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ON THE NEW MULTI-STEP ITERATION PROCESS FOR MULTI-VALUED MAPPINGS IN A COMPLETE GEODESIC SPACE

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Abstract. The purpose of this paper is to prove the strong and $\triangle$-convergence theorems of the new multi-step iteration process for multi-valued quasi-nonexpansive mappings in a complete geodesic space. Our results extend and improve some results in the literature.

1. Introduction

For a real number κ, a CAT(κ) space is defined by a geodesic metric space whose geodesic triangle is sufficiently thinner than the corresponding comparison triangle in a model space with the curvature κ. The concept of this space has been studied by a large number of researchers (see [3, 5, 7, 8, 10, 14, 15]). Since any CAT(κ) space is a CAT($\kappa^{'}$) space for $\kappa^{'} \geq \kappa$ (see [2, p.165]), all results for a CAT(0) space can immediately be applied to any CAT(κ) space with $\kappa \leq 0$. Moreover, a CAT(κ) space with positive κ can be treated as a CAT(1) space by changing the scale of the space. So we are interested in a CAT(1) space.

Panyanak [11] studied the Ishikawa iteration process for multi-valued mappings in a CAT(1) space as follows.

Let K be a nonempty closed convex subset of a CAT(1) space X and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $T : K \to 2^K$ be a multi-valued mapping whose values are nonempty proximinal subsets of K. For each $x \in K$, let $P_T : K \to 2^K$ be a multi-valued mapping defined by $P_T(x) = \{u \in T(x) : d(x,u) = d(x, T(x))\}$.

(A) For $x_1 \in K$, the sequence of Ishikawa iteration is defined by

$$\begin{cases} x_{n+1} = \alpha_n z_n \oplus (1-\alpha_n)x_n, \\ y_n = \beta_n z_n^{'} \oplus (1-\beta_n)x_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $z_n \in T(y_n)$ and $z_n^{'} \in T(x_n)$.

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(B) For $x_1 \in K$, the sequence of Ishikawa iteration is defined as in (1.1) where $z_n \in P_T(y_n)$ and $z_n^{'} \in P_T(x_n)$.

Panyanak [11] proved, under some suitable assumptions, that the sequences defined by (A) and (B) converge strongly to a fixed point of T in a CAT(1) space.

Gürsoy et al. [4] introduced a new multi-step iteration process for single-valued mappings in a Banach space as follows.

For an arbitrary fixed order $k \geq 2$ and $x_1 \in K$,

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n^1 + \alpha_n T(y_n^1), \\ y_n^1 = (1-\beta_n^1)y_n^2 + \beta_n^1 T(y_n^2), \\ y_n^2 = (1-\beta_n^2)y_n^3 + \beta_n^2 T(y_n^3), \\ \vdots \\ y_n^{k-2} = (1-\beta_n^{k-2})y_n^{k-1} + \beta_n^{k-2} T(y_n^{k-1}), \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T(x_n), \quad n \geq 1, \end{cases}$$

or, in short,

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n^1 + \alpha_n T(y_n^1), \\ y_n^i = (1-\beta_n^i)y_n^{i+1} + \beta_n^i T(y_n^{i+1}), \; i = 1, 2, ..., k-2, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T(x_n), \quad n \geq 1. \end{cases} \tag{1.2}$$

By taking $k = 3$ and $k = 2$ in (1.2), we obtain the SP-iteration process of Phuengrattana and Suantai [12] and the two-step iteration process of Thianwan [18], respectively. Recently, Başarır and Şahin [1] studied the iteration process (1.2) for single-valued mappings in a CAT(0) space.

Now, we apply the new multi-step iteration process for multi-valued mappings in a CAT(1) space as follows.

(C) Let T be a multi-valued quasi-nonexpansive mapping from K into 2^K. Then for an arbitrary fixed order $k \geq 2$ and $x_1 \in K$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n^1 \oplus \alpha_n z_n^1, \\ y_n^1 = (1-\beta_n^1)y_n^2 \oplus \beta_n^1 z_n^2, \\ y_n^2 = (1-\beta_n^2)y_n^3 \oplus \beta_n^2 z_n^3, \\ \vdots \\ y_n^{k-2} = (1-\beta_n^{k-2})y_n^{k-1} \oplus \beta_n^{k-2} z_n^{k-1}, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1} z_n^{'}, \quad n \geq 1, \end{cases}$$

or, in short,

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n^1 \oplus \alpha_n z_n^1, \\ y_n^i = (1-\beta_n^i)y_n^{i+1} \oplus \beta_n^i z_n^{i+1}, \; i = 1, 2, ..., k-2, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1} z_n^{'}, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $z_n^i \in T\left(y_n^i\right)$ for each $i = 1, 2, ..., k-1$ and $z_n^{'} \in T(x_n)$.

(D) Let T be a multi-valued quasi-nonexpansive mapping from K into $P(K)$. Then for an arbitrary fixed order $k \geq 2$ and $x_1 \in K$, the sequence $\{x_n\}$ is defined as in (1.3) where $z_n^i \in P_T\left(y_n^i\right)$ for each $i = 1, 2, ..., k-1$ and $z_n^{'} \in P_T\left(x_n\right)$.

In this paper, motivated by the above results, we prove some theorems related to the strong and $\triangle$-convergence of the new multi-step iteration processes defined by (C) and (D) for multi-valued quasi-nonexpansive mappings in a CAT(1) space.

2. Preliminaries and Lemmas

Let K be a nonempty subset of a metric space (X, d). The diameter of K is defined by $\operatorname{diam}(K) = \sup\{d(u, v) : u, v \in K\}$. The set K is called *proximinal* if for each $x \in X$, there exists an element $k \in K$ such that $d(x, k) = d(x, K)$, where $d(x, K) = \inf\{d(x, y) : y \in K\}$. Let $2^K, C(K)$ and $P(K)$ denote the family of nonempty all subsets, nonempty closed all subsets and nonempty proximinal all subsets of K, respectively. The Hausdorff distance on 2^K is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$$

for all $A, B \in 2^K$.

An element $p \in K$ is a fixed point of T if $p \in T(p)$. The set of all fixed points of T is denoted by $F(T)$.

Definition 1. *A multi-valued mapping $T : K \to 2^K$ is said to*

(i) be nonexpansive if $H(T(x), T(y)) \leq d(x, y)$ for all $x, y \in K$;

(ii) be quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(T(x), T(p)) \leq d(x, p)$ for all $x \in K$ and $p \in F(T)$;

(iii) satisfy Condition(I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, T(x)) \geq f(d(x, F(T))) \quad \text{for all} \quad x \in K;$$

(iv) be hemi-compact if for any sequence $\{x_n\}$ in K such that

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = p \in K$;

(v) be $\triangle$-demiclosed if for any $\triangle$-convergent sequence $\{x_n\}$ in K, its $\triangle$-limit belongs to $F(T)$ whenever $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$.

It is clear that each multi-valued nonexpansive mapping with a fixed point is quasi-nonexpansive. But there exist multi-valued quasi-nonexpansive mappings that are not nonexpansive.

Example 1. [16, p.838-839] *Let $K = [0, +\infty)$ be endowed with the usual metric and $T : K \to 2^K$ be defined by*

$$T(x) = \begin{cases} \{0\} & \text{if } x \leq 1, \\ \left[x - \frac{3}{4}, x - \frac{1}{3}\right] & \text{if } x > 1. \end{cases}$$

Then, clearly $F(T)=\{0\}$ *and for any* $x\in K$ *we have* $H(T(x),T(0))\leq|x-0|$ *hence* T *is quasi-nonexpansive. However, if* $x=2, y=1$ *we get that* $H(T(x),T(y))>|x-y|=1$ *and hence not nonexpansive.*

Remark 1. *From the proof of Lemma 5.1 in* [9]*, it is easy to see that if* T *is a multi-valued nonexpansive mapping then* T *is* $\triangle$*-demiclosed.*

The following lemma will be useful in this study.

Lemma 1. [11, Lemma 2.2] *Let* K *be a nonempty subset of a metric space* (X,d) *and* $T:K\to P(K)$ *be a multi-valued mapping. Then*
(i) $d(x,T(x))=d(x,P_T(x))$ *for all* $x\in K$;
(ii) $x\in F(T)\Longleftrightarrow x\in F(P_T)\Longleftrightarrow P_T(x)=\{x\}$;
(iii) $F(T)=F(P_T)$.

Let X be a metric space with a metric d and $x,\ y\in X$ with $d(x,y)=l$. A *geodesic path* from x to y is an isometry $c:[0,l]\subset\mathbb{R}\to X$ such that $c(0)=x$ and $c(l)=y$. The image of c is called a *geodesic segment* joining x and y. A geodesic segment joining x and y is not necessarily unique in general. When it is unique, this geodesic segment is denoted by $[x,y]$. This means that $z\in[x,y]$ if and only if there exists $\alpha\in[0,1]$ such that $d(x,z)=(1-\alpha)d(x,y)$ and $d(y,z)=\alpha d(x,y)$. In this case, we write $z=\alpha x\oplus(1-\alpha)y$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and X is said to be a *uniquely geodesic* if there is exactly one geodesic joining x to y for each $x,y\in X$.

In a geodesic space (X,d), the metric $d:X\times X\to\mathbb{R}$ is *convex* if for any $x,y,z\in X$ and $\alpha\in[0,1]$, one has

$$d(x,\alpha y\oplus(1-\alpha)z)\leq\alpha d(x,y)+(1-\alpha)d(x,z).$$

Let $D\in(0,\infty]$. If for every $x,\ y\in X$ with $d(x,y)<D$, a geodesic from x to y exists, then X is said to be D-*geodesic space.* Moreover, if such a geodesic is unique for each pair of points then X is said to be a D-*uniquely geodesic*. Notice that X is a geodesic space if and only if it is a D-geodesic space.

To define a CAT(κ) space, we use the following concept called *model space.* For $\kappa=0$, the two-dimensional model space $M_\kappa^2=M_0^2$ is the Euclidean space $\mathbb{R}^2$ with the metric induced from the Euclidean norm. For $\kappa>0$, M_κ^2 is the two-dimensional sphere $\left(\frac{1}{\sqrt{\kappa}}\right)\mathbb{S}^2$ whose metric is a length of a minimal great arc joining each two points. For $\kappa<0$, M_κ^2 is the two-dimensional hyperbolic space $\left(\frac{1}{\sqrt{-\kappa}}\right)\mathbb{H}^2$ with the metric defined by a usual hyperbolic distance.

The diameter of M_κ^2 is denoted by

$$D_\kappa=\begin{cases}\frac{\pi}{\sqrt{\kappa}} & \kappa>0,\\ +\infty & \kappa\leq 0.\end{cases}$$

A *geodesic triangle* $\triangle(x,y,z)$ in a geodesic space (X,d) consists of three points x,y,z in X (the vertices of $\triangle$) and three geodesic segments between each pair of vertices (the edges of $\triangle$). We write $p \in \triangle(x,y,z)$ when $p \in [x,y] \cup [y,z] \cup [z,x]$. A *comparison triangle* for $\triangle(x,y,z)$ is a triangle $\triangle(\overline{x},\overline{y},\overline{z})$ in M_κ^2 such that

$$d(x,y) = d_{M_\kappa^2}(\overline{x},\overline{y}), d(y,z) = d_{M_\kappa^2}(\overline{y},\overline{z}) \text{ and } d(z,x) = d_{M_\kappa^2}(\overline{z},\overline{x}).$$

If $\kappa \leq 0$, then such a comparison triangle always exists in M_κ^2. If $\kappa > 0$, then such a triangle exists whenever $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$. A point $\overline{p} \in [\overline{x},\overline{y}]$ is called a *comparison point* for $p \in [x,y]$ if $d(x,p) = d_{M_\kappa^2}(\overline{x},\overline{p})$.

A geodesic triangle $\triangle(x,y,z)$ in X is said to satisfy the CAT(κ) *inequality* if for any $p,q \in \triangle(x,y,z)$ and for their comparison points $\overline{p},\overline{q} \in \triangle(\overline{x},\overline{y},\overline{z})$, one has

$$d(p,q) \leq d_{M_\kappa^2}(\overline{p},\overline{q}).$$

We are ready to introduce the concept of CAT(κ) space in the following definition taken from [2].

Definition 2. *If $\kappa \leq 0$, then X is called a CAT(κ) space if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality. If $\kappa > 0$, then X is called a CAT(κ) space if it is D_κ-geodesic and any geodesic triangle $\triangle(x,y,z)$ in X with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ satisfies the CAT(κ) inequality.*

It follows from [2, p.160] that any CAT(κ) space is D_κ-uniquely geodesic.

Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, we put $r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n)$. The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf_{x\in X} r(x,\{x_n\}).$$

Further, the *asymptotic center* of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

Recall that the sequence $\{x_n\}$ is $\triangle$-*convergent* to $x \in X$ if x is the unique asymptotic center of any subsequence of $\{x_n\}$.

Lemma 2. [3] *Let (X,d) be a complete CAT(1) space and $\{x_n\}$ be a sequence in X. If $r(\{x_n\}) < \pi/2$, then the following statements hold:*
(i) $A(\{x_n\})$ consists of exactly one point,
(ii) $\{x_n\}$ has a $\triangle$-convergent subsequence.

The following lemma is needed for our main result.

Lemma 3. [11, Lemma 2.4] *If (X,d) is a $CAT(1)$ space with diam$(X) < \pi/2$, then there exist a constant $K > 0$ such that*

$$d^2((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)d^2(x,z) + \alpha d^2(y,z) - \frac{K}{2}\alpha(1-\alpha)d^2(x,y)$$

for all $\alpha \in [0,1]$ and $x,y,z \in X$.

3. Main results

We start with the following key lemmas.

Lemma 4. *Let (X,d) be a CAT(1) space with convex metric, K be a nonempty, closed and convex subset of X and let $T : K \to 2^K$ be a multi-valued quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Then, for the sequence $\{x_n\}$ defined by (1.3), $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$.*

Proof. For any $p \in F(T)$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)y_n^1 \oplus \alpha_n z_n^1, p) \\ &\leq (1-\alpha_n)d(y_n^1, p) + \alpha_n d(z_n^1, p) \\ &\leq (1-\alpha_n)d(y_n^1, p) + \alpha_n H\left(T\left(y_n^1\right), T(p)\right) \\ &\leq (1-\alpha_n)d(y_n^1, p) + \alpha_n d(y_n^1, p) \\ &= d(y_n^1, p). \end{aligned}$$

Also, we obtain

$$\begin{aligned} d(y_n^1, p) &= d((1-\beta_n^1)y_n^2 \oplus \beta_n^1 z_n^2, p) \\ &\leq (1-\beta_n^1)d(y_n^2, p) + \beta_n^1 d(z_n^2, p) \\ &\leq (1-\beta_n^1)d(y_n^2, p) + \beta_n^1 H\left(T\left(y_n^2\right), T(p)\right) \\ &\leq (1-\beta_n^1)d(y_n^2, p) + \beta_n^1 d(y_n^2, p) \\ &= d(y_n^2, p). \end{aligned}$$

Continuing the above process, we get

$$d(x_{n+1}, p) \leq d(y_n^1, p) \leq d(y_n^2, p) \leq ... \leq d(y_n^{k-1}, p) \leq d(x_n, p). \tag{3.1}$$

This inequality guarantees that the sequence $\{d(x_n, p)\}$ is non-increasing and bounded below, and so $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(T)$. □

Lemma 5. *Let (X,d) be a CAT(1) space with convex metric and $diam(X) < \pi/2$, K be a nonempty, closed and convex subset of X and let $T : K \to 2^K$ be a multi-valued quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the sequence defined by (1.3) with $\beta_n^{k-1} \in [a,b] \subset (0,1)$. Then $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$.*

Proof. It follows from Lemma 4 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$. We may assume that $\lim_{n\to\infty} d(x_n, p) = r > 0$. Otherwise the proof is trivial. By using

(3.1), we get $\lim_{n\to\infty} d(y_n^{k-1}, p) = r$. By Lemma 3, we also have

$$\begin{aligned} d^2(y_n^{k-1}, p) &= d^2((1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1} z_n^{'}, p) \\ &\leq (1-\beta_n^{k-1})d^2(x_n, p) + \beta_n^{k-1} d^2(z_n^{'}, p) - \frac{K}{2}\beta_n^{k-1}(1-\beta_n^{k-1})d^2(x_n, z_n^{'}) \\ &\leq (1-\beta_n^{k-1})d^2(x_n, p) + \beta_n^{k-1} H^2\left(T\left(x_n\right), T\left(p\right)\right) \\ &\quad -\frac{K}{2}\beta_n^{k-1}(1-\beta_n^{k-1})d^2(x_n, z_n^{'}) \\ &\leq (1-\beta_n^{k-1})d^2(x_n, p) + \beta_n^{k-1} d^2\left(x_n, p\right) - \frac{K}{2}\beta_n^{k-1}(1-\beta_n^{k-1})d^2(x_n, z_n^{'}) \\ &= d^2(x_n, p) - \frac{K}{2}\beta_n^{k-1}(1-\beta_n^{k-1})d^2(x_n, z_n^{'}), \end{aligned}$$

which implies that

$$d^2(x_n, z_n^{'}) \leq \frac{2}{a(1-b)K}\left[d^2(x_n, p) - d^2(y_n^{k-1}, p)\right].$$

Hence, $\lim_{n\to\infty} d(x_n, z_n^{'}) = 0$. Since $d(x_n, T\left(x_n\right)) \leq d(x_n, z_n^{'})$, then we obtain

$$\lim_{n\to\infty} d(x_n, T\left(x_n\right)) = 0.$$

□

We give the $\triangle$-convergence of the iteration process defined by (C) in a CAT(1) space.

Theorem 1. *Let X, K and $\{x_n\}$ satisfy the hypotheses of Lemma 5, X be a complete space and let $T : K \to C(K)$ be a multi-valued quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. If T is $\triangle$-demiclosed, then the sequence $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.*

Proof. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. Since $r(\{u_n\}) \leq r(\{x_n\}) < \pi/2$, by Lemma 2(i), there exists a unique asymptotic center u of $\{u_n\}$. Moreover, by Lemma 2(ii), there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\{v_n\}$ is $\triangle$-convergent to v for some $v \in X$. Further, since $\lim_{n\to\infty} d(v_n, T\left(v_n\right)) = 0$ (by Lemma 5) and T is $\triangle$-demiclosed, we have $v \in F(T)$. By Lemma 4, $\lim_{n\to\infty} d(x_n, v)$ exists. Then we can show that $u = v$. If not, from the uniqueness of the asymptotic center, we have

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, v) \\ &= \lim_{n\to\infty} d(x_n, v) \\ &= \limsup_{n\to\infty} d(v_n, v) \\ &< \limsup_{n\to\infty} d(v_n, u) \\ &\leq \limsup_{n\to\infty} d(u_n, u). \end{aligned} \tag{3.2}$$

This is a contradiction. Hence we get $u = v \in F(T)$. Next, we show that for any subsequence of $\{x_n\}$, its asymptotic center consists of the unique element. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$. Finally, we show that $x = v$. If not, then the existence of $\lim_{n\to\infty} d(x_n, v)$ and the uniqueness of the asymptotic center imply that there exists a contradiction as (3.2). Hence we get $x = v \in F(T)$. Therefore, the sequence $\{x_n\}$ is $\triangle$-convergent to a fixed point of T. □

It is well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space (see [2]). If (X, d) is a CAT(0) space and K is a convex subset of X, then (K, d) is a CAT(0) space and hence it is a CAT(κ) space with $\kappa > 0$. Now we give an example of such mappings which are multi-valued quasi-nonexpansive mappings as in Theorem 1.

Example 2. *Let X be the real line with the usual metric and let $K = [0, 1]$. Define two mappings $S, T : K \to C(K)$ by*

$$S(x) = \left[0, \frac{x}{4}\right] \quad \textit{and} \quad T(x) = \left[0, \frac{x}{2}\right].$$

Obviously, $F(S) = F(T) = \{0\}$. It is proved in [6, Example 2] *that both S and T are multi-valued nonexpansive mappings. Therefore, they are multi-valued quasi-nonexpansive mappings. Additionally, for $0 \in F(S) = F(T)$, we have that $S(0) = T(0) = \{0\}$.*

We prove the strong convergence of the iteration process defined by (C) in a CAT(1) space as follows.

Theorem 2. *Let X, K, T and $\{x_n\}$ be the same as in Theorem 1.*

(i) If T satisfies Condition (I), then the sequence $\{x_n\}$ is convergent strongly to a point in $F(T)$.

(ii) If T is hemi-compact and continuous, then the sequence $\{x_n\}$ is convergent strongly to a point in $F(T)$.

Proof. (i) By Condition (I) and Lemma 5, we have

$$\lim_{n\to\infty} f\left(d\left(x_n, F\left(T\right)\right)\right) \leq \lim_{n\to\infty} d(x_n, T\left(x_n\right)) = 0.$$

That is, $\lim_{n\to\infty} f\left(d(x_n, F(T))\right) = 0$. Since f is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, it follows that $\lim_{n\to\infty} d\left(x_n, F(T)\right) = 0$. The proof of the remaining part follows the proof of Theorem 3.2 in [13], therefore we omit it.

(ii) From hemi-compactness of T and Lemma 5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = q \in K$. Since T is continuous, we have

$$d(q, T(q)) = \lim_{k\to\infty} d(x_{n_k}, T(x_{n_k})) = 0.$$

This implies that $q \in F(T)$ since $T(q)$ is closed. Thus $\lim_{n\to\infty} d(x_n, q)$ exists by Lemma 4. Hence the sequence $\{x_n\}$ is convergent strongly to a fixed point q of T. $\square$

Since every nonexpansive mapping having a fixed point is quasi-nonexpansive, then we get the following corollary.

Corollary 1. *Let X, K and $\{x_n\}$ satisfy the hypotheses of Theorem 1 and $T : K \to C(K)$ be a multi-valued nonexpansive mapping with $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Then the sequence $\{x_n\}$ is $\triangle$-convergent to a fixed point of T. Moreover, if T satisfies Condition (I) or is hemi-compact, then the sequence $\{x_n\}$ is convergent strongly to a fixed point of T.*

To avoid the restriction of T, that is, $T(p) = \{p\}$ for each $p \in F(T)$, we use the iteration process defined by (D). Using this iteration, we give the $\triangle$-convergence result in a CAT(1) space.

Theorem 3. *Let (X, d) be a complete CAT(1) space with convex metric and diam(X) $< \pi/2$, K be a nonempty, closed and convex subset of X and let $T : K \to P(K)$ be a multi-valued mapping with $F(T) \neq \emptyset$ and P_T is quasi-nonexpansive. Let $\{x_n\}$ be the sequence defined by (D) with $\beta_n^{k-1} \in [a, b] \subset (0, 1)$. If T is $\triangle$-demiclosed, then the sequence $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.*

Proof. It follows from Lemma 1 that $d(x, P_T(x)) = d(x, T(x))$ for all $x \in K$, $F(P_T) = F(T)$ and $P_T(p) = \{p\}$ for each $p \in F(P_T)$. The $\triangle$-demiclosedness of P_T follows from the $\triangle$-demiclosedness of T. Then, applying Theorem 1 to the mapping P_T, we can conclude that the sequence $\{x_n\}$ is $\triangle$-convergent to a point $p \in F(P_T) = F(T)$. $\square$

We now present an example of a multi-valued mapping T for which P_T is quasi-nonexpansive.

Example 3. *Let X and K be defined as in Example 2. Define a mapping $T : K \to P(K)$ by*

$$T(x) = \begin{cases} [0, x] & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \left\{\frac{1}{2}\right\} & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Then, we have

$$P_T(x) = \begin{cases} \{x\} & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \left\{\frac{1}{2}\right\} & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Clearly, $F(T) = F(P_T) = \left\{x : 0 \leq x \leq \frac{1}{2}\right\}$. It is proved in [17, Example 5] *that T is not nonexpansive and P_T is quasi-nonexpansive.*

We give several strong convergence results of the iteration process defined by (D) in a CAT(1) space.

Theorem 4. *Let X, K, T and $\{x_n\}$ be the same as in Theorem 3. If T satisfies Condition (I), then the sequence $\{x_n\}$ is convergent strongly to a fixed point of T.*

Proof. By following the same proof of Theorem 3.4 in [11] and using Lemma 1, we can obtain that P_T satisfies Condition (I) and $P_T(x)$ is closed for any $x \in K$. Then, applying Theorem 2(i) to the mapping P_T, we can conclude that the sequence $\{x_n\}$ is convergent strongly to a fixed point of T. □

Theorem 5. *Let X, K and $\{x_n\}$ satisfy the hypotheses of Theorem 3 and $T : K \to P(K)$ be a multi-valued hemi-compact mapping with $F(T) \neq \emptyset$ and P_T is quasi-nonexpansive and continuous. Then the sequence $\{x_n\}$ is convergent strongly to a fixed point of T.*

Proof. From the hemi-compactness of T, we can prove that P_T is hemi-compact. The conclusion follows from Theorem 2(ii). □

Corollary 2. *Let X, K and $\{x_n\}$ be the same as in Theorem 3 and $T : K \to P(K)$ be a multi-valued mapping with $F(T) \neq \emptyset$ and P_T is nonexpansive. Then the sequence $\{x_n\}$ is $\triangle$-convergent to a fixed point of T. Moreover, if T satisfies Condition (I) or is hemi-compact, then the sequence $\{x_n\}$ is convergent strongly to a fixed point of T.*

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