SOME PROPERTIES OF THE GENERALIZED BLEIMANN, BUTZER AND HAHN OPERATORS

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Abstract. In the present paper, we introduce sequences of Bleimann, Butzer and Hahn operators which are based on a function $\tau$. This function is a continuously differentiable function on $\mathbb{R}^+$, such that $\tau(0) = 0$, $\inf \tau'(x) \geq 1$. We give a Korovkin-type theorem and prove uniform approximation of the generalized Bleimann, Butzer and Hahn operator. We also investigate the monotonic convergence property of the sequence of the operators under $\tau$-convexity.

1. Introduction

Let, as usual, $C[0, \infty)$ denote the space of all continuous and real valued functions defined on $[0, \infty)$ and $C_B[0, \infty)$ denote the space of all bounded functions from $C[0, \infty)$. Obviously

$$
\|f\|_{C_B} = \sup_{x \geq 0} |f(x)|
$$

defines a norm on $C_B[0, \infty)$. In [6], Bleimann, Butzer and Hahn proposed a sequence of positive linear operators $L_n$ defined by

$$
L_n(f; x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} f\left( \frac{k}{n+k+1} \right) \binom{n}{k} x^k, \quad x \geq 0, \quad n \in \mathbb{N} \quad (1.1)
$$

for $f \in C[0, \infty)$. Here, the authors proved that $L_n(f; x) \to f(x)$ as $n \to \infty$ pointwise on $[0, \infty)$ when $f \in C_B[0, \infty)$. Moreover, the convergence being uniform on each compact subset of $[0, \infty)$. In [9], using test functions $\left( \frac{1}{1+t^v} \right)^n$ for $v = 0, 1, 2$, Gadjev and Çakar stated a Korovkin-type theorem for the uniform convergence of functions belonging to some suitable function space by some linear positive operators. As an application of this result, they proved uniform approximation of Bleimann, Butzer and Hahn operators.

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Monotonicity properties of the Bleimann, Butzer and Hahn operators was investigated by Della Vecchia in [8]. The operator $L_n$ and its generalizations have been studied by several authors some are in [1], [4], [11], [2].

In [7], Cárdenas-Moreles, Garrancho and Raşa introduce a new type generalization of Bernstein polynomials denoted by $B_n^\tau$ and defined as

$$B_n^\tau (f; x) := B_n \left( f \circ \tau^{-1}; \tau (x) \right)$$

where $B_n$ is the $n$–th Bernstein polynomial, $f \in C [0, 1]$, $x \in [0, 1]$ and $\tau$ being any function that is continuously differentiable of infinite order on $[0, 1]$ such that $\tau (0) = 0$, $\tau (1) = 1$ and $\tau' (x) > 0$ for $x \in [0, 1]$. In this work, the authors studied some shape preserving and convergence properties concerning the generalized Bernstein operators $B_n^\tau (f; x)$. A similar generalization for Szász-Mirakyan operator was stated in [5] by Aral, Inoan and Raşa by taking $\rho$ as continuously differentiable function on $[0, \infty)$, $\rho(0) = 0$, $\inf_{x \in \mathbb{R}_+} \rho'(x) \geq 1$. Here, weighted approximation as well as the degree of the approximation were obtained. Among other results, they also showed that the sequence of the generalized Szász-Mirakyan operators is monotonically nonincreasing under the notion of $\rho$–convexity of the original function.

Now, accordingly, we consider the following generalized Bleimann, Butzer and Hahn operators for $f \in C[0, \infty)$:

$$L_n^\tau (f; x) = \frac{1}{(1 + \tau (x))^n} \sum_{k=0}^{n} \left( \frac{k}{n} \right) \tau (x)^k (f \circ \tau^{-1}) \left( \frac{n - k + 1}{n} \right) \tau (x)^k,$$

where $\tau$ is a continuously differentiable function defined on $[0, \infty)$ such that $\tau (0) = 0$, $\inf_{x \in [0, \infty)} \tau'(x) \geq 1$. (1.4)

An example of such a function $\tau$ is given in [5]. Note that, in the setting of the operator (1.3) we have

$$L_n^\tau f := L_n \left( f \circ \tau^{-1} \right) \circ \tau,$$

where $L_n$ is the $n$–th Bleimann, Butzer and Hahn operator given by (1.1). If $\tau (x) = x$ then $L_n^\tau = L_n$. Obviously, we have

$$L_n^\tau (1; x) = 1,$$

$$L_n^\tau \left( \frac{\tau}{1 + \tau}; x \right) = \frac{n}{n + 1} \frac{\tau (x)}{1 + \tau (x)},$$

$$L_n^\tau \left( \left( \frac{\tau}{1 + \tau} \right)^2; x \right) = \frac{n (n - 1)}{(n + 1)^2} \left( \frac{\tau (x)}{1 + \tau (x)} \right)^2 + \frac{n}{(n + 1)^2} \frac{\tau (x)}{1 + \tau (x)}.$$
The first purpose of this paper is to extend the results of Gadjiev and Çakar in
[9] to the generalized Bleimann, Butzer and Hahn operators (1.3). In this direction,
we first give a generalized Korovkin-type theorem to obtain uniform convergence
by \{L_n^\tau (f; x)\}_{n \in \mathbb{N}} to f(x) on [0, \infty) for f belonging to some suitable subspace
of continuous and bounded functions that we will denote it by \(H_\omega^\tau\). Next, we
study the monotonic convergence under the \(\tau\)-convexity of the function which is
approximated.

For this purpose, we define the following class of functions

Let \(\omega\) be a general functions of modulus of continuity, satisfying the following
properties:
\[(a) \; \omega \text{ is a continuous nonnegative increasing function on } [0, \infty),\]
\[(b) \; \omega (\delta_1 + \delta_2) \leq \omega (\delta_1) + \omega (\delta_2)\]
\[(c) \lim_{\delta \to 0} \omega (\delta) = 0.\]

Suppose that \(H_\omega^\tau\) denote the space of all real valued functions \(f\) defined on \([0, \infty)\)
satisfying
\[
|f(x) - f(y)| = \omega \left( \frac{\tau(x)}{1 + \tau(x)} - \frac{\tau(y)}{1 + \tau(y)} \right) \quad (1.6)
\]
for all \(x, y \in [0, \infty)\). It readily follows from (c) that if \(f \in H_\omega^\tau\), then it is continuous
on \([0, \infty)\). Moreover, if \(f \in H_\omega^\tau\), then we have
\[
|f(x)| \leq |f(0)| + \omega \left( \frac{\tau(x)}{1 + \tau(x)} \right)
\leq |f(0)| + \omega (1), \quad (x \geq 0),
\]
by the assumption on \(\tau\), which clearly gives that \(f\) is bounded on \([0, \infty)\). Therefore
we have the following inclusion:
\[
H_\omega^\tau \subset C_{B}[0, \infty).
\]
When \(\omega(t) = Mt^\alpha, \; 0 < \alpha \leq 1\), the space of \(H_\omega^\tau\) will be denoted by \(H_\alpha^\tau\). From (1.6)
we get that
\[
|f(x) - f(y)| \leq M \frac{|\tau(x) - \tau(y)|^\alpha}{(1 + \tau(x))^{\alpha}(1 + \tau(y))^{\alpha}}.
\]
Hence we reach to
\[
H_\alpha^\tau \subset \text{Lip}_M (\tau(x); \alpha),
\]
where \(\text{Lip}_M (\tau(x); \alpha), \; 0 < \alpha \leq 1, \; M > 0\), is the set of all functions \(f \in C[0, \infty)\)
satisfying the inequality
\[
|f(t) - f(x)| \leq M |\tau(t) - \tau(x)|^\alpha, \; x, \; t \geq 0
\]
(see [10]).
Definition 1. A continuous, real valued function $f$ is said to be convex in $D \subseteq \{0, 1\}$, if

$$f \left( \sum_{i=1}^{m} \alpha_i x_i \right) \leq \sum_{i=1}^{m} \alpha_i f(x_i)$$

for every $x_1, x_2, ..., x_m \in D$ and for every nonnegative numbers $\alpha_1, \alpha_2, ..., \alpha_m$ such that $\alpha_1 + \alpha_2 + ... + \alpha_m = 1$.

In [7] Cárdenas-Morales, Garrancho and Raşa introduced the following definition of $\tau-$convexity of a continuous function.

Definition 2. A continuous, real valued function $f$ is said to be $\tau-$convex in $D$, if $f \circ \tau^{-1}$ is convex in the sense of Definition 1.

Here, we give a Korovkin-type theorem in the sense of Gadjiev and Çakar ([9]).

2. Main Results

Here, we give a Korovkin-type theorem in the sense of Gadjiev and Çakar ([9]).

Theorem 1. Let $\{T_n^\tau(f)\}_{n\in\mathbb{N}}$ be a sequence of linear positive operators from $H_\omega^\tau$ to $C_B[0, \infty)$. If

$$\lim_{n \to \infty} \left\| T_n^\tau \left( \left( \frac{\tau(t)}{1+\tau(t)} \right)^v ; x \right) - \left( \frac{\tau(x)}{1+\tau(x)} \right)^v \right\|_{C_B} = 0, \tag{2.1}$$

is satisfied for $v = 0, 1, 2$, then for $f \in H_\omega^\tau$ we have

$$\lim_{n \to \infty} \|T_n^\tau(f) - f\|_{C_B} = 0.$$

Proof. Supposing that $f \in H_\omega^\tau$, we deduce from (1.6) that for any $\epsilon > 0$ there exist a positive $\delta > 0$ such that

$$|f(t) - f(x)| < \epsilon$$

whenever $\left| \tau(t) \frac{\tau(t)}{1+\tau(t)} - \tau(x) \frac{\tau(x)}{1+\tau(x)} \right| < \delta$. On the other hand, from the boundedness of $f$, we get

$$|f(t) - f(x)| < \frac{2M}{\delta^2} \left[ \frac{\tau(t) - \tau(x)}{(1+\tau(t))(1+\tau(x))} \right]^2$$

when $\left| \tau(t) \frac{\tau(t)}{1+\tau(t)} - \tau(x) \frac{\tau(x)}{1+\tau(x)} \right| \geq \delta$. In this case, for all $t, x \in [0, \infty)$ we can write

$$|f(t) - f(x)| < \epsilon + \frac{2M}{\delta^2} \left[ \frac{\tau(t) - \tau(x)}{(1+\tau(t))(1+\tau(x))} \right]^2. \tag{2.2}$$

Since $T_n^\tau$ is linear and positive, then applying the operator on $f(t) - f(x)$, we obtain

$$\|T_n^\tau(f; x) - f(x)\|_{C_B} \leq \|T_n^\tau(|f(t) - f(x)|; x)\|_{C_B} + \|f(x)\|_{C_B} \|T_n^\tau(1; x) - 1\|_{C_B} = : I_n^1 + I_n^2.$$
From (2.1) and the boundedness of $f$, $\|f\|_{C_B} \leq M$, we get
\[
\lim_{n \to \infty} I_n^2 = 0.
\]
On the other hand, (2.1) gives that
\[
\left\| T_n^r \left( \left( \frac{\tau(t) - \tau(x)}{(1 + \tau(t))(1 + \tau(x))} \right)^2 ; x \right) \right\|_{C_B} < C \epsilon_n
\]
where $\epsilon_n \to 0$ as $n \to \infty$ and $C$ is a positive constant independent of $n$.
Moreover, from (2.1), (2.2) and (2.3) it follows that
\[
I_n^1 \leq \epsilon \|T_n^r (1;x)\|_{C_B} + \frac{2M}{\delta^2} \left\| T_n^r \left( \left( \frac{\tau(t) - \tau(x)}{(1 + \tau(t))(1 + \tau(x))} \right)^2 ; x \right) \right\|_{C_B}
\]
\[
\leq \epsilon (1 + \epsilon_n) + \frac{2M}{\delta^2} C \epsilon_n.
\]
Hence we deduce
\[
\lim_{n \to \infty} I_n^1 = 0,
\]
which completes the proof.

**Theorem 2.** Let $L_n^r$ be the operator defined by (1.3). Then for any $f \in H_n^r$ we have
\[
\lim_{n \to \infty} \|L_n^r f - f\|_{C_B} = 0.
\]

**Proof.** From Theorem 1 it suffices to show that (2.1) hold for $L_n^r$. Indeed, from (1.5) we easily obtain that
\[
L_n^r (1;x) = 1,
\]
\[
\left\| L_n^r \left( \frac{\tau(t)}{1 + \tau(t)} ; x \right) \frac{\tau(x)}{1 + \tau(x)} \right\|_{C_B} = \left\| \left( \frac{n}{n+1} - 1 \right) \frac{\tau(x)}{1 + \tau(x)} \right\|_{C_B}
\]
\[
\leq \frac{1}{n+1},
\]
and
\[
\left\| L_n^r \left( \left( \frac{\tau(t)}{1 + \tau(t)} \right)^2 ; x \right) - \left( \frac{\tau(x)}{1 + \tau(x)} \right)^2 \right\|_{C_B}
\]
\[
\leq \left\| \frac{n}{(n+1)^2} + \frac{n}{(n+1)^2} + 1 \right\|_{C_B}
\]
\[
\leq \frac{4n + 1}{(n+1)^2}.
\]
Therefore, using (2.4) – (2.6) we get that (2.1) holds. By Theorem 1, the proof is completed.

3. Monotonicity Result for $L_n^r$

**Theorem 3.** If $f$ is $\tau$-convex and non-increasing on $[0, \infty)$, then we have

$$L_n^r (f; x) \geq L_{n+1}^r (f; x)$$

for $n \in \mathbb{N}$.

**Proof.** From (1.3), we can write

$$L_n^r (f; x) - L_{n+1}^r (f; x)$$

$$= \frac{1}{(1 + \tau (x))^{n+1}} \sum_{k=0}^{n} (f \circ \tau^{-1}) \left( \frac{k}{n-k+1} \right) \left( \begin{array}{c} n \\ k \end{array} \right) \tau (x)^k$$

$$+ \frac{1}{(1 + \tau (x))^{n+1}} \sum_{k=0}^{n} (f \circ \tau^{-1}) \left( \frac{k}{n-k+1} \right) \left( \begin{array}{c} n \\ k \end{array} \right) \tau (x)^{k+1} \quad (3.1)$$

$$- \frac{1}{(1 + \tau (x))^{n+1}} \sum_{k=0}^{n+1} (f \circ \tau^{-1}) \left( \frac{k}{n-k+2} \right) \left( \begin{array}{c} n+1 \\ k \end{array} \right) \tau (x)^k \quad (3.2)$$

removing, $n$-term and $n+1$-term from (3.1), (3.2) respectively and taking into account of the fact $\frac{n+1}{n+1-k} \left( \begin{array}{c} n \\ k \end{array} \right) = \left( \begin{array}{c} n+1 \\ k \end{array} \right)$, we have

$$L_n^r (f; x) - L_{n+1}^r (f; x)$$

$$= \left( \frac{\tau (x)}{1 + \tau (x)} \right)^{n+1} \left[ (f \circ \tau^{-1}) (n) - (f \circ \tau^{-1}) (n+1) \right]$$

$$+ \frac{1}{(1 + \tau (x))^{n+1}} \left[ (f \circ \tau^{-1}) (0) - (f \circ \tau^{-1}) (0) \right]$$

$$+ \frac{1}{(1 + \tau (x))^{n+1}} \sum_{k=1}^{n} (f \circ \tau^{-1}) \left( \frac{k}{n-k+1} \right) \left( \begin{array}{c} n \\ k \end{array} \right) \tau (x)^k$$

$$+ \frac{1}{(1 + \tau (x))^{n+1}} \sum_{k=0}^{n-1} (f \circ \tau^{-1}) \left( \frac{k}{n-k+1} \right) \left( \begin{array}{c} n \\ k \end{array} \right) \tau (x)^{k+1}$$

$$- \frac{1}{(1 + \tau (x))^{n+1}} \sum_{k=1}^{n} (f \circ \tau^{-1}) \left( \frac{k}{n-k+2} \right) \left( \begin{array}{c} n+1 \\ n+1-k \end{array} \right) \tau (x)^k$$
\[ \begin{align*}
  &\left( \frac{\tau(x)}{1+\tau(x)} \right)^{n+1} [(f \circ \tau^{-1})(n) - (f \circ \tau^{-1})(n+1)] \\
  &\quad + \frac{1}{(1+\tau(x))^{n+1}} \sum_{k=1}^{n} (f \circ \tau^{-1}) \left( \frac{k}{n-k+1} \right) \left( \binom{n}{k} \right) \tau(x)^{k} \\
  &\quad + \frac{1}{(1+\tau(x))^{n+1}} \sum_{k=1}^{n} (f \circ \tau^{-1}) \left( \frac{k-1}{n-k+2} \right) \left( \binom{n}{k-1} \right) \tau(x)^{k} \\
  &\quad - \frac{1}{(1+\tau(x))^{n+1}} \sum_{k=1}^{n} (f \circ \tau^{-1}) \left( \frac{k}{n-k+2} \right) \left( \binom{n+1}{k+1} \right) \tau(x)^{k} \\
  &\left( \frac{\tau(x)}{1+\tau(x)} \right)^{n+1} [(f \circ \tau^{-1})(n) - (f \circ \tau^{-1})(n+1)] \\
  &\quad + \frac{1}{(1+\tau(x))^{n+1}} \sum_{k=1}^{n} \left( \binom{n}{k} \right) \tau(x)^{k} \\
  &\quad \times \left[ (f \circ \tau^{-1}) \left( \frac{k}{n-k+1} \right) + \frac{k}{n+1-k} (f \circ \tau^{-1}) \left( \frac{k-1}{n-k+2} \right) \\
  &\quad - \frac{n+1}{n+1-k} (f \circ \tau^{-1}) \left( \frac{k}{n-k+2} \right) \right].
\end{align*} \]

By taking \( \alpha_1 = \frac{n-k+1}{n+1} \geq 0, \alpha_2 = \frac{k}{n+1} \geq 0, \alpha_1 + \alpha_2 = 1, \) and \( x_1 = \frac{k}{n-k+1}, \)
\( x_2 = \frac{k-1}{n-k+2} \) one has
\[ \alpha_1 x_1 + \alpha_2 x_2 = \frac{k}{n+1} + \frac{k}{n+1} \frac{k-1}{n-k+2} = \frac{k}{n-k+2}. \]

Therefore, we obtain that
\[ L_n^\tau (f; x) - L_{n+1}^\tau (f; x) \geq 0 \]
by \( \tau \)-convexity and non-increasingness of \( f \) for \( x \in [0, \infty) \).

\[ \square \]

References


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