}\lambda_f^{L^*} = {}_{(p)}\rho_g^{L^*}$, *(iii)* ${}_{(p)}\sigma_g^{L^*} < \infty$ *and (iv)* ${}^{(m)}_{(p)}\tau_f^{L^*} > 0$ *where* m *and* p *are any two positive integers. Then*
(a) If $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \le \begin{cases} \frac{{}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\tau_f^{L^*}} & \text{for } m=1 \\ \frac{{}^{(m)}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{{}^{(m)}_{(p)}\tau_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) if $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \le {}^{(m)}_{(p)}\rho_f^{L^*} .$$

Remark 6. *Theorem 36 extends Theorem* 3 *of Datta et. al. { cf.* [4]*}.*

Theorem 37. *Let* f *be a transcendental meromorphic function having the maximum deficiency sum and* g *be entire with (i)* ${}^{(m)}_{(p)}\lambda_f^{L^*} = {}_{(p)}\rho_g^{L^*}$, *(ii)* $0 < {}_{(p)}\sigma_g^{L^*} < \infty$

and (iii) ${}_{(p)}^{(m)}\tau_f^{L^*} > 0$ *where* m *and* p *are any two positive integers. Then*
(a) *If* $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq \begin{cases} \frac{{}_{(p)}\lambda_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\tau_f^{L^*}} & \text{for } m=1 \\ \frac{{}_{(p)}^{(m)}\lambda_f^{L^*} \cdot {}_{(p)}\sigma_g^{L^*}}{{}_{(p)}^{(m)}\tau_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) if $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq {}_{(p)}^{(m)}\lambda_f^{L^*} .$$

Remark 7. *Theorem 37 extends Theorem 2 of Datta et. al. { cf.* [4]*}.*

Theorem 38. *Let* f *be a transcendental meromorphic function with* $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ *and* g *be entire such that (i)* ${}_{(p)}^{(m)}\rho_f^{L^*} = {}_{(p)}\lambda_g^{L^*}$, *(ii)* $0 < {}_{(p)}\overline{\tau}_g^{L^*} < \infty$ *and (iii)* ${}_{(p)}^{(m)}\overline{\sigma}_f^{L^*} > 0$ *where* m *and* p *are any two positive integers. Then*
(a) *If* $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m-1]} T(r, L(f))\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq \begin{cases} \frac{{}_{(p)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\overline{\sigma}_f^{L^*}} & \text{for } m=1 \\ \frac{{}_{(p)}^{(m)}\rho_f^{L^*} \cdot {}_{(p)}\overline{\tau}_g^{L^*}}{{}_{(p)}^{(m)}\overline{\sigma}_f^{L^*}} & \text{for } m>1 \end{cases}$$

and (b) if $\log^{[m-1]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[m-1]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \leq {}_{(p)}^{(m)}\rho_f^{L^*} .$$

Theorem 39. *Let* f *be a transcendental meromorphic function having the maximum deficiency sum and* g *be entire with (i)* ${}_{(p)}^{(m)}\rho_f^{L^*} = {}_{(p)}\lambda_g^{L^*}$, *(ii)* $0 < {}_{(p)}\overline{\tau}_g^{L^*} < \infty$ *and (iii)* ${}_{(p)}^{(m)}\overline{\sigma}_f^{L^*} > 0$ *where* m *and* p *are any two positive integers. Then* *(a) If* $\exp^{[p-1]} L\left(M\left(r,g\right)\right) = o\left\{\log^{[m-1]} T\left(r, L(f)\right)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m-1]} T\left(r, L(f)\right) + \exp^{[p-1]} L\left(M\left(r,g\right)\right)} \leq \begin{cases} \dfrac{{}_{(p)}\lambda_f^{L^*}\cdot {}_{(p)}\overline{\tau}_g^{L^*}}{(1+k-k\delta(\infty;f))\cdot {}_{(p)}\overline{\sigma}_f^{L^*}} & for \;\; m=1 \\ \dfrac{{}^{(m)}_{(p)}\lambda_f^{L^*}\cdot {}_{(p)}\overline{\tau}_g^{L^*}}{{}^{(m)}_{(p)}\overline{\sigma}_f^{L^*}} & for \; m>1 \end{cases}$$

and (b) if $\log^{[m-1]} T\left(r, L(f)\right) = o\left\{\exp^{[p-1]} L\left(M\left(r,g\right)\right)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m-1]} T\left(r, L(f)\right) + \exp^{[p-1]} L\left(M\left(r,g\right)\right)} \leq {}^{(m)}_{(p)}\lambda_f^{L^*} .$$

Theorem 40. *Let f be a transcendental meromorphic function with* $\sum_{a\neq\infty} \delta\left(a;f\right) + \delta\left(\infty;f\right) = 2$ *and* $0 < {}^{(m)}_{(p)}\lambda_f^{L^*} \leq {}^{(m)}_{(p)}\rho_f^{L^*} < \rho_g$ *where m and p are any two positive integers. Also let g be an entire function. Then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m]} T\left(\exp r^{\mu}, L(f)\right)} \geq \frac{{}^{(m)}_{(p)}\lambda_f^{L^*}}{{}^{(m)}_{(p)}\rho_f^{L^*}}$$

where $0 < \mu < \rho_g \leq \infty$.

Proof. In view of Lemma 2, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[m]} T\left(r, f\circ g\right) &\geq \log^{[m]} T\left(\exp r^{\mu}, f\right) \\ i.e.,\;\; \log^{[m]} T\left(r, f\circ g\right) &\geq \left({}^{(m)}_{(p)}\lambda_f^{L^*} - \varepsilon\right)\left[r^{\mu} + \exp^{[p-1]} L\left(\exp r^{\mu}\right)\right] . \end{aligned} \tag{13}$$

Also for any arbitrary $\varepsilon\,(>0)$, it follows from Definition 3 and in view of Lemma 5 for all sufficiently large values of r that

$$\log^{[m]} T\left(\exp r^{\mu}, L(f)\right) \leq \left({}^{(m)}_{(p)}\rho_{L(f)}^{L^*} + \varepsilon\right)\left[r^{\mu} + \exp^{[p-1]} L\left(\exp r^{\mu}\right)\right]$$

$$i.e.,\;\; \log^{[m]} T\left(\exp r^{\mu}, L(f)\right) \leq \left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)\left[r^{\mu} + \exp^{[p-1]} L\left(\exp r^{\mu}\right)\right] . \tag{14}$$

Now from (13) and (14), we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m]} T\left(\exp r^{\mu}, L(f)\right)} \geq \frac{\left({}^{(m)}_{(p)}\lambda_f^{L^*} - \varepsilon\right)\left[r^{\mu} + \exp^{[p-1]} L\left(\exp r^{\mu}\right)\right]}{\left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)\left[r^{\mu} + \exp^{[p-1]} L\left(\exp r^{\mu}\right)\right]} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[m]} T\left(\exp r^{\mu}, L(f)\right)} \geq \frac{{}^{(m)}_{(p)}\lambda_f^{L^*}}{{}^{(m)}_{(p)}\rho_f^{L^*}}\ .$$

Thus the theorem follows. □

Theorem 41. *Let f be meromorphic with ${}^{(m)}_{(p)}\rho_f^{L^*} < \infty$ and g be transcendental entire with finite lower order and $\sum\limits_{a\neq\infty}\delta\left(a;g\right)+\delta\left(\infty;g\right)=2$. Also let there exists entire functions $a_i\ (i=1,2,3,\cdots,q;q\leq\infty)$ such that $T\left(r,a_i\right)=o\left\{T\left(r,g\right)\right\}$ and $\sum\limits_{i=1}^{q}\delta\left(a_i,g\right)=1$. If $\exp^{[p-1]} L\left(M\left(r,g\right)\right)=o\left\{T\left(r,L(g)\right)\right\}$ then*

$$\limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{T\left(r, L(g)\right)} \leq \frac{\pi\cdot\ {}^{(m)}_{(p)}\rho_f^{L^*}}{\left(1+k-k\delta\left(\infty;g\right)\right)}\ ,$$

otherwise

$$\lim_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{T\left(r, L(g)\right)\cdot\exp^{[p-1]} L\left(M\left(r,g\right)\right)}=0$$

where m and p are any two positive integers.

Proof. From (6) we get for all sufficiently large values of r that

$$\frac{\log^{[m]} T\left(r, f\circ g\right)}{T\left(r, L(g)\right)} \leq \qquad .$$

$$\leq \frac{\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)\left(\log M\left(r,g\right)+\exp^{[p-1]} L\left(M\left(r,g\right)\right)\right)+o(1)}{T\left(r, L(g)\right)} \tag{15}$$

$$i.e.,\ \frac{\log^{[m]} T\left(r, f\circ g\right)}{T\left(r, L(g)\right)} \leq +O(1)$$

$$\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)\left[\frac{\log M\left(r,g\right)}{T\left(r,L(g)\right)}+\frac{\exp^{[p-1]} L\left(M\left(r,g\right)\right)}{T\left(r,L(g)\right)}\right]$$

$$i.e.,\ \frac{\log^{[m]} T\left(r, f\circ g\right)}{T\left(r, L(g)\right)} \leq +O(1)$$

$$\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)\left[\frac{\log M\left(r,g\right)}{T\left(r,L(g)\right)}\cdot\frac{T\left(r,g\right)}{T\left(r,L(g)\right)}+\frac{\exp^{[p-1]} L\left(M\left(r,g\right)\right)}{T\left(r,L(g)\right)}\right]\ . \tag{16}$$

Case I. Let $\exp^{[p-1]} L\left(M\left(r,g\right)\right)=o\left\{T\left(r,L(g)\right)\right\}$. Then

$$\lim_{r\to\infty}\frac{\exp^{[p-1]} L\left(M\left(r,g\right)\right)}{T\left(r,L(g)\right)}=0\ . \tag{17}$$

Now combining (17) and (16) and in view of Lemma 3 and Lemma 4, it follows that

$$\limsup_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))}\leq\frac{\pi\cdot{}_{(p)}^{(m)}\rho_f^{L^*}}{(1+k-k\delta(\infty;g))}\ . \tag{18}$$

Case II. Let $\exp^{[p-1]}L(M(r,g))\neq o\{T(r,L(g))\}$. Then from (15) we get for all sufficiently large values of r that

$$\begin{aligned}&\frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))\cdot\exp^{[p-1]}L(M(r,g))}\\ \leq\ &\left({}_{(p)}^{(m)}\rho_f^{L^*}+\varepsilon\right)\cdot\frac{\log M(r,g)}{T(r,L(g))\cdot\exp^{[p-1]}L(M(r,g))}+\frac{O(1)}{T(r,L(g))}\ .\end{aligned}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))\cdot\exp^{[p-1]}L(M(r,g))}=0\ .$$

Thus combining Case I and Case II the theorem follows. □

In the line of Theorem 41 the following theorem can be proved:

Theorem 42. *Let f be meromorphic with ${}_{(p)}^{(m)}\lambda_f^{L^*}<\infty$ where $m\geq 1$ and g be transcendental entire with finite lower order and $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Also let there exists entire functions $a_i\ (i=1,2,3,\cdots,q;q\leq\infty)$ such that $T(r,a_i)=o\{T(r,g)\}$ and $\sum\limits_{i=1}^{q}\delta(a_i,g)=1$. If $\exp^{[p-1]}L(M(r,g))=o\{T(r,L(g))\}$ then*

$$\liminf_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))}\leq\frac{\pi\cdot{}_{(p)}^{(m)}\lambda_f^{L^*}}{(1+k-k\delta(\infty;g))}\ ,$$

otherwise

$$\liminf_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))\cdot\exp^{[p-1]}L(M(r,g))}=0$$

where m and p are any two positive integers.

Remark 8. *Theorem 40 and Theorem 41 respectively extend Theorem* 3.4 *and Theorem* 3.3 *of* [6].

Theorem 43. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be entire such that $0<{}_{(p)}^{(m)}\lambda_f^{L^*}\leq{}_{(p)}^{(m)}\rho_f^{L^*}<\infty$ where m and p are any two positive integers. Then for any positive real number A,*

$$\limsup_{r\to\infty}\frac{\log^{[m+1]}T\left(\exp\left(r^A\right),f\circ g\right)}{\log^{[m]}T(\exp(r^\mu),L(f))+K(r,g;L)}=\infty\ ,$$

where $0<\mu<\rho_g$ and

$$K(r,g;L)=\begin{cases}0\ \textit{if}\ r^\mu=o\left\{\exp^{[p-1]}L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\}\ \textit{as}\ r\to\infty\\ \exp^{[p-1]}L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\ \textit{otherwise}\ .\end{cases}$$

Proof. Let $0 < \mu < \mu' < \rho_g$. Using Definition 3 we obtain in view of Lemma 2 for a sequence of values of r tending to infinity that

$$\log^{[m]} T\left(\exp\left(r^A\right), f \circ g\right) \geq \log^{[m]} T\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}, f\right)$$

$$i.e.,\ \log^{[m]} T\left(\exp\left(r^A\right), f \circ g\right) \geq \left({}_{(p)}^{(m)}\lambda_f^{L^*} - \varepsilon\right) \cdot \left\{\left(\exp\left(r^A\right)\right)^{\mu'} + \exp^{[p-1]} L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)\right\}$$

$$i.e.,\ \log^{[m]} T\left(\exp\left(r^A\right), f \circ g\right) \geq \left({}_{(p)}^{(m)}\lambda_f^{L^*} - \varepsilon\right) \cdot \left\{\left(\exp\left(r^A\right)\right)^{\mu'}\left(1 + \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)}{\left(\exp\left(r^A\right)\right)^{\mu'}}\right)\right\}$$

$$i.e.,\ \log^{[m+1]} T\left(\exp\left(r^A\right), f \circ g\right) \geq O(1) + \mu' r^A + \log\left\{1 + \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)}{\left(\exp\left(r^A\right)\right)^{\mu'}}\right\}$$

$$i.e.,\ \log^{[m+1]} T\left(\exp\left(r^A\right), f \circ g\right) \geq O(1) + \mu' r^A + \log\left[1 + \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu' r^A\right)\right)\right)}{\exp\left(\mu' r^A\right)}\right]$$

$$i.e.,\ \log^{[m+1]} T\left(\exp\left(r^A\right), f \circ g\right) \geq O(1) + \mu' r^A + \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) - \log\left[\exp^{[p]}\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\}\right] + \log\left[1 + \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu' r^A\right)\right)\right)}{\exp\left(\mu' r^A\right)}\right]$$

$$i.e.,\ \log^{[m+1]} T\left(\exp\left(r^A\right), f \circ g\right) \geq O(1) + \mu' r^A + \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) + \log\left[\frac{1}{\exp^{[p]}\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\}} + \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu' r^A\right)\right)\right)}{\exp^{[p]}\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\} \cdot \exp\left(\mu' r^A\right)}\right]$$

$$i.e.,\ \log^{[m+1]} T\left(\exp\left(r^A\right), f \circ g\right) \geq O(1) + \mu' r^{(A-\mu)} \cdot r^{\mu} + \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right). \qquad (19)$$

Again in view of Lemma 5, we have for all sufficiently large values of r that

$$\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) \leq \left({}^{(m)}_{(p)}\rho^{L^*}_{L(f)} + \varepsilon\right) \log\left[\exp\left(r^{\mu}\right) \exp^{[p]} L\left(\exp\left(r^{\mu}\right)\right)\right]$$
$$i.e., \ \log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) \leq \left({}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon\right) \left[r^{\mu} + \exp^{[p-1]} L\left(\exp\left(r^{\mu}\right)\right)\right]$$

$$i.e., \ \frac{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) - \left({}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon\right) \exp^{[p-1]} L\left(\exp\left(r^{\mu}\right)\right)}{\left({}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon\right)} \leq r^{\mu}. \quad (20)$$

Now from (19) and (20), it follows for a sequence of values of r tending to infinity that

$$\log^{[m+1]} T\left(\exp\left(r^{A}\right), f \circ g\right) \geq O(1) +$$
$$\left(\frac{\mu' r^{(A-\mu)}}{{}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon}\right) \left[\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) - \left({}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon\right) \exp^{[p-1]} L\left(\exp\left(r^{\mu}\right)\right)\right]$$
$$+ \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \quad (21)$$

$$i.e., \ \frac{\log^{[m+1]} T\left(\exp\left(r^{A}\right), f \circ g\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right)} \geq \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) + O(1)}{\log T\left(\exp\left(r^{\mu}\right), L(f)\right)}$$
$$+ \frac{\mu' r^{(A-\mu)}}{{}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon} \left\{1 - \frac{\left({}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon\right) \exp^{[p-1]} L\left(\exp\left(r^{\mu}\right)\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right)}\right\}. \quad (22)$$

Again from (21) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[m+1]} T\left(\exp\left(r^{A}\right), f \circ g\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) + \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}$$
$$\geq \frac{O(1) - \mu' r^{(A-\mu)} \cdot \exp^{[p-1]} L\left(\exp\left(r^{\mu}\right)\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) + \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}$$
$$+ \frac{\left(\frac{\mu' r^{(A-\mu)}}{{}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon}\right) \log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) + \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}$$
$$+ \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) + \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}$$

$$i.e., \ \frac{\log^{[m+1]} T\left(\exp\left(r^{A}\right), f \circ g\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) + \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}$$

$$\geq \frac{\frac{O(1) - \mu' r^{(A-\mu)} \cdot \exp^{[p-1]} L(\exp(r^{\mu}))}{\exp^{[p-1]} L(\exp(\exp(\mu r^{A})))}}{\frac{\log^{[m]} T(\exp(r^{\mu}), L(f))}{\exp^{[p-1]} L(\exp(\exp(\mu r^{A})))} + 1} +$$

$$\frac{\left(\frac{\mu' r^{(A-\mu)}}{{}_{(p)}^{(m)}\rho_f^{L^*} + \varepsilon}\right) \log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right)}{1 + \frac{\exp^{[p-1]} L(\exp(\exp(\mu r^{A})))}{\log^{[m]} T(\exp(r^{\mu}), L(f))}} + \frac{1}{1 + \frac{\log^{[m]} T(\exp(r^{\mu}), L(f))}{\exp^{[p-1]} L(\exp(\exp(\mu r^{A})))}} . \tag{23}$$

Case I. If $r^{\mu} = o\left\{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\}$ then it follows from (22) that

$$\limsup_{r \to \infty} \frac{\log^{[m+1]} T\left(\exp\left(r^{A}\right), f \circ g\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right)} = \infty .$$

Case II. $r^{\mu} \neq o\left\{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\}$ then the following two sub cases may arise:

Sub case (a). If $\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) = o\left\{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right)\right\}$, then we get from (23) that

$$\limsup_{r \to \infty} \frac{\log^{[m+1]} T\left(\exp\left(r^{A}\right), f \circ g\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} = \infty .$$

Sub case (b). If $\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \sim \log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right)$ then

$$\lim_{r \to \infty} \frac{\exp^{[p-1]} L\left\{\exp\left(\exp\left(\mu r^{A}\right)\right)\right\}}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right)} = 1$$

and we obtain from (23) that

$$\limsup_{r \to \infty} \frac{\log^{[m+1]} T\left(\exp\left(r^{A}\right), f \circ g\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} = \infty .$$

Combining Case I and Case II we obtain that

$$\limsup_{r \to \infty} \frac{\log^{[m+1]} T\left(\exp\left(r^{A}\right), f \circ g\right)}{\log^{[m]} T\left(\exp\left(r^{\mu}\right), L(f)\right) + K\left(r, g; L\right)} = \infty ,$$

where $K\left(r, g; L\right) = \begin{cases} 0 \text{ if } r^{\mu} = o\left\{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\} \text{ as } r \to \infty \\ \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \text{ otherwise .} \end{cases}$

This proves the theorem. □

Theorem 44. *Let f be a meromorphic function and g be transcendental entire such that ${}_{(p)}^{(m)}\lambda_f^{L^*} > 0$, ${}_{(p)}\rho_g^{L^*} < \infty$ and $\sum_{a \neq \infty} \delta\left(a; g\right) + \delta\left(\infty; g\right) = 2$ where m and p*

are any two positive integers. Then for any positive real number A,

$$\limsup_{r\to\infty}\frac{\log^{[m+1]} T\left(\exp\left(r^{A}\right), f\circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), L(g)\right)+K\left(r, f; L\right)}=\infty\ ,$$

where $0<\mu<\rho_g$ and

$$K\left(r,f;L\right)=\left\{\begin{array}{l} 0 \textit{ if } r^{\mu}=o\left\{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\} \textit{ as } r\to\infty \\ \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \textit{ otherwise .}\end{array}\right.$$

The proof is omitted because it can be carried out in the line of Theorem 43.

Remark 9. *Theorem 43 and Theorem 44 are respectively improve Theorem* 12 *and Theorem* 13 *of Datta et. al. { cf.* [5]*}.*

References

[1] W. Bergweiler : On the Nevanlinna Characteristic of a composite function, Complex Variables, Vol. 10 (1988), pp.225-236.

[2] W. Bergweiler : On the growth rate of composite meromorphic functions, Complex Variables, Vol. 14 (1990) pp.187-196.

[3] S. K. Datta and A. Biswas : Some generalized growth properties of composite entire functions involving their maximum terms on the basis of slowly changing functions, International Journal of Mathematical Analysis, Vol. 5, No.22 (2011), pp. 1093-1101.

[4] S. K. Datta, T. Biswas and S. Ali : Some results on wronskians using slowly changing functions, News Bull. Cal. Math. Soc., Vol. 36, No. 7-9 (2013). pp. 8-24.

[5] S. K. Datta, T. Biswas and S. Ali : Growth analysis of wronskians in terms of slowly changing functions, Journal of Complex Analysis, Vol. 2013, Article ID 395067, 9 pages, http://dx.doi.org/10.1155/2013/395067.

[6] S. K. Datta, T. Biswas and S. Ali : Growth rates of wronskians generated by complex valued functions, Gulf Journal of Mathematics, Vol. 2, Issue 1 (2014), pp. 58-74.

[7] W. K. Hayman : Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[8] T.V. Lakshminarasimhan : A note on entire functions of bounded index, J. Indian Math. Soc. Vol. 38 (1974), pp. 43-49.

[9] Q. Lin and C. Dai : On a conjecture of Shah concerning small functions, Kexue Tong bao (English Ed.), Vol.31, No. 4, (1986), pp. 220-224.

[10] I. Lahiri and N. R. Bhattacharjee : Functions of L-bounded index and of non-uniform L-bounded index, Indian J. Math. Vol. 40, No. 1 (1998), pp. 43-57.

[11] I. Lahiri and A. Banerjee : Value distribution of a Wronskian , Portugaliae Mathematica , Vol.. 61, No. Fasc.2 Nova Série (2004), pp. 161-175.

[12] D. Somasundaram and R. Thamizharasi : A note on the entire functions of L-bounded index and L-type, Indian J. Pure Appl.Math., Vol..19, No. 3 (March 1988), pp. 284-293.

[13] G. Valiron : Lectures on the general theory of integral functions, Chelsea Publishing Company, 1949.

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[0]Başlık: Bazı genelleştirilmiş büyüme belirteçleri ışığında wronskiyenlerin büyüme analizi üzerine.

Anahtar Kelimeler: Transendental tam fonksiyon, transendental meromorfik fonksiyon, bileşke, büyüme, m-y.nci genelleme, p oranlı ${}_pL^*$- basamağı,