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MODIFIED $q-$BASKAKOV OPERATORS

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Abstract. In the present paper, a generalization of the sequences of $q-$ Baskakov operators, which are based on a function τ having continuously differentiable on $[0,\infty)$ with $\tau(0)=0$, $\inf \tau'(x) \geq 1$, has been considered. Uniform approximation of such a sequence has been studied and degree of approximation has been obtained. Moreover, monotonicity properties of the sequence of operators are investigated.

1. Introduction

In [7], Baskakov operator was introduced as

$$B_n(f)(x) = \frac{1}{(1+x)^n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right)$$

for $n \in \mathbb{N}$, $x \in [0,\infty)$ and $f \in C[0,\infty)$ where $C[0,\infty)$ denote the space of all continuous and real valued functions defined on $[0,\infty)$. This operator and its various extentions have been intensively studied. Some are in [1], [6], [8], [15], [16].

Let us recall some notations on $q-$analysis ([10], [17]). The $q-$integer, $[n]$ and the $q-$factorial, $[n]!$ are defined by

$$[n] := [n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases} \quad \text{for } n \in \mathbb{N}$$

$[0] = 0$, and

$$[n]! := \begin{cases} [1]_q [2]_q \ldots [n]_q, & n = 1, 2, \ldots \\ 1 & , n = 0 \end{cases}, \text{ for } n \in \mathbb{N} \text{ and } [0]! = 1,$$

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respectively where $q>0$. For integers $n\geq r\geq 0$ the $q-$binomial coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q!\,[n-r]_q!}.$$

The $q-$derivative of $f(x)$ is denoted by $D_q f(x)$ and defined as

$$D_q f(x) := \frac{f(qx)-f(x)}{(q-1)x},\ x\neq 0,\ D_q f(0) = f^{'}(0),$$

also

$$D_q^0 f := f,\ D_q^n f := D_q(D_q^{n-1} f),\ n=1,2,...$$

$q-$Pochammer formula is given by

$$(x,q)_0 = 1,$$

$$(x,q)_n = \prod_{k=0}^{n-1}\left(1-q^k x\right)$$

with $x\in\mathbb{R}$, $n\in\mathbb{N}\cup\{\infty\}$. The $q-$derivative of the product and quotient of two functions f and g are

$$D_q(f(x)g(x)) = g(x)D_q(f(x)) + f(qx)D_q(g(x))$$

and

$$D_q(\frac{f(x)}{g(x)}) = \frac{g(x)D_q(f(x)) - f(x)D_q(g(x))}{g(x)g(qx)},$$

respectively.

A generalization of the Baskakov operator based on $q-$ integers is defined by Aral and Gupta [4]. The authors constructed the $q-$Baskakov operator as

$$B_{n,q}(f;x) = \sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix} q^{\frac{k(k-1)}{2}} x^k (-x,q)_{n+k}^{-1} f\left(\frac{[k]}{q^{k-1}[n]}\right),\ n\in\mathbb{N}, \quad (1.1)$$

where $x\geq 0$, $q>0$ and f is a real valued continuous function on $[0,\infty)$. They established moments using $q-$derivatives, expressed the operator in terms of divided differences, studied the rate of convergence in a polynomial weighted norm and gave a theorem related to monotonic convergence of the sequence of operators with respect to n.

Finta and Gupta [11] obtained direct estimates for the operators (1.1), using the second order Ditzian-Totik modulus of smoothness. A Voronovskaja-type result for $q-$derivative of $q-$Baskakov operators is given in [2].

Yet, a different type of $q-$Baskakov operator has also been introduced by Aral and Gupta in [3].

Recently, Cárdenas-Morales, Garrancho and Raşa [9] introduced a new type generelization of Bernstein polynomials denoted by B_n^{τ} and defined as

$$\begin{aligned} B_n^{\tau}(f;x) \quad : \quad &= B_n\left(f\circ\tau^{-1};\tau(x)\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}\tau^k(x)\,(1-\tau(x))^{n-k}\,(f\circ\tau^{-1})(\frac{k}{n}), \end{aligned} \tag{1.2}$$

where B_n is the $n-$th Bernstein polynomial, $f\in C[0,1]$, $x\in[0,1]$ and τ is a continuously differentiable of infinite order on $[0,1]$ such that $\tau(0)=0,\ \tau(1)=1$ and $\tau'(x)>0$ for $x\in[0,1]$. Also, the authors studied some shape preserving and convergence properties concerning the generalized Bernstein operators $B_n^{\tau}(f;x)$.

In [5], Aral, Inoan and Raşa constructed sequences of Szasz-Mirakyan operators which are based on a function ρ. They studied weighted approximation properties, Voronovskaja-type result for these operators. They also showed that the sequence of the generalized Szász-Mirakyan operators is monotonically nonincreasing under the $\rho-$convexity of the original function.

In the present paper, we consider a modification of the $q-$ Baskakov operators (1.1) in the sense of [5], we study some approximation and shape preserving properties of the new operators.

Motivated from [5] and [9], we define a new generalization of $q-$Baskakov operators for $f\in C[0,\infty)$ by

$$B_{n,q}^{\rho}(f;x)=\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n]}\right)\begin{bmatrix}n+k-1\\k\end{bmatrix}q^{\frac{k(k-1)}{2}}\rho^k(x)\,(-\rho(x),q)_{n+k}^{-1} \tag{1.3}$$

$q>0$ and ρ is a continuously differentiable function on $[0,\infty)$ such that

$$\rho(0)=0,\quad \inf_{x\in[0,\infty)}\rho'(x)\geq 1.$$

An example of such a function ρ is given in [5]. Note that, in the setting (1.3) we have

$$B_{n,q}^{\rho}f:=B_{n,q}\left(f\circ\rho^{-1}\right)\circ\rho,$$

where the operator $B_{n,q}$ is defined by (1.1). If $\rho=e_1$, then $B_{n,q}^{\rho}=B_{n,q}$. We can write the following equalities that are similar to the corresponding results for the $q-$Baskakov operators (1.1)

$$B_{n,q}^{\rho}(1;x)=1, \tag{1.4}$$

$$B_{n,q}^{\rho}(\rho;x)=\rho(x) \tag{1.5}$$

$$B_{n,q}^{\rho}\left(\rho^2;x\right)=\rho^2(x)+\frac{\rho(x)}{[n]}\left(1+\frac{\rho(x)}{q}\right). \tag{1.6}$$

$$
\begin{aligned}
& B_{n,q}^{\rho}\left(\rho^{3};x\right) \\
= \ & \rho^{3}\left(x\right)+\frac{1}{[n]}\left\{\rho^{2}\left(x\right)\left(1+\frac{\rho\left(x\right)}{q}\right)\left(\frac{2q+1}{q}\right)\right\} \\
& \frac{1}{[n]^{2}}\left\{\rho\left(x\right)\left(1+\frac{\rho\left(x\right)}{q}\right)\left(1+\frac{\rho\left(x\right)}{q^{2}}\right)+\frac{\rho^{2}\left(x\right)}{q}\left(1+\frac{\rho\left(x\right)}{q}\right)\right\}.
\end{aligned} \tag{1.7}
$$

The first purpose of the paper is to investigate uniform convergence of the operators (1.3) on weighted spaces which are defined using the function ρ and obtain the degree of weighted convergence, using weighted modulus of continuity. Next, we study the monotonic convergence under $\rho-$convexity of the function.

Troughout the paper we will consider the following class of functions. Let $\varphi\left(x\right)=1+\rho^{2}\left(x\right)$

$$B_{\varphi}\left(\mathbb{R}^{+}\right)=\left\{f:\mathbb{R}^{+}\rightarrow\mathbb{R},\ \left|f\left(x\right)\right|\leq M_{f}\varphi\left(x\right),\ x\geq0\right\}$$

where M_f is a constant depending on f.

$$C_{\varphi}\left(\mathbb{R}^{+}\right)=\left\{f\in B_{\varphi}\left(\mathbb{R}^{+}\right);\ f \text{ is continuous on } \mathbb{R}^{+}\right\}$$

$$C_{\varphi}^{k}\left(\mathbb{R}^{+}\right)=\left\{f\in C_{\varphi}\left(\mathbb{R}^{+}\right);\ \lim_{x\rightarrow\infty}\frac{f\left(x\right)}{\varphi\left(x\right)}=k_{f}\right\}$$

where k_f is a constant depending on f.

$$U_{\varphi}\left(\mathbb{R}^{+}\right)=\left\{f\in C_{\varphi}\left(\mathbb{R}^{+}\right);\ \frac{f\left(x\right)}{\varphi\left(x\right)} \text{ is uniformly continuous on } \mathbb{R}^{+}\right\}.$$

These spaces are normed spaces with the norm

$$\left\|f\right\|_{\varphi}=\sup_{x\in\mathbb{R}^{+}}\frac{\left|f\left(x\right)\right|}{\varphi\left(x\right)}.$$

Moreover, we shall use the following weighted modulus of continuity

$$\omega_{\rho}\left(f;\delta\right)=\sup_{\substack{x,t\in\mathbb{R}^{+} \\ \left|\rho(t)-\rho(x)\right|\leq\delta}}\frac{\left|f\left(t\right)-f\left(x\right)\right|}{\varphi\left(t\right)+\varphi\left(x\right)}$$

for each $f\in C_{\varphi}\left(\mathbb{R}^{+}\right)$ and for every $\delta>0$ [14]. We observe that $\omega_{\rho}\left(f;0\right)=0$ for every $f\in C_{\varphi}\left(\mathbb{R}^{+}\right)$ and the function $\omega_{\rho}\left(f;\delta\right)$ is nonnegative and nondecreasing with respect to δ for $f\in C_{\varphi}\left(\mathbb{R}^{+}\right)$.

Definition 1. *A continuous, real valued function f is said to be convex in $D\subseteq[0,\infty)$, if*

$$f\left(\sum_{i=1}^{m}\alpha_{i}x_{i}\right)\leq\sum_{i=1}^{m}\alpha_{i}f\left(x_{i}\right)$$

for every $x_1,x_2,...,x_m\in D$ and for every nonnegative numbers $\alpha_1,\alpha_2,...,\alpha_m$ such that $\alpha_1+\alpha_2+...+\alpha_m=1$.

In [9] Cárdenas-Morales, Garrancho and Raşa introduced the following definition of $\rho-$convexity of a continuous function.

Definition 2. *A continuous, real valued function f is said to be $\rho-$convex in D, if $f\circ\rho^{-1}$ is convex in the sense of Definition 1.*

2. Approximation Properties

In this section, we obtain the weighted uniform convergence of $B_{n,q}^{\rho}$ to f and the degree of approximation with the aid of weighted modulus of continuity. Let us recall the weighted form of the Korovkin Theorem ([12], [13]).

Lemma 1. [12] *The positive linear operators L_n, $n\geq 1$, act from $C_{\varphi}(\mathbb{R}^{+})$ to $B_{\varphi}(\mathbb{R}^{+})$ if and only if the inequality*

$$|L_n(\varphi;x)|\leq K_n\varphi(x),\ x\geq 0$$

holds; where K_n is a positive constant.

Theorem 1. [12] *Let the sequence of linear positive operators $(L_n)_{n\geq 1}$ acting from $C_{\varphi}(\mathbb{R}^{+})$ to $B_{\varphi}(\mathbb{R}^{+})$ satisfy the three conditions*

$$\lim_{n\to\infty}\|L_n\rho^{v}-\rho^{v}\|_{\varphi}=0,\ v=0,1,2. \tag{2.1}$$

Then for any function $f\in C_{\varphi}^{k}(\mathbb{R}^{+})$

$$\lim_{n\to\infty}\|L_nf-f\|_{\varphi}=0.$$

Now, we are ready to give the following theorem.

Theorem 2. *Let $B_{n,q}^{\rho}$ be the operator defined by (1.3). Then for any $f\in C_{\varphi}^{k}(\mathbb{R}^{+})$ and $q>1$, we have*

$$\lim_{n\to\infty}\left\|B_{n,q}^{\rho}f-f\right\|_{\varphi}=0.$$

Proof. By Lemma 1 $B_{n,q}^{\rho}$ are linear operators acting from $C_{\varphi}(\mathbb{R}^{+})$ to $B_{\varphi}(\mathbb{R}^{+})$. Indeed, from (1.4) and (1.6) we easily obtain that

$$\left|B_{n,q}^{\rho}(\varphi;x)\right|\leq\left(1+\rho^{2}(x)\right)\left(\frac{q[n]+q+1}{q[n]}\right).$$

On the other hand, using (1.4), (1.5) and (1.6), one can write

$$\left\|B_{n,q}^{\rho}1-1\right\|_{\varphi}=0,$$

$$\left\|B_{n,q}^{\rho}(\rho)-\rho\right\|_{\varphi}=0,$$

and

$$\left\|B_{n,q}^{\rho}(\rho^{2})-\rho^{2}\right\|_{\varphi}=\sup_{x\in\mathbb{R}^{+}}\frac{\rho(x)\left(1+\frac{\rho(x)}{q}\right)}{[n]\,1+\rho^{2}(x)}\leq\frac{2}{[n]}. \tag{2.2}$$

Therefore, the conditions (2.1) are satisfied. By Theorem 1, the proof is completed. □

In [14], the following theorem is given.

Theorem 3. *Let $L_n\colon C_\varphi(\mathbb{R}^+) \to B_\varphi(\mathbb{R}^+)$ be a sequence of positive linear operators with*

$$\|L_n(1) - 1\|_{\varphi^0} = a_n,$$
$$\|L_n(\rho) - \rho\|_{\varphi^{\frac{1}{2}}} = b_n,$$
$$\left\|L_n\left(\rho^2\right) - \rho^2\right\|_{\varphi} = c_n,$$
$$\left\|L_n\left(\rho^3\right) - \rho^3\right\|_{\varphi^{\frac{3}{2}}} = d_n,$$

where a_n, b_n, c_n and d_n tend to zero as $n \to \infty$. Then

$$\|L_n(f) - f\|_{\varphi^{\frac{3}{2}}} \le (7 + 4a_n + 2c_n)\,\omega_\rho(f;\delta_n) + \|f\|_\varphi a_n$$

for all $f \in C_\varphi(\mathbb{R}^+)$, where

$$\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + (a_n + 3b_n + 3c_n + d_n).$$

Applying the above theorem, we obtain the degree of approximation.

Theorem 4. *For all $f \in C_\varphi(\mathbb{R}^+)$ and $q > 1$, we have*

$$\left\|B^\rho_{n,q}(f) - f\right\|_{\varphi^{\frac{3}{2}}} \le \left(7 + \frac{4}{[n]}\right)\omega_\rho\left(f;\frac{2\sqrt{2}}{\sqrt{[n]}} + \frac{18}{[n]}\right).$$

Proof. According to Theorem 3, we shall calculate the sequences a_n, b_n, c_n and d_n. From (1.4), (1.5), (2.2) and (1.7)we get

$$a_n = \left\|B^\rho_{n,q}(1) - 1\right\|_{\varphi^0} = 0,$$
$$b_n = \left\|B^\rho_{n,q}(\rho) - \rho\right\|_{\varphi^{\frac{1}{2}}} = 0,$$
$$c_n = \left\|B^\rho_{n,q}\left(\rho^2\right) - \rho^2\right\|_{\varphi} = \sup_{x\in\mathbb{R}^+}\frac{\rho(x)\left(1 + \frac{\rho(x)}{q_n}\right)}{[n]\left(1 + \rho^2(x)\right)} \le \frac{2}{[n]},$$

$$\begin{aligned} d_n &= \left\|B^\rho_{n,q}\left(\rho^3\right) - \rho^3\right\|_{\varphi^{\frac{3}{2}}} \\ &= \sup_{x\in\mathbb{R}^+}\left\{\frac{1}{[n]}\left[\frac{\rho^2(x)\left(1 + \frac{\rho(x)}{q}\right)\left(\frac{2q+1}{q}\right)}{\left(1 + \rho^2(x)\right)^{\frac{3}{2}}}\right]\right. \\ &\quad \left.\frac{1}{[n]^2}\left[\frac{\rho(x)\left(1 + \frac{\rho(x)}{q}\right)\left(1 + \frac{\rho(x)}{q^2}\right) + \frac{\rho^2(x)}{q}\left(1 + \frac{\rho(x)}{q}\right)}{\left(1 + \rho^2(x)\right)^{\frac{3}{2}}}\right]\right\} \\ &\le \frac{12}{[n]}. \end{aligned}$$

By Theorem 3, the proof is completed. □

3. Monotonicity Properties of $B_{n,q}^{\rho}$

Here, we study the monotonic convergence of the operators (1.3) under the $\rho-$convexity.

Theorem 5. *Let f be a $\rho-$ convex function on $[0,\infty)$. Then we have*

$$B_{n,q}^{\rho}(f;x) \geq B_{n+1,q}^{\rho}(f;x)$$

for $n \in \mathbb{N}$.

Proof. From (1.3), one can write

$$\begin{aligned}
B_{n,q}^{\rho}(f;x)&-B_{n+1,q}^{\rho}(f;x) \\
&= \sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k\end{bmatrix} q^{\frac{k(k-1)}{2}} \frac{\rho^{k}(x)}{(-\rho(x),q)_{n+k}} \left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n]}\right) \\
&\quad -\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k\end{bmatrix} q^{\frac{k(k-1)}{2}} \frac{\rho^{k}(x)}{(-\rho(x),q)_{n+k+1}} \left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n+1]}\right) \\
&= \sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k\end{bmatrix} q^{\frac{k(k-1)}{2}} \frac{\rho^{k}(x)}{(-\rho(x),q)_{n+k}} \left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n]}\right) \\
&\quad -\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k\end{bmatrix} q^{\frac{k(k-1)}{2}} \frac{\rho^{k}(x)}{(-\rho(x),q)_{n+k}} \left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n+1]}\right) \\
&\quad +\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k\end{bmatrix} q^{\frac{k(k-1)}{2}} q^{n+k} \frac{\rho^{k+1}(x)}{(-\rho(x),q)_{n+k+1}} \left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n+1]}\right) \\
&= \sum_{k=1}^{\infty}\begin{bmatrix} n+k-1 \\ k\end{bmatrix} q^{\frac{k(k-1)}{2}} \frac{\rho^{k}(x)}{(-\rho(x),q)_{n+k}} \left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n]}\right) \\
&\quad -\sum_{k=1}^{\infty}\begin{bmatrix} n+k \\ k\end{bmatrix} q^{\frac{k(k-1)}{2}} \frac{\rho^{k}(x)}{(-\rho(x),q)_{n+k}} \left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n+1]}\right) \\
&\quad +\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k\end{bmatrix} q^{\frac{k(k-1)}{2}} q^{n+k} \frac{\rho^{k+1}(x)}{(-\rho(x),q)_{n+k+1}} \left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n+1]}\right).
\end{aligned}$$

Rearranging the above equality, we have

$$\begin{aligned}
&B_{n,q}^{\rho}(f;x)-B_{n+1,q}^{\rho}(f;x)\\
=&\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k+1\end{bmatrix}q^{\frac{k(k-1)}{2}}q^{k}\frac{\rho^{k+1}(x)}{(-\rho(x),q)_{n+k+1}}\left(f\circ\rho^{-1}\right)\left(\frac{[k+1]}{q^{k}[n]}\right)\\
&-\sum_{k=0}^{\infty}\begin{bmatrix}n+k+1\\k+1\end{bmatrix}q^{\frac{k(k-1)}{2}}q^{k}\frac{\rho^{k+1}(x)}{(-\rho(x),q)_{n+k+1}}\left(f\circ\rho^{-1}\right)\left(\frac{[k+1]}{q^{k}[n+1]}\right)\\
&+\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}q^{\frac{k(k-1)}{2}}q^{n+k}\frac{\rho^{k+1}(x)}{(-\rho(x),q)_{n+k+1}}\left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n+1]}\right)\\
=&\sum_{k=0}^{\infty}q^{\frac{k(k-1)}{2}}q^{k}\frac{\rho^{k+1}(x)}{(-\rho(x),q)_{n+k+1}}\left\{\begin{bmatrix}n+k\\k+1\end{bmatrix}\left(f\circ\rho^{-1}\right)\left(\frac{[k+1]}{q^{k}[n]}\right)\right.\\
&-\begin{bmatrix}n+k+1\\k+1\end{bmatrix}\left(f\circ\rho^{-1}\right)\left(\frac{[k+1]}{q^{k}[n+1]}\right)\\
&\left.+\begin{bmatrix}n+k\\k\end{bmatrix}q^{n}\left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n+1]}\right)\right\}
\end{aligned}$$

By the following equalities

$$\begin{aligned}
\begin{bmatrix}n+k+1\\k+1\end{bmatrix}&=\frac{[n+k+1]}{[k+1]}\begin{bmatrix}n+k\\k\end{bmatrix}\\
\begin{bmatrix}n+k\\k+1\end{bmatrix}&=\frac{[n]}{[k+1]}\begin{bmatrix}n+k\\k\end{bmatrix},
\end{aligned}$$

we get

$$\begin{aligned}
&B_{n,q}^{\rho}(f;x)-B_{n+1,q}^{\rho}(f;x)\\
=&\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}q^{\frac{k(k-1)}{2}}q^{k}\frac{\rho^{k+1}(x)}{(-\rho(x),q)_{n+k+1}}\frac{[n+k+1]}{[k+1]}\\
&\times\left\{\frac{[n]}{[n+k+1]}\left(f\circ\rho^{-1}\right)\left(\frac{[k+1]}{q^{k}[n]}\right)-\left(f\circ\rho^{-1}\right)\left(\frac{[k+1]}{q^{k}[n+1]}\right)\right.\\
&\left.+q^{n}\frac{[k+1]}{[n+k+1]}\left(f\circ\rho^{-1}\right)\left(\frac{[k]}{q^{k-1}[n+1]}\right)\right\}.
\end{aligned}$$

By taking, $\lambda_1=\frac{[n]}{[n+k+1]}\geq 0$, $\lambda_2=q^{n}\frac{[k+1]}{[n+k+1]}\geq 0$, $\lambda_1+\lambda_2=1$ and $x_1=\frac{[k+1]}{q^{k}[n]}, x_2=\frac{[k]}{q^{k-1}[n+1]}$ one has

$$\lambda_1x_1+\lambda_2x_2=\frac{[k+1]}{q^{k}[n+1]}.$$

Therefore, we obtain that

$$B_{n,q}^{\rho}(f;x)-B_{n+1,q}^{\rho}(f;x)\geq 0$$

by $\rho-$convexity of f for $x \in [0, \infty)$ and $n \in \mathbb{N}$. This proves the theorem. $\square$

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