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NIL-REFLEXIVE RINGS

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Abstract. In this paper, we deal with a new approach to reflexive property for rings by using nilpotent elements, in this direction we introduce nil-reflexive rings. It is shown that the notion of nil-reflexivity is a generalization of that of nil-semicommutativity. Examples are given to show that nil-reflexive rings need not be reflexive and vice versa, and nil-reflexive rings but not semicommutative are presented. We also proved that every ring with identity is weakly reflexive defined by Zhao, Zhu and Gu. Moreover, we investigate basic properties of nil-reflexive rings and provide some source of examples for this class of rings. We consider some extensions of nil-reflexive rings, such as trivial extensions, polynomial extensions and Nagata extensions.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Mason introduced the reflexive property for ideals, and this concept was generalized by some authors, defining idempotent reflexive right ideals and rings, completely reflexive rings, weakly reflexive rings (see namely, [6], [9], [13]). Let R be a ring and I be a right ideal of R. In [13], I is called *a reflexive right ideal* if for any $x, y \in R$, $xRy \subseteq I$ implies $yRx \subseteq I$. The reflexive right ideal concept is also specialized to the zero ideal of a ring, namely, a ring R is called *reflexive* [13] if its zero ideal is reflexive. Reflexive rings are generalized to weakly reflexive rings in [13]. The ring R is said to be *weakly reflexive* if arb $= 0$ implies bra is nilpotent for a, $b \in R$ and all $r \in R$. Motivated by the works on reflexivity, in this note we study the reflexivity property in terms of nilpotent elements, namely, nil-reflexive rings. It is shown by examples that the class of reflexive rings and the class of nil-reflexive rings are incomparable. In [13], a ring R is called *completely reflexive* if for any a, $b \in R$, $ab = 0$ implies $ba = 0$. Completely reflexive rings are called *reversible* by Cohn in [4] and also studied in [5]. The rings without nonzero nilpotent

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elements are said to be *reduced* rings. Reduced rings are completely reflexive and every completely reflexive ring is semicommutative, i.e. according to [12], a ring R is called *semicommutative* if for all a, $b \in R$, $ab = 0$ implies $aRb = 0$. This is equivalent to the definition that any left (right) annihilator of R is an ideal of R. In [3], semicommutativity of rings is generalized to nil-semicommutativity of rings. A ring R is called *nil-semicommutative* if $a, b \in R$ satisfy that ab is nilpotent, then $arb \in nil(R)$ for any $r \in R$ where $nil(R)$ denotes the set of all nilpotent elements of R. Clearly, every semicommutative ring is nil-semicommutative. In this paper it is proved that the class of nil-reflexive rings lies strictly between the classes of nil-semicommutative rings and weakly reflexive rings.

We first summarize the contents of the sections of this paper. First section is the introduction. In the second section, we investigate the structure of nil-reflexive rings, and some basic characterizations of these rings are obtained. We also deal with relations between nil-reflexive rings and certain classes of rings. We present some examples to illustrate nil-reflexive rings. Examples are given to show that the notions of reflexive rings and nil-reflexive rings do not imply each other. Nil-reflexive rings share a number of important properties with other classes of rings. For instance, among other interesting results, we prove every semicommutative ring is nil-reflexive. For a nil ideal I of a ring R, it is proved that R is nil-reflexive if and only if R/I is nil-reflexive. Also if R is a nil-reflexive ring, then $T_n(R)$, $S_n(R)$ and $V_n(R)$ (see below for the definitions) are nil-reflexive. It is shown that every corner ring of any nil-reflexive ring inherits the nil-reflexive property. On the other hand, we determine abelian semiperfect nil-reflexive rings, they are exactly in the form of a finite direct sum of local nil-reflexive rings. In the third section, we study some extensions of nil-reflexive rings and it is proved that a ring R with a multiplicatively closed subset U consisting of some central elements is nil reflexive if and only if $U^{-1}R$ is nil-reflexive; for a ring R, $R[x]$ is nil-reflexive if and only if $R[x; x^{-1}]$ is nil-reflexive; if R is a nil-reflexive and Armendariz ring, then $R[x]$ is a nil-reflexive ring. Also, R is nil-reflexive if and only if its trivial extension $T(R, R)$ is nil-reflexive. We also deal with the Nagata extension $N[R, R; \alpha]$ of a commutative ring R in terms of nil-reflexivity.

In what follows, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of natural numbers, the ring of integers and the ring of rational numbers, and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. For a positive integer n, let $Mat_n(R)$ denote the ring of all $n \times n$ matrices and $T_n(R)$ the ring of all $n \times n$ upper triangular matrices with entries in R. We write $R[x]$, $U(R)$, $P(R)$, and $S_n(R)$ $(V_n(R))$ for the polynomial ring over a ring R, the set of invertible elements, the prime radical of R, and the subring consisting of all upper triangular matrices over a ring R with equal main diagonal (every diagonal) entries, respectively.

2. Nil-Reflexivity of Rings

In this section, we introduce nil-reflexive rings, and investigate basic properties of this class of rings. We also study the relations between nil-reflexive rings and some certain classes of rings.

Definition 2.1. A ring R is said to be *nil-reflexive* if for any $a, b \in R$, arb being nilpotent implies that bra is nilpotent for all $r \in R$.

In the next, we provide some examples for nil-reflexive rings. The third example in the following also shows that nil-reflexive rings need not be reflexive. In [6, Theorem 2.6], Kwak and Lee proved that R is a reflexive ring if and only if $Mat_n(R)$ is a reflexive ring for all $n \geq 1$. However, this is not the case in nil-reflexivity of R. There are nil-reflexive rings over which matrix rings need not be nil-reflexive as shown below.

Examples 2.2. (1) Let R be a ring with $nil(R)$ an ideal of R. Then R is nil-reflexive.

(2) For any reduced ring S, the ring $T_n(S)$ is nil-reflexive. However, the ring of all 2×2 matrices over any field is not nil-reflexive.

(3) Let R be a reduced ring. Consider the ring

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R; 1 \leq i, j \leq n \right\}.$$

Then $S_n(R)$ is not reflexive when $n \geq 4$, but $S_n(R)$ and R are nil-reflexive for all $n \geq 1$.

Proof. (1) Let a, $b \in R$. Assume that arb is nilpotent for all $r \in R$. Then $ab \in nil(R)$ and so $brabra \in nil(R)$. Hence bra is nilpotent for all $r \in R$. Hence R is nil-reflexive.

(2) For a ring R, by [2], $nil(T_n(R)) = \begin{pmatrix} nil(R) & R & R & \cdots & R \\ 0 & nil(R) & R & \cdots & R \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & nil(R) \end{pmatrix}$. Let S be a reduced ring. Then $nil(S) = 0$ and so $nil(T_n(S))$ is an ideal. By (1), $T_n(S)$ is nil-reflexive. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in Mat_2(F)$ where F is

a field. For any $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat_2(F)$, $ACB = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}$ is nilpotent for all $C \in Mat_2(F)$, but for $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in Mat_2(F)$, $BCA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not nilpotent. Therefore $Mat_2(F)$ is not nil-reflexive.

(3) It is proved in [6] that $S_n(R)$ is not reflexive when $n \geq 4$. Since R is reduced, R is nil-reflexive. Note that

$$nil(S_n(R)) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a \in nil(R), a_{ij} \in R, 1 \leq i, j \leq n \right\}.$$

The ring R being reduced implies that $nil(S_n(R))$ is an ideal. By (1), $S_n(R)$ is nil-reflexive. □

Lemma 2.3. *For a ring R, consider the following conditions.*

1. *R is nil-reflexive.*
2. *If ARB is a nil set, then so is BRA for any subsets A, B of R.*
3. *If IJ is nil, then JI is nil for all right (or left) ideals I, J of R.*

Then (1) ⇒ (2) ⇒ (3).

Proof. (1) ⇒ (2) Assume that R is a nil-reflexive ring and ARB is a nil set. For any $a \in A$, $b \in B$, arb is nilpotent for all $r \in R$, then bra is nilpotent. This implies that BRA is nil.

(2) ⇒ (3) Let I and J be any right ideals of R such that IJ is nil. Since $IR \subseteq I$, IRJ is nil. By (2), JRI is nil. Since $JI \subseteq JRI$, we get JI is nil. Assume that I and J be any left ideals of R such that IJ is nil. Since $RJ \subseteq J$ and then $IRJ \subseteq IJ$, IRJ is nil. By (2), JRI is nil. Since $JI \subseteq JRI$, we get JI is nil. □

Lemma 2.4. *The following conditions are equivalent for a ring R.*

1. *$aR \subseteq nil(R)$ for any $a \in nil(R)$.*
2. *$Ra \subseteq nil(R)$ for any $a \in nil(R)$.*

Proof. (1) ⇒ (2) Assume that $ar \in nil(R)$ for all $r \in R$, for any $a \in nil(R)$. Let $(ar)^n = 0$ for some positive integer n. Then $(ra)^{n+1} = 0$, hence ra is nilpotent. Thus $Ra \subseteq nil(R)$. Similarly, we can show (2) ⇒ (1). □

The next result gives a source of nil-reflexive rings.

Proposition 2.5. *Let R be a ring such that $aR \subseteq nil(R)$ for any $a \in nil(R)$. Then R is nil-reflexive.*

Proof. Assume that for $a, b \in R$, $arb \in nil(R)$ for any $r \in R$. So $ab \in nil(R)$. By hypothesis, $abR \subseteq nil(R)$. Then there exists $m \in \mathbb{N}$ such that $(abr)^m = 0$. Hence $br\underbrace{(abrabrabr\ldots abr)}a = (bra)^{m+1} = 0$. So $bra \in nil(R)$ for any $r \in R$. □

[13, Example 2.1] shows that any semicommutative ring need not be reflexive, but this is not the case when we deal with nil-reflexive rings. It can be observed that every semicommutative ring is nil-reflexive as a consequence of Proposition 2.5. But we give its direct proof in the next.

Lemma 2.6. *If R is a semicommutative ring, then it is nil-reflexive.*

Proof. Assume that R is semicommutative and arb is nilpotent for a, $b \in R$ and for all $r \in R$. Let $r \in R$ with $(arb)^n = 0$ for some positive integer n.

$$\underbrace{(arb)(arb)(arb)\ldots(arb)}_{\text{n-times}} = 0 \tag{2.1}$$

In (2.1) insert b before r, and a after r to have

$$a(bra)ba(bra)ba(bra)b\ldots ba(bra)b = 0 \tag{2.2}$$

Replacing ba in (2.2) by bra to obtain

$$a(bra)(bra)(bra)(bra)(bra)bra\ldots bra(bra)b = 0 \tag{2.3}$$

Multiplying the equation (2.3) by br from the left and by ra from the right, we get

$$(bra)(bra)(bra)(bra)(bra)(bra)br\ldots bra(bra)(bra) = (bra)^{2n+1} = 0 \tag{2.4}$$

Hence R is nil-reflexive. □

In [13], weakly reflexive rings are studied in detail for rings having an identity. However weakly reflexive rings are nothing but all rings with identity as it is shown below.

Lemma 2.7. *Every ring with identity is weakly reflexive.*

Proof. Let $a, b \in R$ with $arb = 0$ for all $r \in R$. Then $ab = 0$ and so $(bra)^2 = br(ab)ra = 0$ for all $r \in R$. Hence bra is nilpotent for all $r \in R$. Thus R is weakly reflexive. □

Lemma 2.6 and the next result show that the class of nil-reflexive rings lies between the classes of semicommutative rings and weakly reflexive rings. It is known that every completely reflexive ring is semicommutative and so nil-reflexive by Lemma 2.6. In the following, we give the direct proof of this fact for the sake of completeness.

Lemma 2.8. *If R is a completely reflexive ring, then it is nil-reflexive.*

Proof. Let R be a completely reflexive ring and $a, b \in R$. Assume that arb is nilpotent for all $r \in R$. Then there exists $n \in \mathbb{N}$ such that $(arb)^n = 0$. For any $r \in R$, we apply successively completely reflexivity of R to get $0 = (ab)(ab)(ab)\dots(ab)abr = br(ab)(ab)(ab)\dots(ab)a = abr(ab)(ab)(ab)\dots(ab)abr = brabr(ab)(ab)(ab)\dots(ab)a = (bra)(bra)b(ab)(ab)\dots(ab)a = a(bra)(bra)b(ab)\dots(ab) = a(bra)(bra)b(ab)\dots(ab)r = (bra)(bra)(bra)b(ab)\dots a = \cdots = (bra)^n$. Therefore R is nil-reflexive. □

Now we shall give an example to show that there exists a nil-reflexive ring which is not reflexive. Also reflexive rings may not be nil-reflexive either as shown below.

Example 2.9. There exists a nil-reflexive ring which is neither reflexive nor semicommutative.

Proof. Let R be a reduced ring. By Examples 2.2(2), $T_2(R)$ is nil-reflexive. On the other hand, $nil(T_2(R)) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R \right\}$. Consider $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in T_2(R)$. Then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R & R \\ 0 & R \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$ for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in R$. This shows that $T_2(R)$ is not reflexive. $T_2(R)$ is also not semicommutative. For if, $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in T_2(R)$, then $AB = 0$ but $ACB \neq 0$. □

Example 2.10. Consider the ring $Mat_2(F)$ where F is a field. Since F is a semiprime ring, $Mat_2(F)$ is also semiprime due to [7, Proposition 10.20]. This implies that $Mat_2(F)$ is a reflexive ring, it is also weakly reflexive. On the other hand, $Mat_2(F)$ is not nil-reflexive by Examples 2.2(2).

Note that there are nil-reflexive rings but not completely reflexive as the following example shows.

Example 2.11. Let F be a field. Then $T_2(F)$ is nil-reflexive which is not completely reflexive.

We now observe some relations among nil-reflexive rings, nil-semicommutative rings and semiprime rings. According to next result, the class of nil-reflexive rings is weaker than that of nil-semicommutative rings.

Proposition 2.12. *Every nil-semicommutative ring is nil-reflexive.*

Proof. Let R be a nil-semicommutative ring. Let $a, b \in R$ with $arb \in nil(R)$ for all $r \in R$. In particular, for $r = 1$, we have $ab \in nil(R)$. So $ba \in nil(R)$. Since R is nil-semicommutative, $bra \in nil(R)$ for any $r \in R$. Thus R is a nil-reflexive ring. □

It is easy to check that every semiprime ring is reflexive (see, for example [6]). We have given an example showing that this is not the case for nil-reflexive rings in Example 2.10. There are also nil-reflexive rings but not semiprime.

Example 2.13. Let D be a division ring. Consider the ring

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in D \right\}.$$

Then R is nil-reflexive but not semiprime.

Proof. R is not semiprime since the set consisting of all main diagonal off elements of R is a nonzero nilpotent ideal. On the other hand, R is nil-reflexive by Examples 2.2(3). □

In the next, we investigate the relations between a ring R and R/I for some ideal I of R in terms of nil-reflexivity. Lambek called a ring R *symmetric* [8] provided that $abc = 0$ implies $acb = 0$ for all a, b, $c \in R$. By Lemma 2.6, every symmetric ring is nil-reflexive.

Proposition 2.14. *Let R be a ring. Then the following hold.*

(1) *Let I be an ideal of R contained in $nil(R)$. Then R is nil-reflexive if and only if R/I is nil-reflexive.*

(2) *If R is symmetric, I is an ideal of R and I is a right annihilator $I = r_R(S)$ for some nonempty subset S of R, then R/I is nil-reflexive.*

Proof. (1) "$\Longrightarrow$" Let $\bar{a}, \bar{b} \in R/I$ with $\bar{a}\bar{r}\bar{b} \in nil(R/I)$ for all $\bar{r} \in R/I$. Then there exists $n \in \mathbb{N}$ such that $(\bar{a}\bar{r}\bar{b})^n = \bar{0}$. So $(arb)^n \in I$. Since $I \subseteq nil(R)$, $(arb)^n \in nil(R)$. Hence $arb \in nil(R)$. Since R is nil-reflexive, $bra \in nil(R)$. Thus $\bar{b}\bar{r}\bar{a} \in nil(R/I)$ for all $\bar{r} \in R/I$. Therefore R/I is nil-reflexive.
"$\Longleftarrow$" Let $a, b \in R$ and suppose that $arb \in nil(R)$ for all $r \in R$. Then $\bar{a}\bar{r}\bar{b} \in nil(R/I)$ and so $\bar{b}\bar{r}\bar{a} \in nil(R/I)$ since R/I is nil-reflexive. There exists $m \in \mathbb{N}$ such that $(\bar{b}\bar{r}\bar{a})^m = \bar{0}$. This shows that $(bra)^m \in I$. Since $I \subseteq nil(R)$, $(bra)^m \in nil(R)$. So there exists $n \in \mathbb{N}$ such that $((bra)^m)^n = 0$ and so $bra \in nil(R)$. This implies that R is nil-reflexive.
(2) Let a, $b \in R$ with $\overline{arb} \in nil\,(R/I)$ for all $\overline{r} \in R/I$. There exists a positive integer t such that $(arb)^t \in I$ and so $S(arb)^t = 0$. Hence

$$\begin{aligned}
0 = S(arb)^t &= S\underbrace{(arb)(arb)(arb)\dots(arb)(arb)(arb)}_{t-times} \\
&= S(arb)(arb)(arb)\dots(arb)(arb)(arb)b \\
&= S(bar)(bar)(bar)b\dots(arb)(arb)(arb) \\
&= S(bar)(bar)(bar)b\dots(arb)(arb)(ar)(bra) \\
&= S(bra)(bar)(bar)(bar)b\dots(arb)(arb)(ar) \\
&= S(bra)(bar)(bar)(bar)b\dots(arb)(ar)(bar) \\
&= S(bra)(bar)(bar)(bar)b\dots(arb)(ar)(bra) \\
&= S(bra)(bra)(bar)(bar)(bar)b\dots(arb)(ar).
\end{aligned}$$

We continue in this way to have $S(bra)^{t+1} = 0$. Hence $(bra)^{t+1} \in I$. This shows that $\overline{bra}$ is nilpotent for each $\overline{r} \in R/I$. Thus R/I is nil-reflexive. □

Now we give some characterizations of nil-reflexivity by using the prime radical of a ring, upper triangular matrix rings and polynomial rings.

Corollary 2.15. *A ring R is nil-reflexive if and only if $R/P(R)$ is nil-reflexive.*

Proof. Since every element of $P(R)$ is nilpotent, it follows from Proposition 2.14. □

Proposition 2.16. *A ring R is nil-reflexive if and only if $T_n(R)$ is nil-reflexive, for any positive integer n.*

Proof. Let $A = (a_{ij})$, $B = (b_{ij}) \in T_n(R)$, with $ACB \in nil(T_n(R))$ for all $C = (c_{ij}) \in T_n(R)$, where $1 \leq i \leq j \leq n$. Then we have $a_{ii}c_{ii}b_{ii} \in nil(R)$ for any $1 \leq i \leq n$. Since R is nil-reflexive, $b_{ii}c_{ii}a_{ii} \in nil(R)$. So it follows from $BCA \in nil(T_n(R))$

that $T_n(R)$ is nil-reflexive. Conversely, let $a, b \in R$ with $arb \in nil(R)$ for all $r \in R$. Then $\begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} r & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} b & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in nil(T_n(R))$. Since $T_n(R)$ is nil-reflexive,

$$\begin{pmatrix} b & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} r & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in nil(T_n(R)).$$

So $bra \in nil(R)$. Therefore R is nil-reflexive. □

Proposition 2.17. *A ring R is nil-reflexive if and only if $R[x]/(x^n)$ is nil-reflexive for any $n \geq 1$ where (x^n) is the ideal generated by x^n in $R[x]$.*

Proof. Note that for $n = 1$, $R[x]/(x) \cong R$ and for $n \geq 2$,

$$R[x]/(x^n) \cong \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \ddots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{pmatrix} \mid a_i \in R, 1 \leq i \leq n \right\} = V_n(R)$$

and $V_n(R)$ is a subring of $T_n(R)$. Therefore

$$nil(V_n(R)) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \ddots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{pmatrix} \mid a_1 \in nil(R), a_2, \ldots, a_n \in R \right\}.$$

The proof is completed as in the proof of Proposition 2.16. □

Theorem 2.18. *Let I and K be ideals of a ring R. Assume that $R = I \oplus K$ is a ring direct sum of I and K. Then R is nil-reflexive if and only if I and K are nil-reflexive rings.*

Proof. Note that $R = I \oplus K$ is a ring direct sum of ideals I and K. Then I and K become rings with identity. Suppose that R is nil-reflexive. Let x, $y \in I$ with xiy nilpotent for all $i \in I$. Then $(x, 0)(i, k)(y, 0)$ is nilpotent for all $(i, k) \in R$. By supposition, $(y, 0)(i, k)(x, 0)$ is nilpotent for all $(i, k) \in R$. Hence yix is nilpotent for all $i \in I$ or I is nil-reflexive. A similar discussion proves that K is also nil-reflexive. Conversely, assume that I and K are nil-reflexive rings. Let (x, y), $(x', y') \in R$ with $(x, y)(x'', y'')(x', y')$ nilpotent for all $(x'', y'') \in R$. Then $xx''x'$ is nilpotent for all $x'' \in I$ and $yy''y'$ is nilpotent for all $y'' \in K$. By assumption, $x'x''x$ is nilpotent for all $x'' \in I$ and $y'y''y$ is nilpotent for all $y'' \in K$. Then $(x', y')(x'', y'')(x, y)$ is nilpotent for all $(x'', y'') \in R$. Hence R is nil-reflexive. □

Proposition 2.19. *Finite product of nil-reflexive rings is nil-reflexive.*

Proof. Let $\{R_i\}_{i \in I}$ be a class of nil-reflexive rings for an indexed set $I = \{1, 2, \ldots, n\}$ where $n \in \mathbb{N}$. By [10, Proposition 2.13], $nil(\prod_{i=1}^{n} R_i) = \prod_{i=1}^{n} nil(R_i)$. Suppose that for any $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \prod_{i=1}^{n} R_i$,

$$(a_1, a_2, \ldots, a_n)(r_1, r_2, \ldots, r_n)(b_1, b_2, \ldots, b_n) \in nil(\prod_{i=1}^{n} R_i)$$

for all $(r_1, r_2, \ldots, r_n) \in \prod_{i=1}^{n} R_i$. Then we have $a_i r_i b_i \in nil(R_i)$ for each $i = 1, 2, \ldots, n$. Since R_i is nil-reflexive, $b_i r_i a_i \in nil(R_i)$ for each $1 \leq i \leq n$. Therefore $(b_1, b_2, \ldots, b_n)(r_1, r_2, \ldots, r_n)(a_1, a_2, \ldots, a_n) \in nil(\prod_{i=1}^{n} R_i)$. □

In the next result it is presented that any corner ring of a nil-reflexive ring inherits the nil-reflexivity property. But the nil-reflexivity property is not Morita invariant because of Examples 2.2(2).

Proposition 2.20. *Let R be a ring and $e^2 = e \in R$. If R is nil-reflexive, then so is eRe.*

Proof. Let $exe, eye \in eRe$ with $(exe)(ere)(eye) \in nil(eRe)$ for all $ere \in eRe$. Then there exists $m \in \mathbb{N}$ such that $((exe)(ere)(eye))^m = 0$. Hence $(exe)r(eye) \in nil(R)$.

Since R is nil-reflexive, we have $(eye)r(exe) \in nil(R)$. Thus $(eye)(ere)(exe) \in nil(eRe)$. □

Corollary 2.21. *For a central idempotent e of a ring R, eR and $(1-e)R$ are nil-reflexive if and only if R is nil-reflexive.*

Proof. Assume that eR and $(1-e)R$ are nil-reflexive. Since the nil-reflexivity property is closed under finite direct products, $R \cong eR \times (1-e)R$ is nil-reflexive. The converse is trivial by Proposition 2.20. □

By Examples 2.2(2), for any positive integer n, there are rings R for which $Mat_n(R)$ can not be nil-reflexive. However the converse statement holds as the next result shows.

Corollary 2.22. *Let R be a ring. If $Mat_n(R)$ is a nil-reflexive ring for some $n \in \mathbb{N}$, then R is a nil-reflexive ring.*

Proof. Let E_{11} denote the matrix unit whose $(1,1)$ entry is 1 and all other entries are zero. Assume that $Mat_n(R)$ is nil-reflexive. Then $R \cong RE_{11} = E_{11}Mat_n(R)E_{11}$ is nil-reflexive by Proposition 2.20. □

Recall that a ring R is said to be *abelian* if every idempotent is central, that is, $ae = ea$ for any $e^2 = e$, $a \in R$. There exists a nil-reflexive ring which is not abelian as shown below.

Example 2.23. Let F be a field. The ring $T_2(F)$ is nil-reflexive by Example 2.2(2). But for an idempotent $E = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} \in T_2(F)$ and for $A = E_{12} \in T_2(F)$, $EA \neq AE$. Thus $T_2(F)$ is not an abelian ring.

We close this section by determining abelian semiperfect nil-reflexive rings.

Theorem 2.24. *Let R be a ring. Consider the following statements.*

(1) *R is a finite direct sum of local nil-reflexive rings.*

(2) *R is a semiperfect nil-reflexive ring.*

Then $(1) \Rightarrow (2)$. *If R is abelian, then* $(2) \Rightarrow (1)$.

Proof. $(1) \Rightarrow (2)$ Assume that R is a finite direct sum of local nil-reflexive rings. Then R is semiperfect because local rings are semiperfect and a finite direct sum of semiperfect rings is semiperfect, and moreover R is nil-reflexive by Proposition

2.19.

(2) $\Rightarrow$ (1) Suppose that R is an abelian semiperfect nil-reflexive ring. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \dots, e_n\}$ of local idempotents whose sum is 1 by [1, Theorem 27.6], say $1 = e_1 + e_2 + \cdots + e_n$ such that each e_iRe_i is a local ring where $1 \leq i \leq n$. The ring R being abelian implies $e_iRe_i = e_iR$. Each e_iR is a nil-reflexive ring by Proposition 2.20. Hence R is nil-reflexive by Proposition 2.19. □

3. Extensions of Nil-Reflexive Rings

In this section, we consider some extensions of nil-reflexive rings and characterize nil-reflexive rings from various aspects. Let R be a ring and U be a multiplicative closed subset of R consisting of some central regular elements, that is, for any element $u \in U$, $ur = 0$ implies that $r = 0$ and u is in the center of R. Consider the ring $U^{-1}R = \{u^{-1}r \mid u \in U, r \in R\}$. In the following, we obtain a characterization of nil-reflexivity of the ring $U^{-1}R$.

Proposition 3.1. *A ring R is nil-reflexive if and only if $U^{-1}R$ is nil-reflexive.*

Proof. Assume that R is a nil-reflexive ring. Let $u^{-1}a$, $v^{-1}b \in U^{-1}R$ be such that $(u^{-1}a)(s^{-1}r)(v^{-1}b)$ is nilpotent for all $s^{-1}r \in U^{-1}R$. Since $(u^{-1}a)(s^{-1}r)(v^{-1}b) = (usv)^{-1}(arb)$ and usv is central and invertible, arb is nilpotent for all $r \in R$. By assumption bra is nilpotent for all $r \in R$. It gives rise to $(v^{-1}b)(s^{-1}r)(u^{-1}a)$ is nilpotent for all $s^{-1}r \in U^{-1}R$. So $U^{-1}R$ is nil-reflexive. Conversely, suppose that $U^{-1}R$ is nil-reflexive. Let a, $b \in R$ with asb nilpotent for each $s \in R$. Then for any $u \in U$, $(u^{-1}a)(u^{-1}s)(u^{-1}b) = u^{-3}asb$ is nilpotent for each $u^{-1}s \in U^{-1}R$. By supposition $(u^{-1}b)(u^{-1}s)(u^{-1}a) = u^{-3}bsa$ is nilpotent for each $u^{-1}s \in U^{-1}R$. Since u is central invertible, bsa is nilpotent for each $s \in R$. This completes the proof. □

Corollary 3.2. *For a ring R, $R[x]$ is nil-reflexive if and only if $R[x; x^{-1}]$ is nil-reflexive.*

Proof. Let $U = \{1, x, x^2, \dots\}$. Clearly, U is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = U^{-1}R[x]$, the ring $R[x; x^{-1}]$ is nil-reflexive by Proposition 3.1. □

In [12], a ring R is said to be *Armendariz* if whenever two polynomials $f(x) = \sum_{i=0}^{n} a_ix^i$, $g(x) = \sum_{j=0}^{m} b_jx^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ where

$0 \le i \le n$, $0 \le j \le m$. There are nil-reflexive rings but not Armendariz as the following example shows.

Example 3.3. Let F be a field and consider the ring $T_2(F)$. Then by Example 2.2(2), $T_2(F)$ is nil-reflexive. On the other hand, let $f(x)$ and $g(x)$ be given by $f(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$, $g(x) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \in T_2(F)[x]$. Then $f(x)g(x) = 0$, but $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$. Therefore $T_2(F)$ is not an Armendariz ring.

Theorem 3.4. *If R is a nil-reflexive Armendariz ring, then $R[x]$ is a nil-reflexive ring.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ such that $f(x)h(x)g(x) \in nil(R[x])$, for all $h(x) = \sum_{k=0}^{t} c_k x^k \in R[x]$. Since R is Armendariz, by [2, Corollary 5.2], we have $nil(R[x]) = nil(R)[x]$. We get $a_i c_k b_j \in nil(R)$, $0 \le i \le n$, $0 \le j \le m$, $0 \le k \le t$. Since R is nil-reflexive, $b_j c_k a_i \in nil(R)$, $0 \le i \le n$, $0 \le j \le m$, $0 \le k \le t$. So $g(x)h(x)f(x) \in nil(R[x])$. □

Let R be a ring and M an R-bimodule. Recall that the *trivial extension of R by M* is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the matrix ring $\left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r \in R, m \in M \right\}$ with the usual matrix operations.

Theorem 3.5. *A ring R is nil-reflexive if and only if $T(R, R)$ is nil-reflexive.*

Proof. "$\Longrightarrow$" Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, $X = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$, $B = \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \in T(R, R)$ with AXB nilpotent. Then axu is nilpotent for all $x \in R$. By hypothesis, uxa is nilpotent, say $(uxa)^t = 0$. Then $(BXA)^t = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. Hence $((BXA)^t)^2 = 0$. Thus $T(R, R)$ is nil-reflexive.

"$\Longleftarrow$" Suppose that $T(R, R)$ is a nil-reflexive ring. Let $a, b \in R$ with arb nilpotent for all $r \in R$. Then for $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in T(R, R)$, $ACB = \begin{pmatrix} arb & atb \\ 0 & arb \end{pmatrix}$ is nilpotent for all $C = \begin{pmatrix} r & t \\ 0 & r \end{pmatrix} \in T(R, R)$. By supposition

$BCA = \begin{pmatrix} bra & bta \\ 0 & bra \end{pmatrix}$ is nilpotent for all $C = \begin{pmatrix} r & t \\ 0 & r \end{pmatrix} \in T(R,R)$. It follows that bra is nilpotent for all $r \in R$. This completes the proof. □

We end this paper by studying the Nagata extension of a ring in terms of the nil-reflexive property. Let R be a commutative ring, M be an R-module, and α be an endomorphism of R. Let $R \oplus M$ be a direct sum of R and M. Define componentwise addition and multiplication given by $(r_1, m_1)(r_2, m_2) = (r_1 r_2, \alpha(r_1)m_2 + r_2 m_1)$, where $r_1, r_2 \in R$ and m_1, $m_2 \in M$. This extension is called *Nagata extension* of R by M and α, and denoted by $N[R, M; \alpha]$ (see [11]).

Theorem 3.6. *Let R be a ring. If R is commutative, then the Nagata extension $N[R, R; \alpha]$ is nil-reflexive.*

Proof. Assume R is commutative. Let (a, n), $(b, m) \in N[R, R; \alpha]$. If $(a, n)(x, y)(b, m)$ is nilpotent for all $(x, y) \in N[R, R; \alpha]$, then axb is nilpotent for all $x \in R$. By assumption bxa, therefore $(b, m)(x, y)(a, n)$ is nilpotent for all $(x, y) \in N[R, R; \alpha]$. □

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