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PROLONGATIONS OF GOLDEN STRUCTURES TO TANGENT BUNDLES OF ORDER r

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Abstract. Our purpose in this paper is to focus on some applications in differential geometry of golden structure. We study $r-$lift of the golden structure in tangent bundle of order r and we obtain integrabilitiy conditions of golden structure in T_rM.

1. Introduction

In differential geometry, the lift method has an important role. This method allows to generalize differentiable structures on any manifold. The extended manifold is significant since geometric structures of an extended manifold has coincided more knowledge than geometric structures of a manifold. The lifts from M ($n-$dimensional differentiable manifold) to its tangent bundle of order r are found in the literature [1, 5, 9, 10, 17].

We give some information about references which are the basis of our paper. Hretcanu [6] studied the golden structure on a manifold M in 2007. Then, Hretcanu and Crasmareanu [2] introduced the geometry of the golden structure on a manifold M by using a corresponding almost product structure. Golden structures were studied by various authors [3, 7, 8, 12, 15, 16]. Based on these studies, Özkan [13] investigated prolongations of golden structure to tangent bundles. The aim of this paper is to generalize the former prolongations by considering the tangent bundle of order r (which is the tangent bundle of higher order). In particular, we follow the spirit of [13].

The outline of this paper is as follows: In section 2, we remind significant definitions and features about the golden structure. In section 3, we introduce the $r-$lift of golden structures in tangent bundle of order r. In section 4, integrability and parallelism of golden structures in tangent bundle of order r are showed. Section 5 deals with golden semi-Riemannian manifold in tangent bundle of order r.

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2. Golden structures on manifolds

Definition 1 ([2, 6]). *A tensor field Φ of type $(1,1)$ on M and of class C^∞ providing*

$$\Phi^2 - \Phi - I = 0 \tag{2.1}$$

is called a golden structure on M.

Recall ([2], Theorem 1.1) that if P is an almost product structure on M, then

$$\Phi = \frac{1}{2}\left(I + \sqrt{5}P\right) \tag{2.2}$$

is a golden structure on M. Conversely, if Φ is a golden structure on M then

$$P = \frac{1}{\sqrt{5}}\left(2\Phi - I\right) \tag{2.3}$$

is an almost product structure on M.

Now we define the operators k and s as follows [2]:

$$k = \frac{1}{2}\left(I + P\right),\ s = \frac{1}{2}\left(I - P\right)$$

where P is an almost product structure.

By using $\Phi = \frac{1}{2}\left(I + \sqrt{5}P\right)$, we have

$$k = \frac{1}{\sqrt{5}}\Phi - \frac{1-\phi}{\sqrt{5}}I,\ s = -\frac{1}{\sqrt{5}}\Phi + \frac{\phi}{\sqrt{5}}I \tag{2.4}$$

where ϕ is a solution of the equation $x^2 - x - 1 = 0$, and it is called the golden ratio. Then we get

$$k + s = I,\ ks = sk = 0,\ k^2 = k,\ s^2 = s. \tag{2.5}$$

Equation (2.5) shows that there exist two complementary distributions K and S in M corresponding to the projection operators k and s.

k and s are operators providing following relations [2]:

$$\begin{aligned} \Phi k = k\Phi = \phi k = \tfrac{\phi}{\sqrt{5}}\Phi + \tfrac{1}{\sqrt{5}}I, \\ \Phi s = s\Phi = (1-\phi)\, s = \tfrac{\phi - 1}{\sqrt{5}}\Phi - \tfrac{1}{\sqrt{5}}I. \end{aligned} \tag{2.6}$$

3. $r-$lift of Golden structures in tangent bundle

Firstly, we give information about the tangent bundle of order r, which is the bundle of $r-$jets.

Let M be a differentiable C^∞ manifold, $dimM = n$, $r \geq 1$ be an integer and $\mathbb{R}$ be the real line. $C^\infty(M)$ is an algebra of all differentiable functions on M.

We introduce an equivalence relation $\sim$ in the set of all differentiable mappings. We denote these mappings by $S(M)$. If the mappings $\varphi : \mathbb{R} \to M$ and $\psi : \mathbb{R} \to M$ satisfy the following conditions

$$\varphi^h\left(0\right) = \psi^h\left(0\right), \frac{d\varphi^h\left(0\right)}{dt} = \frac{d\psi^h\left(0\right)}{dt}, ..., \frac{d^r\varphi\left(0\right)}{dt^r} = \frac{d^r\psi\left(0\right)}{dt^r},$$

where φ and ψ are indicated respectively by $x^h = \varphi^h(t)$ and $x^h = \psi^h(t)$ $(t \in \mathbb{R})$ with respect to local coordinates x^h in a coordinate neighborhood of $\{U, x^h\}$ containing the point $\varphi(0) = \psi(0) = p \in U$, then we say that the mapping φ is equivalent to ψ and denoted by $\varphi \underset{r}{\sim} \psi$. Each equivalence relation is called $r-$jet of M and shown by $j_P^r(\varphi)$. The set of all $r-$jets of M is called the tangent bundle of order r and denoted by T_rM.

Let $\{U, x^h\}$ be a coordinate neighborhood of M. The local coordinates of T_rM are indicated by the set $\left(x^h, y^{(1)h}, y^{(2)h}, ..., y^{(r)h}\right)$, x^h being coordinates of p in U, and $y^{(1)h}, y^{(2)h}, ..., y^{(r)h}$ are defined respectively by

$$y^{(1)h} = \frac{d\varphi^h(0)}{dt}, y^{(2)h} = \frac{1}{2!}\frac{d^2\varphi^h(0)}{dt^2}, ..., y^{(r)h} = \frac{1}{r!}\frac{d^r\varphi^h(0)}{dt^r},$$

where φ has the local expression $x^h = \varphi^h(t)$ $(t \in \mathbb{R})$ with the point $p = \varphi(0)$. In such a way T_rM becomes a differentiable manifold of dimension $(r+1)n$ [1, 9, 17].

The $r-$lift of a tensor field Φ of type $(1,1)$ with local components Φ_i^h in M to T_rM has components in the following form [17]

$$\Phi^{(r)} : \begin{pmatrix} 0 & 0 & 0 & ... & ... & ... & ... & 0 \\ 0 & 0 & 0 & ... & ... & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ \left(\Phi_i^h\right)^{(0)} & 0 & 0 & ... & ... & ... & ... & 0 \\ \left(\Phi_i^h\right)^{(1)} & \left(\Phi_i^h\right)^{(2)} & 0 & ... & ... & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ \left(\Phi_i^h\right)^{(r)} & \left(\Phi_i^h\right)^{(r-1)} & ... & ... & \left(\Phi_i^h\right)^{(0)} & 0 & ... & 0 \end{pmatrix}.$$

Let Φ and G be tensor fields of type (1,1) on M. We get [17]

$$(\Phi G)^{(r)} = \Phi^{(r)} G^{(r)}. \tag{3.1}$$

For the case $G = \Phi$ in (3.1), we have

$$\left(\Phi^2\right)^{(r)} = \left(\Phi^{(r)}\right)^2. \tag{3.2}$$

By using equation (2.1), we obtain $\left(\Phi^2 - \Phi - I\right)^{(r)} = 0$. By the help of (3.2) and $I^{(r)} = I$, we get

$$\left(\Phi^{(r)}\right)^2 - \Phi^{(r)} - I = 0. \tag{3.3}$$

Then we have the following proposition.

Proposition 1. *Let $\Phi \in \Im_1^1(M)$. Φ is a golden structure if and only if the $r-$lift $\Phi^{(r)}$ of Φ is a golden structure in T_rM.*

Let Φ be a golden structure on a manifold M. The $r-$ lifts of k, s are $k^{(r)}$ and $s^{(r)}$, respectively, which are complementary projection tensors in T_rM. Thus, there are complementary distributions $K^{(r)}$ and $S^{(r)}$, which are defined by $k^{(r)}$ and $s^{(r)}$, respectively.

Proposition 2. *i) If Φ is a golden structure on M, then the golden structure $\Phi^{(r)}$ is an isomorphism on the tangent space of the tangent manifold, $T_q(T_rM)$ for every $q \in T_rM$.*

ii) $\Phi^{(r)}$ is invertible and its inverse $\hat{\Phi}^{(r)} = \left(\Phi^{(r)}\right)^{-1}$ satisfies

$$\left(\hat{\Phi}^{(r)}\right)^2 + \hat{\Phi}^{(r)} - I = 0.$$

Proposition 3. *If Φ is a golden structure on M, then $\Phi^{(r)}$ is a golden structure, and $\tilde{\Phi}^{(r)} = I - \Phi^{(r)}$ is also a golden structure in T_rM.*

Remark 1. *a) If T is an almost tangent structure on M, then $T^{(r)}$ is an almost tangent structure in T_rM, and $-T^{(r)}$ is also an almost tangent structure* [17].

b) If P is an almost product structure on M, then $P^{(r)}$ is an almost product structure in T_rM, and $-P^{(r)}$ is also an almost product structure [11].

c) If J is an almost complex structure on M, then $J^{(r)}$ is an almost complex structure in T_rM, and $-J^{(r)}$ is also an almost complex structure [17].

By using (2.2), (2.3) and taking into account Remark 3, we have the following theorem.

Theorem 1. *If P is an almost product structure on M, then almost product structure $P^{(r)}$ yields a golden structure in T_rM as follows:*

$$\Phi^{(r)} = \frac{1}{2}\left(I + \sqrt{5}P^{(r)}\right). \tag{3.4}$$

Contrarily, let Φ be a golden structure on M, then golden structure $\Phi^{(r)}$ induces an almost product structure in T_rM

$$P^{(r)} = \frac{1}{\sqrt{5}}\left(2\Phi^{(r)} - I\right).$$

Remark 2. *Taking into account $\Phi^{(r)} \longleftrightarrow P^{(r)}$ in Theorem 1, we get*

$$\tilde{\Phi}^{(r)} = I - \Phi^{(r)} \longleftrightarrow \tilde{P}^{(r)} = -P^{(r)}.$$

Thus we have

I) Let (M, T) be an almost tangent manifold. The tensor field $\Phi_t^{(r)}$ on T_rM which is defined by

$$\Phi_t^{(r)} = \frac{1}{2}\left(I + \sqrt{5}T^{(r)}\right)$$

is called tangent golden structure on $\left(T_rM, T^{(r)}\right)$.

II) Let (M, J) be an almost complex manifold. The tensor field $\Phi_j^{(r)}$ on T_rM which is defined by

$$\Phi_j^{(r)} = \frac{1}{2}\left(I + \sqrt{5}J^{(r)}\right)$$

is called complex golden structure on $\left(T_rM, J^{(r)}\right)$.

Example 1 (Triple structures in terms of golden structures on T_rM)**.** *From (3.4) and Example 2.4 of* [2] *we get*

$$\Phi_{F^{(r)}} = \frac{1}{2}\left(I+\sqrt{5}F^{(r)}\right),\ \Phi_{P^{(r)}} = \frac{1}{2}\left(I+\sqrt{5}P^{(r)}\right),\ \Phi_{J^{(r)}} = \frac{1}{2}\left(I+\sqrt{5}J^{(r)}\right)$$

where $F, P \in \Im_1^1(M)$ *and* $J = P \circ F$*. Hence we obtain*

$$\sqrt{5}\Phi_{J^{(r)}} = 2\Phi_{P^{(r)}}\Phi_{F^{(r)}} - \Phi_{P^{(r)}} - \Phi_{F^{(r)}} + \phi I$$

and $(\Phi_{F^{(r)}}, \Phi_{P^{(r)}}, \Phi_{J^{(r)}})$ *is:*

1) An (ahp)-structure in T_rM *if and only if* (Φ_F, Φ_P, Φ_J) *is (ahp)-structure on* M*.*

2) An (abpc)-structure in T_rM *if and only if* (Φ_F, Φ_P, Φ_J) *is (abpc)-structure on* M*.*

3) An (apbc)-structure in T_rM *if and only if* (Φ_F, Φ_P, Φ_J) *is (apbc)-structure on* M*.*

4) An (ahc)-structure if in T_rM *and only if* (Φ_F, Φ_P, Φ_J) *is (ahc)-structure on* M*.*

4. Integrability and parallelism of Golden structures in tangent bundle of order r

Let P be an almost product structure and Φ be a golden structure on M. N_P and N_Φ are Nijenhuis tensors of P and Φ, respectively, given by [2, 17] as follows

$$\begin{aligned} N_P(X,Y) &= [PX,PY] - P[PX,Y] - P[X,PY] + P^2[X,Y], \\ N_\Phi(X,Y) &= [\Phi X,\Phi Y] - \Phi[\Phi X,Y] - \Phi[X,\Phi Y] + \Phi^2[X,Y] \end{aligned} \tag{4.1}$$

for any $X, Y \in \Im_0^1(M)$.

By noticing $\Phi = \frac{1}{2}\left(I+\sqrt{5}P\right)$, the following relations are verified [2]

$$N_P(X,Y) = \frac{4}{5}N_\Phi(X,Y). \tag{4.2}$$

For $\Phi \in \Im_1^1(M)$, we have [17]

$$\begin{aligned} (X+Y)^{(r)} &= X^{(r)} + Y^{(r)}, \\ \left[X^{(r)}, Y^{(r)}\right] &= [X,Y]^{(r)}, \\ \Phi^{(r)}X^{(r)} &= (\Phi X)^{(r)}. \end{aligned} \tag{4.3}$$

From (2.4), (2.5), (2.6), (3.1) and (3.2), we obtain

$$\begin{gathered} k^{(r)} = \frac{1}{\sqrt{5}}\Phi^{(r)} - \frac{1-\phi}{\sqrt{5}}I,\ s^{(r)} = -\frac{1}{\sqrt{5}}\Phi^{(r)} + \frac{\phi}{\sqrt{5}}I, \\ k^{(r)} + s^{(r)} = I,\ k^{(r)}s^{(r)} = s^{(r)}k^{(r)} = 0,\ \left(k^{(r)}\right)^2 = k^{(r)},\ \left(s^{(r)}\right)^2 = s^{(r)}, \\ \Phi^{(r)}k^{(r)} = k^{(r)}\Phi^{(r)} = \phi k^{(r)},\ \Phi^{(r)}s^{(r)} = s^{(r)}\Phi^{(r)} = (1-\phi)s^{(r)}. \end{gathered} \tag{4.4}$$

Let $N_{P^{(r)}}$, $N_{\Phi^{(r)}}$ be the Nijenhuis tensor of $\Phi^{(r)}$ and $P^{(r)}$ in T_rM, respectively. By the help of (3.2), we get

$$N_{P^{(r)}}\left(X^{(r)},Y^{(r)}\right)=\left[P^{(r)}X^{(r)},P^{(r)}Y^{(r)}\right]-P^{(r)}\left[P^{(r)}X^{(r)},Y^{(r)}\right]$$
$$-P^{(r)}\left[X^{(r)},P^{(r)}Y^{(r)}\right]+\left(P^2\right)^{(r)}\left[X^{(r)},Y^{(r)}\right], \quad (4.5)$$

$$N_{\Phi^{(r)}}\left(X^{(r)},Y^{(r)}\right)=\left[\Phi^{(r)}X^{(r)},\Phi^{(r)}Y^{(r)}\right]-\Phi^{(r)}\left[\Phi^{(r)}X^{(r)},Y^{(r)}\right]$$
$$-\Phi^{(r)}\left[X^{(r)},\Phi^{(r)}Y^{(r)}\right]+\left(\Phi^2\right)^{(r)}\left[X^{(r)},Y^{(r)}\right] \quad (4.6)$$

for any $X,Y\in\Im_0^1(M)$.

Proposition 4. *The $r-$lift $K^{(r)}$ of a distribution K in T_rM is integrable if and only if K is integrable in M.*

Proof. For any $X,Y\in\Im_0^1(M)$, the distribution K is integrable if and only if [2]

$$s\left[kX,kY\right]=0. \quad (4.7)$$

Taking $r-$lift on both sides of equation (4.7) and using (4.3), we get

$$s^{(r)}\left[k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right]=0 \quad (4.8)$$

where $s^{(r)}=(I-k)^{(r)}=I-k^{(r)}$ is the projection tensor complementary to $k^{(r)}$. Thus, equations (4.7) and (4.8) are equivalent. This completes the proof. □

So, we have the following proposition.

Proposition 5. *Let the distribution K be integrable in M, that is $sN_\Phi(kX,kY)=0$ [2] for any $X,Y\in\Im_0^1(M)$. Then the distribution $K^{(r)}$ is integrable in T_rM if and only if*

$$s^{(r)}N_{\Phi^{(r)}}\left(k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right)=0.$$

Proof. Let $N_{\Phi^{(r)}}$ be the Nijenhuis tensor of $\Phi^{(r)}$ in T_rM. Then in the view of (3.2), we have

$$N_{\Phi^{(r)}}\left(k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right)=$$

$$\left[\Phi^{(r)}k^{(r)}X^{(r)},\Phi^{(r)}k^{(r)}Y^{(r)}\right]-\Phi^{(r)}\left[\Phi^{(r)}k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right]$$
$$-\Phi^{(r)}\left[k^{(r)}X^{(r)},\Phi^{(r)}k^{(r)}Y^{(r)}\right]+\left(\Phi^2\right)^{(r)}\left[k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right]. \quad (4.9)$$

According to (4.9) and with the help of (3.3) and (4.4),

$$\begin{aligned} N_{\Phi^{(r)}}\left(k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right) &= (2\phi-1)\,\Phi^{(r)}\left[k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right] \\ &\quad +(3-\phi)\left[k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right]. \end{aligned}$$

Multiplying throughout by $\frac{1}{5}s^{(r)}$ and from (4.4), we obtain

$$\frac{1}{5}s^{(r)}N_{\Phi^{(r)}}\left(k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right)=s^{(r)}\left[k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right]=\left(sN_{\Phi}\left(kX,kY\right)\right)^{(r)}.$$

By using (4.8) or $sN_{\Phi}(kX,kY)=0$, we obtain

$$s^{(r)}N_{\Phi^{(r)}}\left(k^{(r)}X^{(r)},k^{(r)}Y^{(r)}\right)=0.$$

Hence Proposition 5 is proved. □

Proposition 6. *The $r-$lift $S^{(r)}$ of a distribution S in T_rM is integrable if and only if S is integrable in M.*

Proof. The distribution S is integrable if and only if [2]

$$k\left[sX,sY\right]=0 \tag{4.10}$$

for any $X,Y\in\Im_0^1(M)$.

Taking $r-$lift on both sides of equation (4.10) and using (4.3), we get

$$k^{(r)}\left[s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right]=0. \tag{4.11}$$

where $k^{(r)}=(I-s)^{(r)}=I-s^{(r)}$ is the projection tensor complementary to $s^{(r)}$. Thus, the equations (4.10) and (4.11) are equivalent. This completes the proof. □

Proposition 7. *Let the distribution S be integrable in M, that is $kN_{\Phi}(sX,sY)=0$ [2], for any $X,Y\in\Im_0^1(M)$. Then, the distribution $S^{(r)}$ is integrable in T_rM if and only if*

$$k^{(r)}N_{\Phi^{(r)}}\left(s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right)=0.$$

Proof. Taking into account the Nijenhuis tensor $\Phi^{(r)}$, we obtain

$$\begin{aligned} &N_{\Phi^{(r)}}\left(s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right)= \\ &\qquad\left[\Phi^{(r)}s^{(r)}X^{(r)},\Phi^{(r)}s^{(r)}Y^{(r)}\right]-\Phi^{(r)}\left[\Phi^{(r)}s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right] \\ &\qquad-\Phi^{(r)}\left[s^{(r)}X^{(r)},\Phi^{(r)}s^{(r)}Y^{(r)}\right]+\left(\Phi^{2}\right)^{(r)}\left[s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right] \end{aligned} \tag{4.12}$$

According to (4.12) and with the help of (3.3) and (4.4),

$$\begin{aligned} N_{\Phi^{(r)}}\left(s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right) &= (1-2\phi)\,\Phi^{(r)}\left[s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right] \\ &\quad +(2+\phi)\left[s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right]. \end{aligned}$$

Multiplying throughout by $\frac{1}{5}k^{(r)}$ and from (4.4), we obtain,

$$\frac{1}{5}k^{(r)}N_{\Phi^{(r)}}\left(s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right)=k^{(r)}\left[s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right]=\left(kN_{\Phi}\left(sX,sY\right)\right)^{(r)}.$$

By using (4.11) or $kN_{\Phi}\left(sX,sY\right)=0$, we obtain

$$k^{(r)}N_{\Phi^{(r)}}\left(s^{(r)}X^{(r)},s^{(r)}Y^{(r)}\right)=0.$$

Hence Proposition 7 is proved. □

Proposition 8. *For any $X,Y\in\Im_0^1\left(M\right)$ and $\Phi^{(r)}=\frac{1}{2}\left(I+\sqrt{5}P^{(r)}\right)$, the relation between $N_{P^{(r)}}$ and $N_{\Phi^{(r)}}$ is satisfying*

$$N_{P^{(r)}}\left(X^{(r)},Y^{(r)}\right)=\frac{4}{5}N_{\Phi^{(r)}}\left(X^{(r)},Y^{(r)}\right).$$

Proof. By the help of (4.2), (4.3) and (4.5), we have

$$\begin{aligned}N_{P^{(r)}}\left(X^{(r)},Y^{(r)}\right)&=\left(N_P\left(X,Y\right)\right)^{(r)}\\&=\left(\frac{4}{5}N_{\Phi}\left(X,Y\right)\right)^{(r)}.\end{aligned}$$

Using (4.1) and (4.3), we have

$$N_{P^{(r)}}\left(X^{(r)},Y^{(r)}\right)=\frac{4}{5}N_{\Phi^{(r)}}\left(X^{(r)},Y^{(r)}\right).$$

This proves the proposition. □

Proposition 9. *Let P be an almost product structure on M and the r−lift $\Phi^{(r)}$ of Φ is golden structure in T_rM. Then, $\Phi^{(r)}$ is integrable in T_rM if and only if P is integrable in M.*

Proposition 10. *Let the golden structure Φ be integrable in M. Then the golden structure $\Phi^{(r)}$ is integrable in T_rM if and only if*

$$N_{\Phi^{(r)}}\left(X^{(r)},Y^{(r)}\right)=0.$$

Proof. In view of equations (4.3) and (4.6), we have

$$N_{\Phi^{(r)}}\left(X^{(r)},Y^{(r)}\right)=\left(N_{\Phi}\left(X,Y\right)\right)^{(r)}=0$$

because the golden structure Φ is integrable in M. □

Recall ([2], Proposition 4.1) that if the golden structure Φ is integrable, then the distributions K and S are integrable. Hence we have:

Proposition 11. *If the r−lift $\Phi^{(r)}$ of Φ is integrable in T_rM, then the distributions $K^{(r)}$ and $S^{(r)}$ are integrable on T_rM.*

Let ∇ be a linear connection on M. Then there exists a unique linear connection $\nabla^{(r)}$ in T_rM which verifies

$$\nabla^{(r)}_{X^{(r)}}Y^{(r)} = (\nabla_X Y)^{(r)}$$

for any $X, Y \in \Im^1_0(M)$ [17]. Thus, for the pair $\left(\Phi^{(r)}, \nabla^{(r)}\right)$ we obtain two other linear connections in T_rM:

i) The Schouten connection

$$\tilde{\nabla}^{(r)}_{X^{(r)}}Y^{(r)} = k^{(r)}\left(\nabla^{(r)}_{X^{(r)}}k^{(r)}Y^{(r)}\right) + s^{(r)}\left(\nabla^{(r)}_{X^{(r)}}s^{(r)}Y^{(r)}\right).$$

ii) The Vrănceanu connection

$$\begin{aligned}\check{\nabla}^{(r)}_{X^{(r)}}Y^{(r)} &= k^{(r)}\left(\nabla^{(r)}_{k^{(r)}X^{(r)}}k^{(r)}Y^{(r)}\right) + s^{(r)}\left(\nabla^{(r)}_{s^{(r)}X^{(r)}}s^{(r)}Y^{(r)}\right)\\ &\quad + k^{(r)}\left[s^{(r)}X^{(r)}, k^{(r)}Y^{(r)}\right] + s^{(r)}\left[k^{(r)}X^{(r)}, s^{(r)}Y^{(r)}\right].\end{aligned}$$

Proposition 12. *The projectors $k^{(r)}$ and $s^{(r)}$ are parallels with respect to Schouten and Vrănceanu connections for every linear connection $\nabla^{(r)}$ on T_rM. Similarly, $\Phi^{(r)}$ is parallel with respect to Schouten and Vrănceanu connections.*

We know from [2] that a distribution D on M is called parallel with respect to the linear connection ∇ if $X \in \Im^1_0(M)$ and $Y \in D$ imply $\nabla_X Y \in D$.

By the help of this knowledge, a distribution $D^{(r)}$ on T_rM is called parallel with respect to the linear connection $\nabla^{(r)}$ if $X^{(r)} \in \Im^1_0(T_rM)$ and $Y^{(r)} \in D^{(r)}$ imply $\nabla^{(r)}_{X^{(r)}}Y^{(r)} \in D^{(r)}$.

Proposition 13. *For the linear connection $\nabla^{(r)}$ in T_rM, the distributions $K^{(r)}$ and $S^{(r)}$ are parallel with respect to Schouten and Vrănceanu connections.*

Proof. Let $X \in \Im^1_0(M)$ and $Y \in K$. Thus, $X^{(r)} \in \Im^1_0(T_rM)$ and $Y^{(r)} \in K^{(r)}$. Since $s^{(r)}Y^{(r)} = (sY)^{(r)} = 0$, $k^{(r)}Y^{(r)} = (kY)^{(r)} = Y^{(r)}$, we have

$$\begin{aligned}\tilde{\nabla}^{(r)}_{X^{(r)}}Y^{(r)} &= k^{(r)}\left(\nabla^{(r)}_{X^{(r)}}Y^{(r)}\right) \in K^{(r)},\\ \check{\nabla}^{(r)}_{X^{(r)}}Y^{(r)} &= k^{(r)}\left(\nabla^{(r)}_{k^{(r)}X^{(r)}}Y^{(r)}\right) + k^{(r)}\left[s^{(r)}X^{(r)}, Y^{(r)}\right] \in K^{(r)}.\end{aligned}$$

Similar relations are satisfied for $S^{(r)}$. □

5. Golden semi-Riemannian metrics in tangent bundle of order r

Definition 2 ([4, 14])**.** *A semi-Riemannian almost product structure is a pair (g, P) with g a semi-Riemannian metric on M, and P is an almost product structure related by*

$$g(PX, PY) = g(X, Y)$$

or equivalently, P is a $g-$symmetric endomorphism

$$g(PX,Y) = g(X,PY)$$

for every $X,Y \in \Im_0^1(M)$.

Proposition 14 ([17])**.** *If g is a semi-Riemannian metric in M, then $g^{(r)}$ is a semi-Riemannian metric in T_rM.*

Let g be a semi-Riemannian metric and P is an almost product structure on M, then the pair $\left(g^{(r)}, P^{(r)}\right)$ is a semi-Riemannian almost product structure on T_rM if and only if (g,P) is so in M. So, we get

$$g^{(r)}\left(P^{(r)}X^{(r)}, P^{(r)}Y^{(r)}\right) = g^{(r)}\left(X^{(r)}, Y^{(r)}\right)$$

or equivalently,

$$g^{(r)}\left(P^{(r)}X^{(r)}, Y^{(r)}\right) = g^{(r)}\left(X^{(r)}, P^{(r)}Y^{(r)}\right).$$

From equations (2.2) and (3.4), we have:

Proposition 15. *The almost product structure P is a $g-$symmetric endomorphism if and only if golden structure $\Phi^{(r)}$ is a $g^{(r)}-$symmetric endomorphism.*

Definition 3 ([2], Definition 5.1.)**.** *A golden Riemannian structure on M is a pair (g,Φ) with*

$$g(\Phi X,Y) = g(X,\Phi Y).$$

The triple (M,g,Φ) is a golden Riemannian manifold.

Definition 4 ([13])**.** *A golden semi-Riemannian structure on M is a pair (g,Φ) with*

$$g(\Phi X,Y) = g(X,\Phi Y).$$

The triple (M,g,Φ) is a golden semi-Riemannian manifold.

Proposition 16. *Let $\Phi \in \Im_1^1(M)$ then the $r-$lift $\Phi^{(r)}$ of Φ is a golden semi-Riemannian structure in T_rM if Φ is a golden semi-Riemannian structure in M.*

Corollary 1. *Let (M,g,Φ) be a golden semi-Riemannian manifold, then on a golden semi-Riemannian manifold $\left(T_rM, g^{(r)}, \Phi^{(r)}\right)$ we have the following results:*
i) The projectors $k^{(r)}$, $s^{(r)}$ are $g^{(r)}-$symmetric endomorphism, i.e.

$$\begin{aligned} g^{(r)}\left(k^{(r)}X^{(r)}, Y^{(r)}\right) &= g^{(r)}\left(X^{(r)}, k^{(r)}Y^{(r)}\right), \\ g^{(r)}\left(s^{(r)}X^{(r)}, Y^{(r)}\right) &= g^{(r)}\left(X^{(r)}, s^{(r)}Y^{(r)}\right). \end{aligned}$$

ii) The distribution $K^{(r)}$, $S^{(r)}$ are $g^{(r)}-$orthogonal, i.e.

$$g^{(r)}\left(k^{(r)}X^{(r)}, s^{(r)}Y^{(r)}\right) = 0.$$

iii) The golden structure $\Phi^{(r)}$ is $N_{\Phi^{(r)}}-$symmetric, i.e.

$$N_{\Phi^{(r)}}\left(\Phi^{(r)}X^{(r)},Y^{(r)}\right)=N_{\Phi^{(r)}}\left(X^{(r)},\Phi^{(r)}Y^{(r)}\right).$$

Proposition 17. *If $P^{(r)}$ is parallel with respect to the Levi-Civita connection $\overset{g^{(r)}}{\nabla^{(r)}}$ of $g^{(r)}$, i.e. $\overset{g^{(r)}}{\nabla^{(r)}}P^{(r)}=0$, then a semi-Riemannian almost product structure is a locally product structure. If $\nabla^{(r)}$ is a symmetric linear connection, then the Nijenhuis tensor of $P^{(r)}$ satisfies*

$$\begin{aligned}N_{P^{(r)}}\left(X^{(r)},Y^{(r)}\right) &= \left(\nabla^{(r)}_{P^{(r)}X^{(r)}}P^{(r)}\right)Y^{(r)}-\left(\nabla^{(r)}_{P^{(r)}Y^{(r)}}P^{(r)}\right)X^{(r)}\\ &\quad -P^{(r)}\left(\nabla^{(r)}_{X^{(r)}}P^{(r)}\right)Y^{(r)}+P^{(r)}\left(\nabla^{(r)}_{Y^{(r)}}P^{(r)}\right)X^{(r)}.\end{aligned}$$

Proposition 18. *On a locally product golden semi-Riemannian manifold, the golden structure $\Phi^{(r)}$ is integrable.*

By using Proposition 18 and from ([2], Theorem 5.1), we get the following theorem.

Theorem 2. *If a linear connection*

$$\begin{aligned}\nabla^{(r)}_{X^{(r)}}Y^{(r)} &= \frac{1}{5}\left[3\tilde{\nabla}^{(r)}_{X^{(r)}}Y^{(r)}+\Phi^{(r)}\left(\tilde{\nabla}^{(r)}_{X^{(r)}}\Phi^{(r)}Y^{(r)}\right)-\Phi^{(r)}\left(\tilde{\nabla}^{(r)}_{X^{(r)}}Y^{(r)}\right)\right.\\ &\quad\left.-\tilde{\nabla}^{(r)}_{X^{(r)}}\Phi^{(r)}Y^{(r)}\right]+O_{P^{(r)}}Q^{(r)}\left(X^{(r)},Y^{(r)}\right)\end{aligned}$$

where $\tilde{\nabla}^{(r)}$ is $r-$lift of a linear connection $\tilde{\nabla}$ and $Q^{(r)}$ is $r-$lift of an $(1,2)-$tensor field Q for which O_PQ is a related Obata operator

$$O_PQ\left(X,Y\right)=\frac{1}{2}\left[Q\left(X,Y\right)+PQ\left(X,PY\right)\right]$$

for the corresponding almost product structure (2.3), then $\Phi^{(r)}$ is parallel with respect to $\nabla^{(r)}$ linear connection, i.e. $\nabla^{(r)}\Phi^{(r)}=0$.

From ([2], Example 5.6), we have the following example.

Example 2.

$$\begin{cases}K^{(r)}=Span\left\{\sum\limits_{\mu=0}^{r}\left(x^1\right)^{(\mu)}\frac{\partial}{\partial y^{(\mu)1}}+\frac{\partial}{\partial y^{(0)2}}\right\}\\ S^{(r)}=Span\left\{\frac{\partial}{\partial y^{(0)1}}-\sum\limits_{\mu=0}^{r}\left(x^1\right)^{(\mu)}\frac{\partial}{\partial y^{(\mu)2}}\right\}\end{cases}$$

where $\frac{\partial}{\partial y^{(\mu)1}}=\left(\frac{\partial}{\partial x^1}\right)^{(r-\mu)}$ and $\frac{\partial}{\partial y^{(\mu)2}}=\left(\frac{\partial}{\partial x^2}\right)^{(r-\mu)}$. $K^{(r)}$ and $S^{(r)}$ are defined complementary distributions orthogonal with respect to $r-$lift of the Euclidean metric *of* $\mathbb{R}^2$. *These distributions are related to the golden structure*

$$\begin{cases} \Phi^{(r)}\left(\left(\frac{\partial}{\partial x^1}\right)^{(r)}\right) = \sum\limits_{\mu=0}^{r} \frac{\phi\left(\left(x^1\right)^{(\mu)}\right)^2+(1-\phi)}{\left(\left(x^1\right)^{(\mu)}\right)^2+1} \frac{\partial}{\partial y^{(\mu)1}} + \sum\limits_{\mu=0}^{r} \frac{\sqrt{5}\left(x^1\right)^{(\mu)}}{\left(\left(x^1\right)^{(\mu)}\right)^2+1} \frac{\partial}{\partial y^{(\mu)2}} \\ \\ \Phi^{(r)}\left(\left(\frac{\partial}{\partial x^2}\right)^{(r)}\right) = \sum\limits_{\mu=0}^{r} \frac{\sqrt{5}\left(x^1\right)^{(r)}}{\left(\left(x^1\right)^{(r)}\right)^2+1} \frac{\partial}{\partial y^{(\mu)1}} + \sum\limits_{\mu=0}^{r} \frac{(1-\phi)\left(\left(x^1\right)^{(r)}\right)^2+\phi}{\left(\left(x^1\right)^{(r)}\right)^2+1} \frac{\partial}{\partial y^{(\mu)2}} \end{cases}$$

which is integrable since $N_{\Phi^{(r)}}\left(\left(\frac{\partial}{\partial x^1}\right)^{(r)}, \left(\frac{\partial}{\partial x^2}\right)^{(r)}\right) = 0.$

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