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UNIT DUAL SPLIT QUATERNIONS AND ARCS OF DUAL HYPERBOLIC SPHERICAL TRIANGLES

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Abstract. In this paper we obtain the cosine hyperbolic and sine hyperbolic rules for a dual hyperbolic spherical triangle $T(\tilde{A},\tilde{B},\tilde{C})$ whose arcs are represented by dual split quaternions.

1. INTRODUCTION

The dual hyperbolic unit sphere $\tilde{H}_0^2$ is the set of all time-like unit vectors in the dual Lorentzian space $\mathbb{D}_1^3$ with signature $(-,+,+)$. Dual hyperbolic spherical geometry which is studied by means of dual time-like unit vectors is analogous to real hyperbolic spherical geometry which is studied by means of real time-like unit vectors. Quaternions and split quaternions have many applications in mathematics (see [1], [2], [3]). Some of the recent works include [4], [5]. Great circle arcs on a unit sphere represented by a unit quaternion and sine and cosine rules are obtained by J. P. Ward (see [6] pp.98-102). A similar correspondence is possible with unit dual split quaternions and arcs of a dual hyperbolic spherical triangle on the dual hyperbolic unit sphere $\tilde{H}_0^2$. The hyperbolic sine and hyperbolic cosine rules for dual and real spherical trigonometry have been well known for a long time (see [7], [8], [9], [10], [11]). Here in this paper we obtain hyperbolic sine and hyperbolic cosine rules by means of the correspondence between arcs of the dual hyperbolic spherical triangle on $\tilde{H}_0^2$ and dual split quaternions.

2. PRELIMINARIES

A dual number has the form $q=\lambda+\varepsilon\lambda^*$, where λ and λ^* are real numbers and ε is the dual unit which satisfies the rules:

$$\varepsilon\neq 0,\ 0\varepsilon=\varepsilon 0=0,\ 1\varepsilon=\varepsilon 1=\varepsilon,\ \varepsilon^2=0.$$

The set of dual numbers is a ring denoted by $\mathbb{D}$.
The Lorentzian inner product of dual vectors $\tilde{A}=r+\varepsilon r^*$, $\tilde{B}=s+\varepsilon s^*$ is defined

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by

$$\langle \tilde{A}, \tilde{B} \rangle = \langle r, s \rangle + \varepsilon(\langle r, s^* \rangle + \langle r^*, s \rangle)$$

where $\langle r, s \rangle$ is the Lorentzian inner product of the vectors r and s in $\mathbb{R}^3_1$, (see [12]) given by

$$\langle r, s \rangle = -r_1 s_1 + r_2 s_2 + r_3 s_3$$

A dual vector $\tilde{A} = r + \varepsilon r^*$ is said to be time-like if the vector r is time-like, (resp. space-like if the vector r is space-like and light-like if the vector r is light-like). The set $\mathbb{D}^3_1 = \{\tilde{A} = r + \varepsilon r^* | r, r^* \in \mathbb{R}^3_1\}$ is called dual Lorentzian space. The Lorentzian cross product of dual vectors $\tilde{A}$ and $\tilde{B} \in \mathbb{D}^3_1$ is given by

$$\tilde{A} \Lambda \tilde{B} = r \Lambda s + \varepsilon(r^* \Lambda s + r \Lambda s^*)$$

where $r \Lambda s$ is the Lorentzian cross product in $\mathbb{R}^3_1$ given by

$$r \Lambda s = \begin{vmatrix} -e_1 & e_2 & e_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix}.$$

Lemma 1. *Let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathbb{D}^3_1$. Then we have*

$$\langle \tilde{A} \Lambda \tilde{B}, \tilde{C} \rangle = \det(\tilde{A}, \tilde{B}, \tilde{C}) \tag{2.1}$$

$$\tilde{A} \Lambda \tilde{B} = -\tilde{B} \Lambda \tilde{A} \tag{2.2}$$

$$(\tilde{A} \Lambda \tilde{B}) \Lambda \tilde{C} = -\langle \tilde{A}, \tilde{C} \rangle \tilde{B} + \langle \tilde{B}, \tilde{C} \rangle \tilde{A} \tag{2.3}$$

$$\langle \tilde{A} \Lambda \tilde{B}, \tilde{C} \Lambda \tilde{D} \rangle = -\langle \tilde{A}, \tilde{C} \rangle \langle \tilde{B}, \tilde{D} \rangle + \langle \tilde{A}, \tilde{D} \rangle \langle \tilde{B}, \tilde{C} \rangle \tag{2.4}$$

$$\langle \tilde{A} \Lambda \tilde{B}, \tilde{A} \rangle = 0 \textit{ and } \langle \tilde{A} \Lambda \tilde{B}, \tilde{B} \rangle = 0. \tag{2.5}$$

The set of all dual time-like unit vectors is called dual hyperbolic unit sphere and denoted by $\tilde{H}^2_0$. There are two components of the dual hyperbolic unit sphere $\tilde{H}^2_0$.

$$\widetilde{H}^{+2}_0 = \left\{ \tilde{A} = r + \varepsilon r^* \in \mathbb{D}^3_1 : |\tilde{A}| = 1,\ r,\ r^* \in \mathbb{R}^3_1 \text{ and } r \text{ is future pointing time-like vector} \right\}$$

is called future dual hyperbolic unit sphere,

$$\widetilde{H}^{-2}_0 = \left\{ \tilde{A} = r + \varepsilon r^* \in \mathbb{D}^3_1 : |\tilde{A}| = 1,\ r,\ r^* \in \mathbb{R}^3_1 \text{ and } r \text{ is past pointing time-like vector} \right\}$$

is called past dual hyperbolic unit sphere (see [12])., where $\|\tilde{A}\| = \sqrt{\langle \tilde{A}, \tilde{A} \rangle} = |r| + \varepsilon \frac{\langle r, r^* \rangle}{|r|^2}$ with $|r| \neq 0$. Since we work on $\tilde{H}^{+2}_0$ in this paper without loss of generality we use $\widetilde{H}^2_0$ instead of $\widetilde{H}^{+2}_0$.

Definition 1. *Let x and y be distinct points of H^n. Then x and y span a $2-$dimensional time-like subspace $V(x,y)$ of R^{n+1} and so*

$$L(x,y) = H^n \cap V(x,y)$$

is the unique hyperbolic line of H^n containing x and y. Note that $L(x,y)$ is a branch of a hyperbola (see [13] *pp. 68)*

Definition 2. *Let x, y, z be three hyperbolically non-collinear points of H^2. Let $L(x,y)$ be the unique hyperbolic line of H^2 containing x and y, and let $H(x,y,z)$ be the closed half-plane of H^2 with $L(x,y)$ as its boundary and z in its interior. The hyperbolic triangle with vertices x, y, z is defined to be*

$$T(x,y,z) = H(x,y,z) \cap H(y,z,x) \cap H(z,x,y).$$

(see [13] *pp. 83)*

Definition 3. *Let $\tilde{A}$ and $\tilde{B}$ be distinct points of $\widetilde{H_0^2}$. Then $\tilde{A}$ and $\tilde{B}$ span a dual timelike subspace $V(\tilde{A},\tilde{B})$ of $\mathbb{D}_1^3$ and so*

$$L(\tilde{A},\tilde{B}) = \widetilde{H_0^2} \cap V(\tilde{A},\tilde{B})$$

is called a dual hyperbolic line of $\widetilde{H_0^2}$. Note that this is the unique dual hyperbolic line containing $\tilde{A}$ and $\tilde{B}$.

Definition 4. *Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ be hyperbolically non -colinear points of $\widetilde{H_0^2}$. Let $L(\tilde{A},\tilde{B})$ be the unique dual hyperbolic line of $\widetilde{H_0^2}$ containing $\tilde{A}$ and $\tilde{B}$, and let $H(\tilde{A},\tilde{B},\tilde{C})$ be the closed half-plane of $\widetilde{H_0^2}$ with $L(\tilde{A},\tilde{B})$ as its boundary and $\tilde{C}$ is in its interior. The dual hyperbolic triangle with vertices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ is defined to be*

$$T(\tilde{A},\tilde{B},\tilde{C}) = H(\tilde{A},\tilde{B},\tilde{C}) \cap H(\tilde{B},\tilde{C},\tilde{A}) \cap H(\tilde{C},\tilde{A},\tilde{B})$$

(See figure 1).

3. DUAL SPLIT QUATERNIONS AND ARCS

The elements of

$$H'_{\mathbb{D}} = \{Q = q_0 + q_1\overrightarrow{e_1} + q_2\overrightarrow{e_2} + q_3\overrightarrow{e_3} | q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{D}\}$$

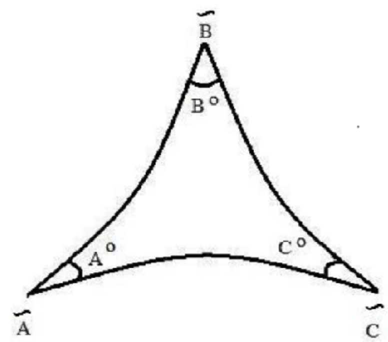


FIGURE 1. Dual hyperbolic triangle.

with the following multiplication table

	1	$\overrightarrow{e_1}$	$\overrightarrow{e_2}$	$\overrightarrow{e_3}$
1	1	$\overrightarrow{e_1}$	$\overrightarrow{e_2}$	$\overrightarrow{e_3}$
$\overrightarrow{e_1}$	$\overrightarrow{e_1}$	-1	$\overrightarrow{e_3}$	$-\overrightarrow{e_2}$
$\overrightarrow{e_2}$	$\overrightarrow{e_2}$	$-\overrightarrow{e_3}$	1	$-\overrightarrow{e_1}$
$\overrightarrow{e_3}$	$\overrightarrow{e_3}$	$\overrightarrow{e_2}$	$\overrightarrow{e_1}$	1

are called dual split quaternions (see [14]).
A dual split quaternion can be expressed as $Q = S_Q + \overrightarrow{V}_Q$, where $S_Q = q_0$, $\overrightarrow{V}_Q = q_1\overrightarrow{e_1} + q_2\overrightarrow{e_2} + q_3\overrightarrow{e_3}$. The product rule of dual split quaternions is given as follows:

$$QP = S_Q S_P + \langle \overrightarrow{V}_Q, \overrightarrow{V}_P \rangle + S_Q \overrightarrow{V}_P + S_P \overrightarrow{V}_Q + \overrightarrow{V}_Q \Lambda \overrightarrow{V}_P,$$

where $\langle \overrightarrow{V}_Q, \overrightarrow{V}_P \rangle = -q_1p_1 + q_2p_2 + q_3p_3$, and $\overrightarrow{V}_Q \Lambda \overrightarrow{V}_P = (q_3p_2 - q_2p_3)\overrightarrow{e_1} + (q_3p_1 - q_1p_3)\overrightarrow{e_2} + (q_1p_2 - q_2p_1)\overrightarrow{e_3}$. If

$$\begin{aligned} Q &= q_0 + q_1\overrightarrow{e_1} + q_2\overrightarrow{e_2} + q_3\overrightarrow{e_3} \\ &= q + \varepsilon q^* \end{aligned}$$

where $q = \lambda_0 + \lambda_1 \overrightarrow{e_1} + \lambda_2 \overrightarrow{e_2} + \lambda_3 \overrightarrow{e_3}$ and $q^* = \lambda_0^* + \lambda_1^* \overrightarrow{e_1} + \lambda_2^* \overrightarrow{e_2} + \lambda_3^* \overrightarrow{e_3}$, λ_i, $\lambda_i^* \in \mathbb{R}$, $0 \le i \le 3$, then the norm of Q is given as

$$\begin{aligned} N_Q &= q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ &= -\langle q, q\rangle - 2\varepsilon \langle q, q^*\rangle . \end{aligned}$$

If $N_Q = 1$ then Q is said to be a unit dual split quaternion (see [14]). unit dual split quaternion may be expressed as:

$$\begin{aligned} Q &= \langle \tilde{A}, \tilde{B}\rangle - \tilde{A}\Lambda\tilde{B} \\ &= -\cosh\phi + \hat{Q}\sinh\phi \\ &= -\cosh(\varphi + \varepsilon\varphi^*) + \hat{Q}\sinh(\varphi + \varepsilon\varphi^*) \end{aligned}$$

where $\tilde{A}$, $\tilde{B}$ are unit dual time-like vectors. We may associate this unit dual split quaternion to the dual hyperbolic arc $L(\tilde{A}, \tilde{B}) = arc\tilde{A}\tilde{B}$ which is obtained by dual time-like subspace $V(\tilde{A}, \tilde{B})$ with normal $\hat{Q}$ intersect the unit dual hyperbolic sphere $\widetilde{H}_0^2$. Let $\tilde{A}, \tilde{B}$ and $\tilde{C}$ be future pointing time-like unit dual vectors in dual Lorentzian space $\mathbb{D}_1^3$ and $Q = \langle \tilde{A}, \tilde{B}\rangle - \tilde{A}\Lambda\tilde{B}$ and $P = \langle \tilde{B}, \tilde{C}\rangle - \tilde{B}\Lambda\tilde{C}$ be unit dual split quaternions. The dual split quaternion product of P and Q is

$PQ = \langle \tilde{B}, \tilde{C}\rangle\langle \tilde{A}, \tilde{B}\rangle + \langle \tilde{B}\Lambda\tilde{C}, \tilde{A}\Lambda\tilde{B}\rangle - \langle \tilde{B}, \tilde{C}\rangle\tilde{A}\Lambda\tilde{B} - \langle \tilde{A}, \tilde{B}\rangle\tilde{B}\Lambda\tilde{C} + (-\tilde{B}\Lambda\tilde{C})\Lambda(-\tilde{A}\Lambda\tilde{B})$.

From Lemma 1, (2.4) and since $\tilde{B}$ is a unit dual vector we have

$$\langle \tilde{B}\Lambda\tilde{C}, \tilde{A}\Lambda\tilde{B}\rangle = -\langle \tilde{C}, \tilde{B}\rangle\langle \tilde{B}, \tilde{A}\rangle + \langle \tilde{C}, \tilde{A}\rangle.$$

From Lemma 1 (2.3), (2.5) and (2.1) we get,

$$\left(\tilde{C}\Lambda\tilde{B}\right)\Lambda\left(\tilde{B}\Lambda\tilde{A}\right) = -\langle \tilde{C}\Lambda\tilde{B}, \tilde{A}\rangle\tilde{B} = -\det\left(\tilde{B}, \tilde{A}, \tilde{C}\right)\tilde{B} = -\langle \tilde{B}\Lambda\tilde{A}, \tilde{C}\rangle\tilde{B}.$$

From lemma 1 (2.3) and (2.2) we obtain,

$$-\langle \tilde{B}\Lambda\tilde{A}, \tilde{C}\rangle\tilde{B} + \langle \tilde{B}, \tilde{C}\rangle\tilde{B}\Lambda\tilde{A} = \left[\left(\tilde{B}\Lambda\tilde{A}\right)\Lambda\tilde{B}\right]\Lambda\tilde{C} = -\tilde{A}\Lambda\tilde{C} + \langle \tilde{A}, \tilde{B}\rangle\tilde{B}\Lambda\tilde{C}.$$

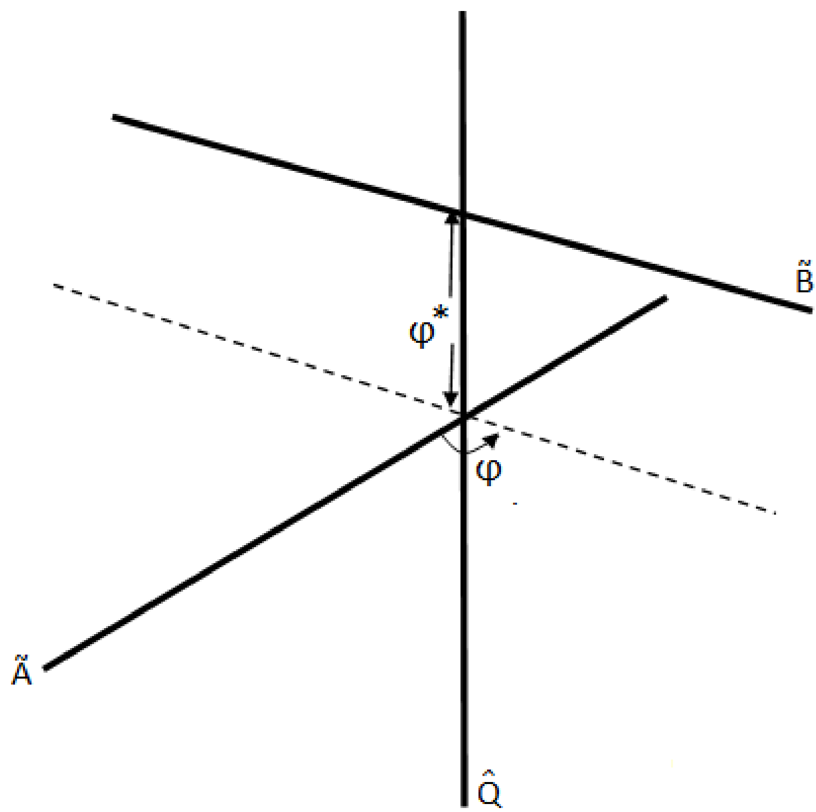
Thus from the above calculations we have,

$$PQ = \langle \tilde{A}, \tilde{C}\rangle - \tilde{A}\Lambda\tilde{C}.$$

3.0.1. *Geometrical Interpretation of Q.* Let $\tilde{A} = \overrightarrow{r} + \varepsilon\overrightarrow{r^*}$, $\tilde{B} = \overrightarrow{s} + \varepsilon\overrightarrow{s^*}$, with $\left\|\tilde{A}\right\|$ $= \left\|\tilde{B}\right\| = 1$, where $\overrightarrow{r}$, $\overrightarrow{r^*} \in \mathbb{R}^3$, $\overrightarrow{s}$, $\overrightarrow{s^*} \in \mathbb{R}^3$, i.e. $\tilde{A}$, $\tilde{B} \in \widetilde{H}_0^2$. Using the product rule for dual split quaternions we have,

$$\tilde{B}\tilde{A} = \left\langle \tilde{A}, \tilde{B}\right\rangle - \tilde{A}\Lambda\tilde{B} = Q.$$

Note that $Q\tilde{A} = \tilde{B}$. Since $Q = -\cosh\phi + \hat{Q}\sinh\phi = -\cosh(\varphi + \varepsilon\varphi^*) + \hat{Q}\sinh(\varphi + \varepsilon\varphi^*)$, then $\hat{Q} = \frac{\tilde{B}\Lambda\tilde{A}}{\left\|\tilde{B}\Lambda\tilde{A}\right\|}$ is the unit dual vector.($Q\tilde{A} = \tilde{B}$ means that the operator Q transforms the line $\tilde{A}$ into the line $\tilde{B}$. See the below figure.)



Remark 1. *Multiplying the unit dual vector $\tilde{A}$ from the left by Q means that, rotating the line corresponding to $\tilde{A}$ by φ about (E. Study) the line corresponding to $\hat{Q}$ and translating it by φ^* along the line corresponding to $\hat{Q}$.*

Considering that P and Q are unit dual split quaternions, they are screw operators at the same time. Hence $Q(\tilde{A}) = \tilde{B}$, $P(\tilde{B}) = \tilde{C}$ implies that $PQ(\tilde{A}) = \tilde{C}$. This means that the line d_1 which corresponds $\tilde{A}$ transforms into the line d_2 which corresponds $\tilde{C}$. Consider a unit dual split quaternion $Q = \langle \tilde{A}, \tilde{B} \rangle - \tilde{A} \Lambda \tilde{B} = -\cosh \phi + \hat{Q} \sinh \phi$. We may associate this unit dual split quaternion to a great circle arc (hyperbolic line) of $\widetilde{H_0^2}$ which is obtained when diametral plane with normal $\hat{Q}$ intersect $\widetilde{H_0^2}$. Clearly the position of the arc along the circle is arbitrary and so the $arc\tilde{A}\tilde{B}$ is free to slide on this great circle as long as its length (which is the measure of the dual angle corresponding to the arc) and direction are maintained. For fixed $\hat{Q}$ algebra of dual split quaternions is identical to the algebra of quaternions.

We write using $\sim$ to specify the geometrical correspondence,

$$\begin{aligned} arc\tilde{A}\tilde{B} \sim Q &= \langle \tilde{A}, \tilde{B} \rangle - \tilde{A}\Lambda\tilde{B} \\ &= -\cosh\phi + \hat{Q}\sinh\phi \\ &= -\cosh(\varphi + \varepsilon\varphi^*) + \hat{Q}\sinh(\varphi + \varepsilon\varphi^*) \end{aligned}$$

$$\hat{Q} = -\frac{(\tilde{A}\Lambda\tilde{B})}{\|\tilde{A}\Lambda\tilde{B}\|}$$

is a dual space-like vector and

$$\begin{aligned} arc\tilde{B}\tilde{C} \sim P &= \langle \tilde{B}, \tilde{C} \rangle - \tilde{B}\Lambda\tilde{C} \\ &= -\cosh\psi + \hat{P}\sinh\psi \\ &= -\cosh(\theta + \varepsilon\theta^*) + \hat{P}\sinh(\theta + \varepsilon\theta^*) \end{aligned}$$

$$arc\tilde{A}\tilde{C} \sim PQ = \langle \tilde{A}, \tilde{C} \rangle - \tilde{A}\Lambda\tilde{C}.$$

Therefore

$$arc\tilde{A}\tilde{B} + arc\tilde{B}\tilde{C} = arc\tilde{A}\tilde{C} \text{ or } arcQ + arcP = arcPQ.$$

As a trivial consequence of the argument above we have the following theorem:

Theorem 1. *Let $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ be future pointing dual time-like unit vectors. Then*

$$arc\tilde{A}_1\tilde{A}_2 + arc\tilde{A}_2\tilde{A}_3 + \cdots + arc\tilde{A}_{n-1}\tilde{A}_n = arc\tilde{A}_1\tilde{A}_n.$$

4. THE HYPERBOLIC SINE AND COSINE RULES FOR DUAL HYPERBOLIC SPHERICAL TRIANGLES

Let $\tilde{A}, \tilde{B}, \tilde{C}$ be three points on the dual hyperbolic unit sphere $\widetilde{H}_0^2$, given by the linearly independent dual time-like unit vectors,

$$\tilde{r} = r + \varepsilon r^*, \ \tilde{s} = s + \varepsilon s^*, \ \tilde{t} = t + \varepsilon t^*.$$

These points together with $arc\tilde{A}\tilde{B}$, $arc\tilde{B}\tilde{C}$, $arc\tilde{C}\tilde{A}$ form a dual hyperbolic spherical triangle $T(\tilde{A}, \tilde{B}, \tilde{C})$. Having defined a dual hyperbolic spherical triangle there is naturally defined six dual angles

$$a^o = a + \varepsilon a^*, \ b^o = b + \varepsilon b^*, \ c^o = c + \varepsilon c^*$$

called arc angles and

$$A^o = u + \varepsilon u^*, B^o = v + \varepsilon v^*, C^o = w + \varepsilon w^*$$

called vertex angles (see figure 2).

We represent arcs of a dual hyperbolic spherical triangle by dual split quaternions. If $Q = -\cosh a^o + \hat{Q}\sinh a^o$, $P = -\cosh c^o + \hat{P}\sinh c^o$ then

$$\begin{aligned} QP = {} & \cosh a^o \cosh c^o + \langle \hat{Q}, \hat{P} \rangle \sinh a^o \sinh c^o \\ & -\hat{P}\cosh a^o \sinh c^o - \hat{Q}\cosh c^o \sinh a^o + \hat{Q}\Lambda\hat{P}\sinh a^o \sinh c^o. \end{aligned}$$

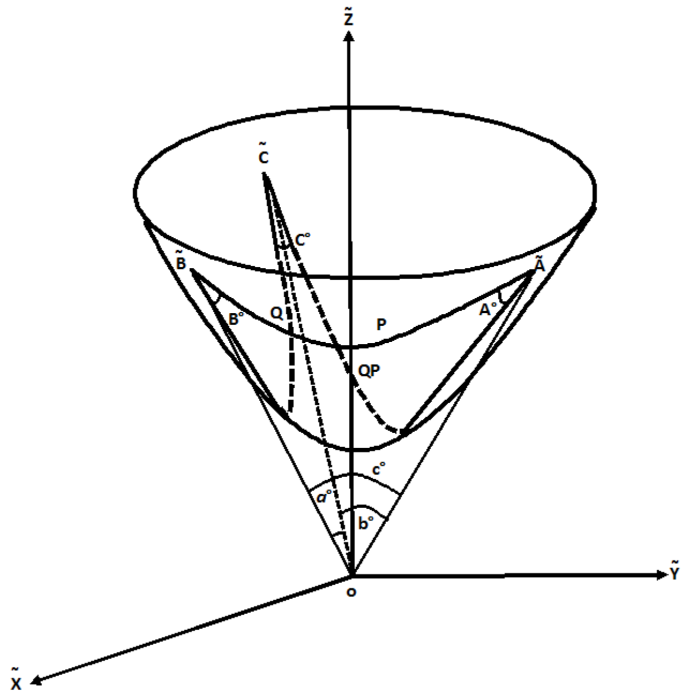


FIGURE 2. *Dual hyperbolic spherical triangle on* $\widetilde{H}_0^2$.

On the other hand $arc\tilde{A}\tilde{B} \sim P$, $arc\tilde{B}\tilde{C} \sim Q$, $arc\tilde{A}\tilde{C} \sim QP$ and writing

$$arc\tilde{A}\tilde{C} \sim -\cosh b^o + \hat{M}\sinh b^o,$$

we get, by equating scalar and vector parts,

$$\cosh a^o \cosh c^o + \langle \hat{Q}, \hat{P} \rangle \sinh a^o \sinh c^o = -\cosh b^o \tag{4.1}$$

$$-\hat{P}\cosh a^o \sinh c^o - \hat{Q}\cosh c^o \sinh a^o + \hat{Q}\Lambda\hat{P}\sinh a^o \sinh c^o = \hat{M}\sinh b^o. \tag{4.2}$$

Note that $\hat{P}, \hat{Q}, \hat{M}$ are space-like unit dual vectors in the direction of $\tilde{A}\Lambda\tilde{B}, \tilde{B}\Lambda\tilde{C}$ and $\tilde{A}\Lambda\tilde{C}$ respectively, i.e.

$$\hat{P} = -\frac{\tilde{A}\Lambda\tilde{B}}{\|\tilde{A}\Lambda\tilde{B}\|},\ \hat{Q} = -\frac{\tilde{B}\Lambda\tilde{C}}{\|\tilde{B}\Lambda\tilde{C}\|},\ \hat{M} = -\frac{\tilde{A}\Lambda\tilde{C}}{\|\tilde{A}\Lambda\tilde{C}\|}.$$

We have $\langle \tilde{A}, \tilde{C} \rangle = -\cosh b^o$, $\tilde{A}\Lambda\tilde{C} = \hat{M}\sinh b^o$, $\langle \tilde{B}, \tilde{C} \rangle = -\cosh a^o$, $\tilde{B}\Lambda\tilde{C} = \hat{Q}\sinh a^o$, $\langle \tilde{A}, \tilde{B} \rangle = -\cosh c^o$, $\tilde{A}\Lambda\tilde{B} = \hat{P}\sinh c^o$.

The dual angles A^o, B^o, C^o satisfy:

$$\langle \hat{P}, \hat{M} \rangle = -\cosh A^o,\ \ \hat{P}\Lambda\hat{M} = \tilde{A}\sinh A^o,$$
$$\langle \hat{P}, \hat{Q} \rangle = -\cosh B^o, \hat{P}\Lambda\hat{Q} = \tilde{B}\sinh B^o,$$
$$\langle \hat{M}, \hat{Q} \rangle = -\cosh C^o,\ \ \hat{M}\Lambda\hat{Q} = \tilde{C}\sinh C^o.$$

Now (4.1) implies

$$\cosh a^o \cosh c^o - \cosh B^o \sinh a^o \sinh c^o = -\cosh b^o.$$

Thus we have the hyperbolic cosine rule as follows:

Theorem 2. *Let $T(\tilde{A}, \tilde{B}, \tilde{C})$ be a dual hyperbolic spherical triangle on $\widetilde{H}_0^2$, then*

$$\cosh B^o = \frac{\cosh a^o \cosh c^o + \cosh b^o}{\sinh a^o \sinh c^o} \tag{4.3}$$
$$\cosh A^o = \frac{\cosh b^o \cosh c^o + \cosh a^o}{\sinh b^o \sinh c^o} \tag{4.4}$$
$$\cosh C^o = \frac{\cosh a^o \cosh b^o + \cosh c^o}{\sinh a^o \sinh b^o} \tag{4.5}$$

Corollary 1. *The real and dual parts of the Formula*(4.3), (4.4), (4.5) *are given by*

$$\cosh v = \frac{\cosh a \cosh c + \cosh b}{\sinh a \sinh c}$$
$$\sinh v = \frac{\sinh b}{v^* \sinh a \sinh c}\left(-a^* \cosh w - c^* \cosh u + b^*\right),$$
$$\cosh u = \frac{\cosh b \cosh c + \cosh a}{\sinh b \sinh c}$$
$$\sinh u = \frac{\sinh a}{u^* \sinh b \sinh c}\left(-b^* \cosh w - c^* \cosh v + a^*\right),$$
$$\cosh w = \frac{\cosh a \cosh b + \cosh c}{\sinh a \sinh b}$$
$$\sinh w = \frac{\sinh c}{w^* \sinh a \sinh b}\left(-a^* \cosh v - b^* \cosh u + c^*\right),$$

By taking scalar product with $\tilde{B}$, since $\langle \hat{P}, \tilde{B} \rangle = 0$, $\langle \hat{Q}, \tilde{B} \rangle = 0$, $\hat{Q}\Lambda\hat{P} = -\tilde{B}\sinh B^o$, then we get from (4.2)

$$-\sinh B^o \sinh a^o \sinh c^o = \langle \hat{M}, \tilde{B} \rangle \sinh b^o.$$

Therefore

$$\begin{aligned} \frac{\sinh B^o}{\sinh b^o} &= \frac{\langle \tilde{B}, \tilde{A}\Lambda\tilde{C} \rangle}{\sinh a^o \sinh b^o \sinh c^o} \\ &= \frac{-\langle \tilde{A}, \tilde{B}\Lambda\tilde{C} \rangle}{\sinh a^o \sinh b^o \sinh c^o} \\ &= \frac{-\det(\tilde{A}, \tilde{B}, \tilde{C})}{\sinh a^o \sinh b^o \sinh c^o}. \end{aligned}$$

But the right hand side is unchanged on cyclic interchange and so we deduce

Theorem 3. *Let $T(\tilde{A}, \tilde{B}, \tilde{C})$ be a dual hyperbolic spherical triangle on $\widetilde{H_0^2}$, then*

$$\frac{\sinh A^o}{\sinh a^o} = \frac{\sinh B^o}{\sinh b^o} = \frac{\sinh C^o}{\sinh c^o} \tag{4.6}$$

Corollary 2. *The real and dual part of the Formula* (4.6) *is given by,*

$$\frac{\sinh u}{\sinh a} = \frac{\sinh v}{\sinh b} = \frac{\sinh w}{\sinh c},$$

and

$$\begin{aligned} u^* \frac{\cosh u}{\sinh a} - a^* \coth a \frac{\sinh u}{\sinh a} &= v^* \frac{\cosh v}{\sinh b} - b^* \coth b \frac{\sinh v}{\sinh b} \\ &= w^* \frac{\cosh w}{\sinh c} - c^* \coth c \frac{\sinh w}{\sinh c}. \end{aligned}$$

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[0]Başlık: Birim dual ayrık kuaternions ve dual hiperbolik küresel üçgenlerin yayları
Anahtar Kelimeler: Hiperbolik küresel trigonometri, ayrık kuaterniyon, Lorentz uzayı.