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ESTIMATION OF TIME VARYING PARAMETERS IN AN OPTIMAL CONTROL PROBLEM

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ABSTRACT. In this paper, we employ a non-linear state space model and the extended Kalman filter to simultaneously estimate the time-varying parameters in an optimal control problem, where the objective (loss) function is quadratic. Our methodology also allows us to derive the difference between the optimal control and the observed control variable. A simulation exercise based on a simple intertemporal model shows that the estimated parameter values are very close to their population values, which provide further support for the estimation methodology introduced in this paper.

1. INTRODUCTION

Although there has been an ever-growing increase in the number of studies, which utilize optimal control techniques in economics, the literature is relatively silent in the estimation of the parameters in an optimal control set-up [1]. In other words, given the observation, state and the control variables, developing estimation techniques for the parameters of a system needs to be explored in details.

Especially when the parameters in an optimal control problem are assumed to vary over time, which is a general characteristic of the models that employ large data sets, estimation of the time-varying parameters become even more important and complex.

This study tries to fulfill the above-mentioned gap in the literature by introducing an estimation algorithm for the time-varying parameters of an optimal linear regulator problem, which has a quadratic objective (loss) function and a linear system of constraints. As it will be clearer, estimating the state vector and the time-varying parameters simultaneously will cause a non-linearity in the system, which leads us to cast the model in a state space form and employ a non-linear filter, namely the extended Kalman filter. This algorithm is very useful method to parameter estimation in nonlinear state-space models but very small using in economic literature. In economic literature, Grillenzoni (1993), Bacchetta and Gerlac (1997), Ozbek and Ozlale (2005) used this method in their studies.

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[1]Ozlale (2003) and Salemi (1995) are some of the exceptions in this context.

In order to test the estimation accuracy of the above mentioned method, we also conduct a simulation exercise and estimate the parameters for a simple intertemporal consumption-saving model of a representative household. The exercise shows that the estimated parameters using the non-linear state space framework and the extended Kalman filter are very close to their generated population values, supporting the estimation method introduced in this paper.
The outline of the paper is as follows. First, we briefly introduce the linear and the non-linear state space models, which are frequently used in an optimal control framework. Then, we will define the optimal control problem, for which we show the estimation methodology of the parameters. The simulation exercise follows. Finally, the last section concludes the paper.

2. THE LINEAR DISCRETE-TIME STATE SPACE MODEL

A linear state space model can be defined as:

$$x_{n+1}=\Phi_n x_n + B_n u_n + G_n w_n \tag{1}$$

$$x_n \in \Re^n, \quad y_n = H_n x_n + v_n \tag{2}$$

where $x_n \in \Re^n$ is the state vector, $y_n \in \Re^m$ is the observation vector and $u_n \in \Re^r$ is the control variable vector. In addition, Φ_n and H_n show the $n \times n$ dimensional transition matrix and the $m \times n$ dimensional observation matrix, respectively. Finally, $w_n \in \Re^n$ and $v_n \in \Re^m$ represent the zero-mean white noise processes, which are the disturbance terms of the model. The assumptions about these white noise processes can be written as:

$$E\left[v_n\right]=0 \tag{3}$$

$$E\left[w_n\right]=0 \tag{4}$$

$$E\left[v_n v_j'\right]=R_n\delta_{nj} \tag{5}$$

$$E\left[w_n w_j'\right]=Q_n\delta_{nj} \tag{6}$$

$$E\left[v_n w_j'\right]=0 \tag{7}$$

$$E\left[x_0\right]=\bar{x}_0 \tag{8}$$

$$E\left[(x_0-\bar{x}_0)(x_0-\bar{x}_0)'\right]=P_0 \tag{9}$$

$$E\left[x_0 w_n'\right]=0 \tag{10}$$

$$E\left[x_0 v_n'\right]=0 \tag{11}$$

It is further assumed that for $n = 0,1,2,...$ the matrices Φ_n, H_n, Q_n and R_n are known with certainty. Given these conditions and the observation vector $Y_n = \{y_0, y_1, ..., y_n\}$, the Kalman filter, which is stated in Kalman (1960), emerges as a possible algorithm to estimate the state vector x_n (see Appendix 1).

3. NON-LINEAR STATE SPACE MODELS AND THE EXTENDED KALMAN FILTER

On the other hand, a non-linear state space model can be represented as:

$$x_{n+1} = F(\theta_n)(x_n) + G(\theta_n)u_n + w_n \quad (12)$$

$$y_n = H(\theta_n)x_n + e_n \quad (13)$$

where, F, G and H represent the vector-valued functions. In addition, w_n and e_n are the white noise processes with covariance matrices, $R_1(\theta_n)$ and $R_2(\theta_n)$, respectively. The vector θ_n represents the time varying parameter vector to be estimated. It is important to note that, the time varying parameters and the state vector is in multiplicative form, which rules out the assumption of linearity and makes it necessary to use the extended Kalman filter, as mentioned in Ljung and Söderström(1985) (see Appendix 2).

4. THE OPTIMAL CONTROL PROBLEM

Finally, the form of the dynamic quadratic loss function, for which the parameters will be estimated, can be defined as:

$$J_N = \mathrm{E}_0 \sum_{n=0}^{\infty} \beta^n \left(x_n^{'} R_1 x_n + u_n^{'} Q_1 u_n + 2x_n^{'} W u_n \right), \quad 0 < \beta < 1 \quad (14)$$

The problem of determining the control sequence $u = [u_0, u_1, ..., u_{N-1}]$ is known as the *discounted stochastic regulator problem*. In the above function, R_1 and Q_1 are non-negative symmetric matrices and. β is discount factor. As stated in Ljungqvist and Sargent (2000) an explicit solution of the above form is given as:

$$u_n = -F_n x_n \quad (15)$$

$$F_n = \left(Q_1 + \beta B'(P_1)_{n+1} B\right)^{-1} \left(\beta B'(P_1)_{n+1} \Phi + W'\right) \quad (16)$$

$$\begin{aligned}(P_1)_{n+1} &= R_1 + \beta\ \Phi'(P_1)_n \Phi \\ &-\left(W + \beta\ \Phi'(P_1)_n B\right)\left(Q_1 + \beta B'(P_1)_n B\right)^{-1}\left(\beta B'(P_1)_n \Phi + W'\right)\end{aligned} \quad (17)$$

When the state vector is unknown, the Kalman filter is executed to estimate the state vector, which leads us to obtain $u_n = -F_n x_n$.

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5. ESTIMATION OF THE PARAMETERS AND THE STATE VARIABLES IN THE CONTROL PROBLEM

This section introduces the methodology to estimate the parameters in the above mentioned optimal control problem. We start by assuming that the representative agent or the policymaker minimizes the loss function by using the control variable u_n, which is obtained from the control algorithm, defined by equations (15) to (17). Then, in order to estimate the parameters, we proceed as follows:

Let e_n be the difference (control error) between the observed control variable and the optimal control variable, which is obtained from the solution of the control problem. Formally;

$$u_n^{optimal} - u_n^{observed} = e_n$$

Then, in the Kalman filtering algorithm, the estimate for the state vector can be stated as:

$$\hat{x}_{n|n-1} = \Phi_{n-1}\hat{x}_{n-1|n-1} + B_{n-1}e_{n-1}$$

which can also be written as:

$$\hat{x}_{n|n-1} = \Phi_{n-1}\hat{x}_{n-1|n-1} + B_{n-1}(u_{n-1}^{optimal} - u_{n-1}^{observed})$$

Since the optimal feedback rule for the linear regulator is

$$u_n = -F_{n-1}\hat{x}_{n-1|n-1}$$

We can write the equation for the state vector as:

$$\hat{x}_{n|n-1} = \Phi_{n-1}\hat{x}_{n-1|n-1} - B_{n-1}F_{n-1}\hat{x}_{n-1|n-1} - B_{n-1}u_{n-1}^{observed} \tag{18}$$

$$\hat{x}_{n|n-1} = (\Phi_{n-1} - B_{n-1}F_{n-1})\hat{x}_{n-1|n-1} - B_{n-1}u_{n-1}^{observed} \tag{19}$$

For simplicity, let $A_{n-1} = (\Phi_{n-1} - B_{n-1}F_{n-1})$. Then, the problem reduces down to obtaining the elements of A_{n-1} at each step. Since the matrix A_{n-1} includes parameters to be estimated, the model can be cast in a non-linear state space model, where the extended Kalman filter is used. As a result, both the optimal control sequence and the time-varying parameters in the model are simultaneously obtained.

6. APPLICATION OF THE METHODOLOGY

In the previous sections, we showed how the parameters in an optimal linear regulator problem can be estimated. In this section, we apply our methodology to an optimal control problem, which includes a linear quadratic loss function and linear constraints.

For application purposes, we focus on the consumption and saving decision of a representative household, who is assumed to face the following loss function Ljungqvist and Sargent (2000)

$$\sum_{n=1}^{\infty}\beta^n\left\{\left(c_n - b\right)^2 + \gamma i_n^2\right\} \tag{20}$$

where c_n and i_n are consumption and investment at time *n*, respectively. Thus, at each period, the representative household has to make a choice between present and future consumption. The economy is characterized by the following system of equations:

$$\begin{aligned} c_n + i_n &= ra_n + y_n \\ a_{n+1} &= a_n + i_n \\ y_{n+1} &= \rho_1 y_n + \rho_2 y_{n-1} \end{aligned} \tag{21}$$

In the above system, a_n, y_n, r denote the accumulated asset, the exogenous labor income and the interest rate, respectively. The parameters of the model are assumed to be as $b > 0, \gamma > 0$. Finally, the control variable u_n can be written as $i_n = a_{n+1} - a_n$. The system of equations can be cast in the state space form as follows Ljungqvist et al (2001):

$$\begin{bmatrix} a_{n+1} \\ y_{n+1} \\ y_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho_1 & \rho_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ y_n \\ y_{n-1} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_n + w_n \tag{22}$$

As it can be seen in (22), the state vector includes unknown parameters to be estimated.
On the other hand, in order to incorporate the loss function into the state space form, the matrices R_1, Q_1 and W should be defined as follows:

$$R_1 = \begin{bmatrix} r^2 & r & 0 & -br \\ r & 1 & 0 & -b \\ 0 & 0 & 0 & 0 \\ -br & -b & 0 & b\wedge 2 \end{bmatrix}, \;\; Q_1 = 1+\gamma, \;\; W = \begin{bmatrix} -r & -1 & 0 & b \end{bmatrix}' \tag{23}$$

Accordingly, we can write:

$$\left(c_n - b\right)^2 + \gamma i_n^2 = x_n^{'} R_1 x_n + u_n^{'} Q_1 u_n + 2 x_n^{'} W u_n \tag{24}$$

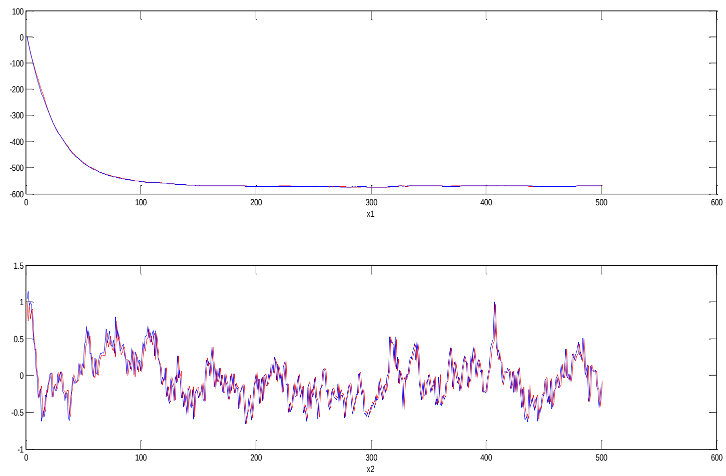
It should be remarked that R_1, Q_1 and W also include some unknown parameters to be estimated. Finally, the observation equation will be as:

$$z_n = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{bmatrix} x_n + v_n \tag{25}$$

Before running the optimal control algorithm and the extended Kalman filter, let the population parameters take the following values $\left(\rho_1, \rho_2, r, b, \gamma\right) = \left(1.2, -0.3, -0.05, 30, 1\right)$ and discount factor

$\beta = 1$. Then, the parameters in the state space model and the loss function can be estimated by using the extended Kalman filter, which is also shown in Appendix 2. Figure 1 compares the actual and the estimated values of the variables in the observation matrix. As it can be seen, these two series are almost identical.

Figure 1: Comparison of the estimated and the actual values of the observation matrix (Actual: Blue, Estimate: Red)



In addition, Figure 2 shows how observed and the estimated control variable path evolve over time. Due to insufficient number of observations, the extended Kalman filter, which is a recursive algorithm, provides inaccurate result at the very first phase of the sample. However, as the number of observations increases, the two series become almost identical. Such a finding leads us to conclude that, as long as there is sufficient number of observations, the extended Kalman filter provides accurate estimates.

Other than comparison of the observation and the control variables, it is also important to see whether the estimated parameters converge to their population values. Figure 3 states that the estimated parameters in the exogenous labor income process, which follows AR(2), converge to their parameter values, which are 1.2 and -0.3, respectively.

The estimated values for the real interest rate and the targeted consumption in the loss function can be seen in Figure 4. As it can be seen, while the interest rate is not very close, the targeted consumption is almost identical with its population value.

Figure 2: Comparison of the estimated and the actual values of the control variable (Actual: Blue, Estimate: Red)

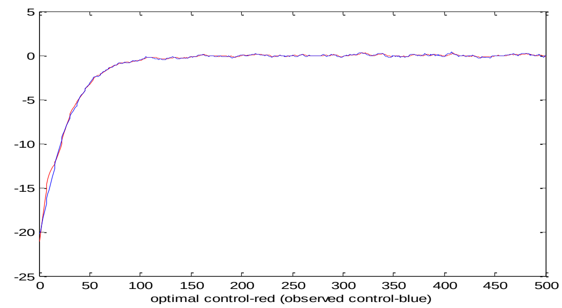


Figure 3: The estimated parameter values in the exogenous labor income process

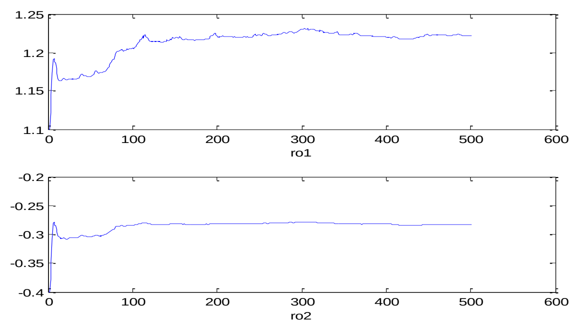
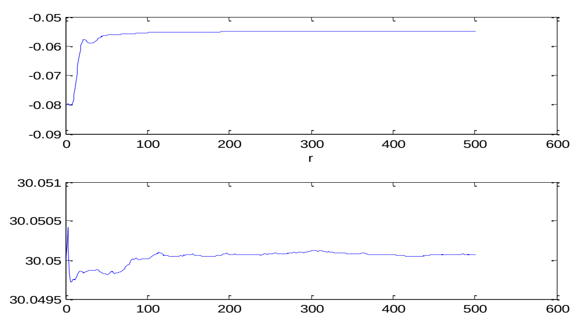
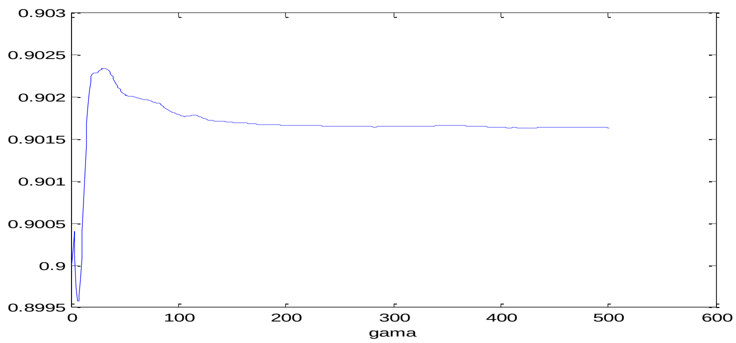


Figure 4: The estimated values for the real interest rate and the targeted consumption



Finally, the evolution of the estimated value for the parameter in front of the investment in the loss function can be seen in Figure 5. While the population value is chosen to be as 1, the estimated value becomes steady at about 0.94, which is very close to its actual value.

Figure 5: The estimated value for the parameter in front of investment in the loss function



7. CONCLUSION

In this study, we develop a method to estimate the unknown parameters in an optimal control problem, where the objective (loss) function is quadratic and the constraints are linear. We show that, when the model is represented in state space form, the unknown parameters in the loss function and the system of equations can be simultaneously estimated by employing the extended Kalman filter.

In the second part of the paper, we conduct a simulation exercise to see the estimation accuracy of the introduced method. Using a simple intertemporal consumption-saving model for a representative household, we see that the estimated parameters are very close to the generated population values, which support our estimation method.

As a result, the estimation method, which is described in this paper, can be conveniently used for the purpose of simultaneously estimating the time-varying parameters in an optimal linear regulator problem.

Appendix 1

Kalman Filter Algorithm

Suppose that, when the observations $Y_n = \{y_0, y_1, ..., y_n\}$ are given, forecasting the state vector x_n is denoted by $\hat{x}_{n|n} = E\left[x_n \left| y_0, y_1, ..., y_n\right.\right] = E\left[x_n \left| Y_n\right.\right]$, and the covariance matrix of the disturbance term is denoted by $P_{n|n} = E\left[(x_n - \hat{x}_{n|n})(x_n - \hat{x}_{n|n})' \left| Y_n\right.\right]$

In this circumstance, depending on the initial values

$$P_{0|-1} = P_0$$

$$\hat{x}_{0|-1} = \bar{x}_0$$

The Kalman filter algorithm is given by the following algorithm:

$$\hat{x}_{n|n-1} = \Phi_{n-1}\hat{x}_{n-1|n-1} + B_{n-1}u_{n-1} \tag{A1}$$

$$\hat{x}_{n|n} = \hat{x}_{n|n-1} + K_n\left[y_n - H_n\hat{x}_{n|n-1}\right] \tag{A2}$$

$$K_n = P_{n(n-1}H'_n\left[H_nP_{n(n-1}H'_n + R_n\right]^{-1} \tag{A3}$$

$$P_{n(n} = \left[I - K_nH_n\right]P_{n(n-1} \tag{A4}$$

$$P_{n|n-1} = \Phi_{n-1}P_{n-1|n-1}\Phi'_{n-1} + G_{n-1}Q_{n-1}G'_{n-1} \tag{A5}$$

Equation (3) is also known as the Kalman gain.

Appendix 2

Extended Kalman Filter Algorithm

Suppose that

$$X_n = \begin{pmatrix} x_n \\ \theta_{n-1} \end{pmatrix}, \quad \overline{K}_n = \begin{pmatrix} K_n \\ L_n \end{pmatrix}, \quad \overline{P}_n = \begin{pmatrix} P_1(n) & P_2(n) \\ P_2^T(n) & P_3(n) \end{pmatrix}$$

where $\overline{K}$ and $\overline{P}$ are Kalman gain and the covariance matrix of the extended state, respectively, as stated in Ljung and Söderström (1985). Then, the updating equations will be:

$$x_{n+1} = F_nx_n + G_nu_n + K_n\left(y_n - H_nx_n\right) \tag{A6}$$

$$\hat{x}_0 = 0$$

$$\theta_n = \theta_{n-1} + L_n\left(y_n - H_{n-1}x_n\right) \tag{A7}$$

$$\theta_0 = \theta_0$$

$$K_n = \left(F_nP_1(n)H_n^T + M_nP_1^T(n)H_n^T + F_nP_2(n)D_n^T + M_nP_2(n)D_n^T + R_{12}\right)S_n^{-1} \tag{A8}$$

$$S_n = H_nP_1(n)H_n^T + H_nP_2(n)D_n^T + D_nP_2^T(n)H_n^T + D_nP_3(n)D_n^T + R_2 \tag{A9}$$

$$L(n) = \left(P_2^T(n)H_{n-1}^T + P_3(n)D_n^T\right)S_n^{-1} \tag{A10}$$

$$P_1(n+1) = F_nP_1(n)F_n^T + F_nP_2(n)M_n^T + M_nP_2^T(n)F_n^T + M_nP_3(n)M_n^T - K_nS_nK_n^T + R_1 \tag{A11}$$

$$P_1(0) = \Pi_0(\theta_0)$$

$$P_2(n+1) = F_nP_2(n) + M_nP_3(n) - K_nS_nL_n^T \tag{A12}$$

$$P_2(0) = 0$$

$$P_3(n+1) = P_3(n) - L_n S_n L_n^T \tag{A13}$$
$$P_3(0) = P_0$$

Here, it is assumed that

$$F_n = F(\theta_n)$$
$$G_n = G(\theta_n) \tag{A14}$$
$$H_n = H(\theta_n)$$
$$M_n = M(\theta_n, x_n, u_n)$$

and

$$M(\theta, x, u) = \frac{\partial}{\partial\theta}\left[F(\theta)x + G(\theta)u\right]\Big|_{\theta=\theta} \tag{A15}$$

and

$$D_n = D(\theta_{n-1}, x_n)$$
$$D(\theta, x) = \frac{\partial}{\partial\theta}\left[H(\theta)x\right]\Big|_{\theta=\theta} \tag{A16}$$

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