ON THE NEW MULTI-STEP ITERATION PROCESS FOR
MULTI-VALUED MAPPINGS IN A COMPLETE GEODESIC
SPACE

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ABSTRACT. The purpose of this paper is to prove the strong and \( \Delta \)-convergence theorems of the new multi-step iteration process for multi-valued quasi-nonexpansive mappings in a complete geodesic space. Our results extend and improve some results in the literature.

1. INTRODUCTION

For a real number \( \kappa \), a CAT(\( \kappa \)) space is defined by a geodesic metric space whose geodesic triangle is sufficiently thinner than the corresponding comparison triangle in a model space with the curvature \( \kappa \). The concept of this space has been studied by a large number of researchers (see [3, 5, 7, 8, 10, 14, 15]). Since any CAT(\( \kappa \)) space is a CAT(\( \kappa' \)) space for \( \kappa' \geq \kappa \) (see [2, p.165]), all results for a CAT(0) space can immediately be applied to any CAT(\( \kappa \)) space with \( \kappa \leq 0 \). Moreover, a CAT(\( \kappa \)) space with positive \( \kappa \) can be treated as a CAT(1) space by changing the scale of the space. So we are interested in a CAT(1) space.


Let \( K \) be a nonempty closed convex subset of a CAT(1) space \( X \) and \( \{ \alpha_n \}, \{ \beta_n \} \subset [0, 1] \) and \( T : K \to 2^K \) be a multi-valued mapping whose values are nonempty proximinal subsets of \( K \). For each \( x \in K \), let \( P_T : K \to 2^K \) be a multi-valued mapping defined by \( P_T(x) = \{ u \in T(x) : d(x, u) = d(x, T(x)) \} \).

(A) For \( x_1 \in K \), the sequence of Ishikawa iteration is defined by

\[
\begin{align*}
    x_{n+1} &= \alpha_n z_n \oplus (1 - \alpha_n)x_n, \\
    y_n &= \beta_n z'_n \oplus (1 - \beta_n)x_n, \quad n \geq 1, \\
\end{align*}
\]

where \( z_n \in T(y_n) \) and \( z'_n \in T(x_n) \).

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mappings in a Banach space as follows. Panyanak [11] proved, under some suitable assumptions, that the sequences defined by (A) and (B) converge strongly to a fixed point of $z_n \in P_T(y_n)$ and $z_n \in P_T(x_n)$.


$$x_{n+1} = (1 - \alpha_n)y_n^1 + \alpha_n T(y_n^1),$$
$$y_n^1 = (1 - \beta_n^1)y_n^1 + \beta_n^1 T(y_n^1),$$
$$y_n^2 = (1 - \beta_n^2)y_n^3 + \beta_n^2 T(y_n^3),$$
$$\vdots$$
$$y_n^{k-2} = (1 - \beta_n^{k-2})y_n^{k-1} + \beta_n^{k-2} T(y_n^{k-1}),$$
$$y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1} T(x_n), \quad n \geq 1,$$

or, in short,

$$x_{n+1} = (1 - \alpha_n)y_n^1 + \alpha_n T(y_n^1),$$
$$y_n^i = (1 - \beta_n^i)y_n^i + \beta_n^i T(y_n^i), \quad i = 1, 2, \ldots, k - 2,$$
$$y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1} T(x_n), \quad n \geq 1. \quad (1.2)$$

By taking $k = 3$ and $k = 2$ in (1.2), we obtain the SP-iteration process of Phuengrattana and Suantai [12] and the two-step iteration process of Thianwan [18], respectively. Recently, Başarır and Şahin [1] studied the iteration process (1.2) for single-valued mappings in a CAT(0) space.

Now, we apply the new multi-step iteration process for multi-valued mappings in a CAT(1) space as follows.

(C) Let $T$ be a multi-valued quasi-nonexpansive mapping from $K$ into $2^K$. Then for an arbitrary fixed order $k \geq 2$ and $x_1 \in K$, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1 - \alpha_n)y_n^1 + \alpha_n z_n^1,$$
$$y_n^1 = (1 - \beta_n^1)y_n^1 + \beta_n^1 z_n^2,$$
$$y_n^2 = (1 - \beta_n^2)y_n^3 + \beta_n^2 z_n^3,$$
$$\vdots$$
$$y_n^{k-2} = (1 - \beta_n^{k-2})y_n^{k-1} + \beta_n^{k-2} z_n^{k-1},$$
$$y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1} z_n', \quad n \geq 1,$$

or, in short,

$$x_{n+1} = (1 - \alpha_n)y_n^1 + \alpha_n z_n^1,$$
$$y_n^i = (1 - \beta_n^i)y_n^i + \beta_n^i z_n^i, \quad i = 1, 2, \ldots, k - 2,$$
$$y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1} z_n', \quad n \geq 1. \quad (1.3)$$

where $z_n^i \in T(y_n^i)$ for each $i = 1, 2, \ldots, k - 1$ and $z_n' \in T(x_n)$. 
(D) Let $T$ be a multi-valued quasi-nonexpansive mapping from $K$ into $P(K)$. Then for an arbitrary fixed order $k \geq 2$ and $x_1 \in K$, the sequence \{x_n\} is defined as in (1.3) where $z^n_i \in Pr\ y^n_i$ for each $i = 1, 2, ..., k - 1$ and $z^n_k \in Pr\ y^n_k$.

In this paper, motivated by the above results, we prove some theorems related to the strong and $\Delta$-convergence of the new multi-step iteration processes defined by (C) and (D) for multi-valued quasi-nonexpansive mappings in a CAT(1) space.

2. Preliminaries and Lemmas

Let $K$ be a nonempty subset of a metric space $(X, d)$. The diameter of $K$ is defined by $\text{diam}(K) = \sup \{d(u, v) : u, v \in K\}$. The set $K$ is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that $d(x, k) = d(x, K)$, where $d(x, K) = \inf \{d(x, y) : y \in K\}$. Let $2^K, C(K)$ and $P(K)$ denote the family of nonempty all subsets, nonempty closed all subsets and nonempty proximinal all subsets of $K$, respectively. The Hausdorff distance on $2^K$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}$$

for all $A, B \in 2^K$.

An element $p \in K$ is a fixed point of $T$ if $p \in T(p)$. The set of all fixed points of $T$ is denoted by $F(T)$.

**Definition 1.** A multi-valued mapping $T : K \to 2^K$ is said to

(i) be nonexpansive if $H(T(x), T(y)) \leq d(x, y)$ for all $x, y \in K$;

(ii) be quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(T(x), T(p)) \leq d(x, p)$ for all $x \in K$ and $p \in F(T)$;

(iii) satisfy Condition(I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, T(x)) \geq f(d(x, F(T))) \quad \text{for all} \quad x \in K;$$

(iv) be hemi-compact if for any sequence $\{x_n\}$ in $K$ such that

$$\lim_{n \to \infty} d(x_n, T(x_n)) = 0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k \to \infty} x_{n_k} = p \in K$;

(v) be $\Delta$-demiclosed if for any $\Delta$-convergent sequence $\{x_n\}$ in $K$, its $\Delta$-limit belongs to $F(T)$ whenever $\lim_{n \to \infty} d(x_n, T(x_n)) = 0$.

It is clear that each multi-valued nonexpansive mapping with a fixed point is quasi-nonexpansive. But there exist multi-valued quasi-nonexpansive mappings that are not nonexpansive.

**Example 1.** [16, p.838-839] Let $K = [0, +\infty)$ be endowed with the usual metric and $T : K \to 2^K$ be defined by

$$T(x) = \begin{cases} \{0\} & \text{if} \ x \leq 1, \\ [x - \frac{3}{4}, x - \frac{1}{4}] & \text{if} \ x > 1. \end{cases}$$
Then, clearly \( F(T) = \{0\} \) and for any \( x \in K \) we have \( H(T(x), T(0)) \leq |x - 0| \) hence \( T \) is quasi-nonexpansive. However, if \( x = 2, y = 1 \) we get that \( H(T(x), T(y)) > |x - y| = 1 \) and hence not nonexpansive.

**Remark 1.** From the proof of Lemma 5.1 in [9], it is easy to see that if \( T \) is a multi-valued nonexpansive mapping then \( T \) is \( \triangle \)-demiclosed.

The following lemma will be useful in this study.

**Lemma 1.** [11, Lemma 2.2] Let \( K \) be a nonempty subset of a metric space \((X, d)\) and \( T : K \to P(K) \) be a multi-valued mapping. Then
\[
\begin{align*}
(i) \quad & d(x, T(x)) = d(x, P_T(x)) \text{ for all } x \in K; \\
(ii) \quad & x \in F(T) \iff x \in F(P_T) \iff P_T(x) = \{x\}; \\
(iii) \quad & F(T) = F(P_T).
\end{align*}
\]

Let \( X \) be a metric space with a metric \( d \) and \( x, y \in X \) with \( d(x, y) = l \). A geodesic path from \( x \) to \( y \) is an isometry \( c : [0, l] \subseteq \mathbb{R} \to X \) such that \( c(0) = x \) and \( c(l) = y \). The image of \( c \) is called a geodesic segment joining \( x \) and \( y \). A geodesic segment joining \( x \) and \( y \) is not necessarily unique in general. When it is unique, this geodesic segment is denoted by \([x, y] \). This means that \( z \in [x, y] \) if and only if there exists \( \alpha \in [0, 1] \) such that \( d(x, z) = (1 - \alpha)d(x, y) \) and \( d(y, z) = \alpha d(x, y) \). In this case, we write \( z = \alpha x \oplus (1 - \alpha)y \). The space \((X, d)\) is said to be a geodesic space if every two points of \( X \) are joined by a geodesic and \( X \) is said to be a uniquely geodesic if there is exactly one geodesic joining \( x \) to \( y \) for each \( x, y \in X \).

In a geodesic space \((X, d)\), the metric \( d : X \times X \to \mathbb{R} \) is convex if for any \( x, y, z \in X \) and \( \alpha \in [0, 1] \), one has
\[
d(x, \alpha y \oplus (1 - \alpha)z) \leq \alpha d(x, y) + (1 - \alpha)d(x, z).
\]

Let \( D \in (0, \infty) \). If for every \( x, y \in X \) with \( d(x, y) < D \), a geodesic from \( x \) to \( y \) exists, then \( X \) is said to be \( D \)-geodesic space. Moreover, if such a geodesic is unique for each pair of points then \( X \) is said to be a \( D \)-uniquely geodesic. Notice that \( X \) is a geodesic space if and only if it is a \( D \)-geodesic space.

To define a CAT(\( \kappa \)) space, we use the following concept called model space. For \( \kappa = 0 \), the two-dimensional model space \( M^2_0 = M^2_0 \) is the Euclidean space \( \mathbb{R}^2 \) with the metric induced from the Euclidean norm. For \( \kappa > 0 \), \( M^2_\kappa \) is the two-dimensional sphere \( \left( \frac{1}{\sqrt{-\kappa}} \right) S^2 \) whose metric is a length of a minimal great arc joining each two points. For \( \kappa < 0 \), \( M^2_\kappa \) is the two-dimensional hyperbolic space \( \left( \frac{1}{\sqrt{-\kappa}} \right) \mathbb{H}^2 \) with the metric defined by a usual hyperbolic distance.

The diameter of \( M^2_\kappa \) is denoted by
\[
D_\kappa = \begin{cases} 
\frac{\pi}{\sqrt{-\kappa}} & \kappa > 0, \\
+\infty & \kappa \leq 0.
\end{cases}
\]

A geodesic triangle \( \triangle(x, y, z) \) in a geodesic space \((X, d)\) consists of three points \( x, y, z \) in \( X \) (the vertices of \( \triangle \)) and three geodesic segments between each pair of
vertices (the edges of \(\triangle\)). We write \(p \in \triangle(x, y, z)\) when \(p \in [x, y] \cup [y, z] \cup [z, x]\). A comparison triangle for \(\triangle(x, y, z)\) is a triangle \(\triangle(\overline{x}, \overline{y}, \overline{z})\) in \(M^n_2\) such that

\[
d(x, y) = d_{M^2}(\overline{x}, \overline{y}), d(y, z) = d_{M^2}(\overline{y}, \overline{z}) \quad \text{and} \quad d(z, x) = d_{M^2}(\overline{z}, \overline{x}).
\]

If \(\kappa \leq 0\), then such a comparison triangle always exists in \(M^n_2\). If \(\kappa > 0\), then such a triangle exists whenever \(d(x, y) + d(y, z) + d(z, x) < 2D_\kappa\). A point \(\overline{p} \in [\overline{x}, \overline{y}]\) is called a comparison point for \(p \in [x, y]\) if \(d(x, p) = d_{M^2}(\overline{x}, \overline{p})\).

A geodesic triangle \(\triangle(x, y, z)\) in \(X\) is said to satisfy the CAT(\(\kappa\)) inequality if for any \(p, q \in \triangle(x, y, z)\) and for their comparison points \(\overline{p}, \overline{q} \in \triangle(\overline{x}, \overline{y}, \overline{z})\), one has

\[
d(p, q) \leq d_{M^2}(\overline{p}, \overline{q}).
\]

We are ready to introduce the concept of CAT(\(\kappa\)) space in the following definition taken from [2].

**Definition 2.** If \(\kappa \leq 0\), then \(X\) is called a CAT(\(\kappa\)) space if \(X\) is a geodesic space such that all of its geodesic triangles satisfy the CAT(\(\kappa\)) inequality. If \(\kappa > 0\), then \(X\) is called a CAT(\(\kappa\)) space if it is \(D_\kappa\)-geodesic and any geodesic triangle \(\triangle(x, y, z)\) in \(X\) with \(d(x, y) + d(y, z) + d(z, x) < 2D_\kappa\) satisfies the CAT(\(\kappa\)) inequality.

It follows from [2, p.160] that any CAT(\(\kappa\)) space is \(D_\kappa\)-uniquely geodesic.

Let \(\{x_n\}\) be a bounded sequence in a metric space \(X\). For \(x \in X\), we put \(r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n)\). The asymptotic radius \(r(\{x_n\})\) of \(\{x_n\}\) is defined by

\[
r(\{x_n\}) = \inf_{x \in X} r(x, \{x_n\}).
\]

Further, the asymptotic center of \(\{x_n\}\) is defined by

\[
A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.
\]

Recall that the sequence \(\{x_n\}\) is \(\triangle\)-convergent to \(x \in X\) if \(x\) is the unique asymptotic center of any subsequence of \(\{x_n\}\).

**Lemma 2.** [3] Let \((X, d)\) be a complete CAT(1) space and \(\{x_n\}\) be a sequence in \(X\). If \(r(\{x_n\}) < \pi/2\), then the following statements hold:

(i) \(A(\{x_n\})\) consists of exactly one point,

(ii) \(\{x_n\}\) has a \(\triangle\)-convergent subsequence.

The following lemma is needed for our main result.

**Lemma 3.** [11, Lemma 2.4] If \((X, d)\) is a CAT(1) space with \(\text{diam}(X) < \pi/2\), then there exist a constant \(K > 0\) such that

\[
d^2((1 - \alpha)x \oplus \alpha y, z) \leq (1 - \alpha)d^2(x, z) + \alpha d^2(y, z) - \frac{K}{2} \alpha(1 - \alpha)d^2(x, y)
\]

for all \(\alpha \in [0, 1]\) and \(x, y, z \in X\).
3. Main results

We start with the following key lemmas.

Lemma 4. Let \((X, d)\) be a CAT(1) space with convex metric, \(K\) be a nonempty, closed and convex subset of \(X\) and let \(T : K \to 2^K\) be a multi-valued quasi-nonexpansive mapping with \(F(T) \neq \emptyset\) and \(T(p) = \{p\}\) for each \(p \in F(T)\). Then, for the sequence \(\{x_n\}\) defined by (1.3), \(\lim_{n \to \infty} d(x_n, p)\) exists for each \(p \in F(T)\).

Proof. For any \(p \in F(T)\), we have

\[
d(x_{n+1}, p) = d((1 - \alpha_n)y_n^1 \oplus \alpha_n z_n^1, p) \\
\leq (1 - \alpha_n)d(y_n^1, p) + \alpha_n d(z_n^1, p) \\
\leq (1 - \alpha_n)d(y_n^1, p) + \alpha_n H (T(y_n^1), T(p)) \\
\leq (1 - \alpha_n)d(y_n^1, p) + \alpha_n d(y_n^1, p) \\
= d(y_n^1, p).
\]

Also, we obtain

\[
d(y_n^1, p) = d((1 - \beta_n^1)y_n^2 \oplus \beta_n^1 z_n^2, p) \\
\leq (1 - \beta_n^1)d(y_n^2, p) + \beta_n^1 d(z_n^2, p) \\
\leq (1 - \beta_n^1)d(y_n^2, p) + \beta_n^1 H (T(y_n^2), T(p)) \\
\leq (1 - \beta_n^1)d(y_n^2, p) + \beta_n^1 d(y_n^2, p) \\
= d(y_n^2, p).
\]

Continuing the above process, we get

\[
d(x_{n+1}, p) \leq d(y_n^1, p) \leq d(y_n^2, p) \leq \ldots \leq d(y_n^{k-1}, p) \leq d(x_n, p).
\]

This inequality guarantees that the sequence \(\{d(x_n, p)\}\) is non-increasing and bounded below, and so \(\lim_{n \to \infty} d(x_n, p)\) exists for any \(p \in F(T)\). \(\square\)

Lemma 5. Let \((X, d)\) be a CAT(1) space with convex metric and \(\text{diam}(X) < \pi/2\), \(K\) be a nonempty, closed and convex subset of \(X\) and let \(T : K \to 2^K\) be a multi-valued quasi-nonexpansive mapping with \(F(T) \neq \emptyset\) and \(T(p) = \{p\}\) for each \(p \in F(T)\). Let \(\{x_n\}\) be the sequence defined by (1.3) with \(\beta_n^{k-1} \in [a, b] \subset (0, 1)\). Then \(\lim_{n \to \infty} d(x_n, T(x_n)) = 0\).

Proof. It follows from Lemma 4 that \(\lim_{n \to \infty} d(x_n, p)\) exists for each \(p \in F(T)\). We may assume that \(\lim_{n \to \infty} d(x_n, p) = r > 0\). Otherwise the proof is trivial. By using
(3.1), we get \( \lim_{n \to \infty} d(y_{n}^{k-1}, p) = r \). By Lemma 3, we also have
\[
d^2(y_{n}^{k-1}, p) = d^2((1 - \beta_{n}^{k-1})x_n \oplus \beta_{n}^{k-1}z_n', p)
\leq (1 - \beta_{n}^{k-1})d^2(x_n, p) + \beta_{n}^{k-1}d^2(z_n', p) - \frac{K}{2} \beta_{n}^{k-1}(1 - \beta_{n}^{k-1})d^2(x_n, z_n')
\leq (1 - \beta_{n}^{k-1})d^2(x_n, p) + \beta_{n}^{k-1}H^2(T(x_n), T(p)) - \frac{K}{2} \beta_{n}^{k-1}(1 - \beta_{n}^{k-1})d^2(x_n, z_n')
\leq (1 - \beta_{n}^{k-1})d^2(x_n, p) + \beta_{n}^{k-1}d^2(x_n, p) - \frac{K}{2} \beta_{n}^{k-1}(1 - \beta_{n}^{k-1})d^2(x_n, z_n')
= d^2(x_n, p) - \frac{K}{2} \beta_{n}^{k-1}(1 - \beta_{n}^{k-1})d^2(x_n, z_n'),
\]
which implies that
\[
d^2(x_n, z_n') \leq \frac{2}{a(1-b)K} [d^2(x_n, p) - d^2(y_{n}^{k-1}, p)].
\]
Hence, \( \lim_{n \to \infty} d(x_n, z_n') = 0 \). Since \( d(x_n, T(x_n)) \leq d(x_n, z_n') \), then we obtain
\[
\lim_{n \to \infty} d(x_n, T(x_n)) = 0.
\]

We give the \( \Delta \)-convergence of the iteration process defined by (C) in a CAT(1) space.

**Theorem 1.** Let \( X, K \) and \( \{x_n\} \) satisfy the hypotheses of Lemma 5, \( X \) be a complete space and let \( T : K \to C(K) \) be a multi-valued quasi-nonexpansive mapping with \( F(T) \neq \emptyset \) and \( T(p) = \{p\} \) for each \( p \in F(T) \). If \( T \) is \( \Delta \)-demiclosed, then the sequence \( \{x_n\} \) is \( \Delta \)-convergent to a fixed point of \( T \).

**Proof.** Let \( \{u_n\} \) be a subsequence of \( \{x_n\} \). Since \( r(\{u_n\}) \leq r(\{x_n\}) < \pi/2 \), by Lemma 2(i), there exists a unique asymptotic center \( u \) of \( \{u_n\} \). Moreover, by Lemma 2(ii), there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \{v_n\} \) is \( \Delta \)-convergent to \( v \) for some \( v \in X \). Further, since \( \lim_{n \to \infty} d(v_n, T(v_n)) = 0 \) (by Lemma 5) and \( T \) is \( \Delta \)-demiclosed, we have \( v \in F(T) \). By Lemma 4, \( \lim_{n \to \infty} d(x_n, v) \) exists. Then we can show that \( u = v \). If not, from the uniqueness of the asymptotic center, we have
\[
\limsup_{n \to \infty} d(u_n, u) < \limsup_{n \to \infty} d(u_n, v) = \lim_{n \to \infty} d(x_n, v) = \limsup_{n \to \infty} d(v_n, v) < \limsup_{n \to \infty} d(v_n, u) \leq \limsup_{n \to \infty} d(u_n, u). \tag{3.2}
\]
This is a contradiction. Hence we get \( u = v \in F(T) \). Next, we show that for any subsequence of \( \{x_n\} \), its asymptotic center consists of the unique element. Let \( \{u_n\} \) be a subsequence of \( \{x_n\} \) with \( A(\{u_n\}) = \{u\} \) and let \( A(\{x_n\}) = \{x\} \). We have already seen that \( u = v \). Finally, we show that \( x = v \). If not, then the existence of \( \lim_{n \to \infty} d(x_n, v) \) and the uniqueness of the asymptotic center imply that there exists a contradiction as (3.2). Hence we get \( x = v \in F(T) \). Therefore, the sequence \( \{x_n\} \) is \( \triangle \)-convergent to a fixed point of \( T \).

It is well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space (see [2]). If \((X, d)\) is a CAT(0) space and \( K \) is a convex subset of \( X \), then \((K, d)\) is a CAT(0) space and hence it is a CAT(\( \kappa \)) space with \( \kappa > 0 \). Now we give an example of such mappings which are multi-valued quasi-nonexpansive mappings as in Theorem 1.

**Example 2.** Let \( X \) be the real line with the usual metric and let \( K = [0, 1] \). Define two mappings \( S, T : K \to C(K) \) by

\[
S(x) = \left[ 0, \frac{x}{4} \right] \quad \text{and} \quad T(x) = \left[ 0, \frac{x}{2} \right].
\]

Obviously, \( F(S) = F(T) = \{0\} \). It is proved in [6, Example 2] that both \( S \) and \( T \) are multi-valued nonexpansive mappings. Therefore, they are multi-valued quasi-nonexpansive mappings. Additionally, for \( 0 \in F(S) = F(T) \), we have that \( S(0) = T(0) = \{0\} \).

We prove the strong convergence of the iteration process defined by (C) in a CAT(1) space as follows.

**Theorem 2.** Let \( X, K, T \) and \( \{x_n\} \) be the same as in Theorem 1.

(i) If \( T \) satisfies Condition (I), then the sequence \( \{x_n\} \) is convergent strongly to a point in \( F(T) \).

(ii) If \( T \) is hemi-compact and continuous, then the sequence \( \{x_n\} \) is convergent strongly to a point in \( F(T) \).

Proof. (i) By Condition (I) and Lemma 5, we have

\[
\lim_{n \to \infty} f(d(x_n, F(T))) = \lim_{n \to \infty} d(x_n, T(x_n)) = 0.
\]

That is, \( \lim_{n \to \infty} f(d(x_n, F(T))) = 0 \). Since \( f \) is a non-decreasing function satisfying \( f(0) = 0 \) and \( f(r) > 0 \) for all \( r \in (0, \infty) \), it follows that \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \).

The proof of the remaining part follows the proof of Theorem 3.2 in [13], therefore we omit it.

(ii) From hemi-compactness of \( T \) and Lemma 5, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \lim_{k \to \infty} x_{n_k} = q \in K \). Since \( T \) is continuous, we have

\[
d(q, T(q)) = \lim_{k \to \infty} d(x_{n_k}, T(x_{n_k})) = 0.
\]
This implies that \( q \in F(T) \) since \( T(q) \) is closed. Thus \( \lim_{n \to \infty} d(x_n, q) \) exists by Lemma 4. Hence the sequence \( \{x_n\} \) is convergent strongly to a fixed point \( q \) of \( T \).

Since every nonexpansive mapping having a fixed point is quasi-nonexpansive, then we get the following corollary.

**Corollary 1.** Let \( X, K \) and \( \{x_n\} \) satisfy the hypotheses of Theorem 1 and \( T: K \to C(K) \) be a multi-valued nonexpansive mapping with \( F(T) \neq \emptyset \) and \( T(p) = \{p\} \) for each \( p \in F(T) \). Then the sequence \( \{x_n\} \) is \( \Delta \)-convergent to a fixed point of \( T \). Moreover, if \( T \) satisfies Condition (I) or is hemi-compact, then the sequence \( \{x_n\} \) is convergent strongly to a fixed point of \( T \).

To avoid the restriction of \( T \), that is, \( T(p) = \{p\} \) for each \( p \in F(T) \), we use the iteration process defined by (D). Using this iteration, we give the \( \Delta \)-convergence result in a CAT(1) space.

**Theorem 3.** Let \( (X, d) \) be a complete CAT(1) space with convex metric and \( \text{diam}(X) < \pi/2 \), \( K \) be a nonempty, closed and convex subset of \( X \) and let \( T: K \to P(K) \) be a multi-valued mapping with \( F(T) \neq \emptyset \) and \( P_T \) is quasi-nonexpansive. Let \( \{x_n\} \) be the sequence defined by (D) with \( \beta_n^{-1} \in [a, b] \subset (0, 1) \). If \( T \) is \( \Delta \)-demiclosed, then the sequence \( \{x_n\} \) is \( \Delta \)-convergent to a fixed point of \( T \).

**Proof.** It follows from Lemma 1 that \( d(x, P_T(x)) = d(x, T(x)) \) for all \( x \in K \), \( F(P_T) = F(T) \) and \( P_T(p) = \{p\} \) for each \( p \in F(P_T) \). The \( \Delta \)-demiclosedness of \( P_T \) follows from the \( \Delta \)-demiclosedness of \( T \). Then, applying Theorem 1 to the mapping \( P_T \), we can conclude that the sequence \( \{x_n\} \) is \( \Delta \)-convergent to a point \( p \in F(P_T) = F(T) \).

We now present an example of a multi-valued mapping \( T \) for which \( P_T \) is quasi-nonexpansive.

**Example 3.** Let \( X \) and \( K \) be defined as in Example 2. Define a mapping \( T: K \to P(K) \) by

\[
T(x) = \begin{cases} 
[0, x] & \text{if } x \in [0, \frac{1}{2}], \\
\{\frac{1}{2}\} & \text{if } x \in \left(\frac{1}{2}, 1\right].
\end{cases}
\]

Then, we have

\[
P_T(x) = \begin{cases} 
\{x\} & \text{if } x \in \left[0, \frac{1}{2}\right], \\
\{\frac{1}{2}\} & \text{if } x \in \left(\frac{1}{2}, 1\right].
\end{cases}
\]

Clearly, \( F(T) = F(P_T) = \{x : 0 \leq x \leq \frac{1}{2}\} \). It is proved in [17, Example 5] that \( T \) is not nonexpansive and \( P_T \) is quasi-nonexpansive.

We give several strong convergence results of the iteration process defined by (D) in a CAT(1) space.

**Theorem 4.** Let \( X, K, T \) and \( \{x_n\} \) be the same as in Theorem 3. If \( T \) satisfies Condition (I), then the sequence \( \{x_n\} \) is convergent strongly to a fixed point of \( T \).
Proof. By following the same proof of Theorem 3.4 in [11] and using Lemma 1, we can obtain that $P_T$ satisfies Condition (I) and $P_T(x)$ is closed for any $x \in K$. Then, applying Theorem 2(i) to the mapping $P_T$, we can conclude that the sequence $\{x_n\}$ is convergent strongly to a fixed point of $T$.

Theorem 5. Let $X, K$ and $\{x_n\}$ satisfy the hypotheses of Theorem 3 and $T : K \rightarrow P(K)$ be a multi-valued hemi-compact mapping with $F(T) \neq \emptyset$ and $P_T$ is quasi-nonexpansive and continuous. Then the sequence $\{x_n\}$ is convergent strongly to a fixed point of $T$.

Proof. From the hemi-compactness of $T$, we can prove that $P_T$ is hemi-compact. The conclusion follows from Theorem 2(ii).

Corollary 2. Let $X, K$ and $\{x_n\}$ be the same as in Theorem 3 and $T : K \rightarrow P(K)$ be a multi-valued mapping with $F(T) \neq \emptyset$ and $P_T$ is nonexpansive. Then the sequence $\{x_n\}$ is $\Delta$-convergent to a fixed point of $T$. Moreover, if $T$ satisfies Condition (I) or is hemi-compact, then the sequence $\{x_n\}$ is convergent strongly to a fixed point of $T$.

References


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