Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.
Volume 65, Number 1, Pages 49–52 (2016)
DOI: 10.1501/Commua1_0000000743
ISSN 1303–5991

# A WEAKER FORM OF CONNECTEDNESS

S. MODAK AND T. NOIRI

Abstract. In this paper, we introduce the notion of $Cl-Cl$ - separated sets and $Cl-Cl$ - connected spaces. We obtain several properties of the notion analogous to these of connectedness. We show that $Cl-Cl$ - connectedness is preserved under continuous functions.

## 1. Introduction

In this paper, we introduce a weaker form of connectedness. This form is said to be $Cl-Cl$ - connected. We investigate several properties of $Cl-Cl$ - connected spaces analogous to connected spaces. And also, we show that every connected space is $Cl-Cl$ - connected . Furthermore we present a $Cl-Cl$ - connected space which is not a connected space. Among them we interrelate with $Cl-Cl$ - connections of semi-regularization topology [4], Velicko's $\theta$ - topology [2] and $V-\theta$-connection [3]. We show that $Cl-Cl$ - connectedness is preserved under continuous functions. Let $(X,\tau)$ be a topological space and $A$ be a subset of $X$. The closure of $A$ is denoted by $Cl(A)$. A topological space is briefly called a space.

## 2. $Cl-Cl$ - separated sets

**Definition 1.** *Let $X$ be a space. Nonempty subsets $A$, $B$ of $X$ are called $Cl-Cl$ - separated sets if $Cl(A)\cap Cl(B)=\emptyset$.*

It is obvious that every $Cl-Cl$ - separated sets are separated sets. But the converse need not hold in general.

**Example 1.** *In $\Re$ with the usual topology on $\Re$ the sets $A=(0,1)$ and $B=(1,2)$ are separated sets but not $Cl-Cl$ - separated sets.*

**Theorem 1.** *Let $A$ and $B$ be $Cl-Cl$ - separated in a space $X$. If $C\subset A$ and $D\subset B$, then $C$ and $D$ are also $Cl-Cl$ - separated.*

**Definition 2.** *A subset $A$ of a space $X$ is said to be $Cl-Cl$ - connected if $A$ is not the union of two $Cl-Cl$ - separated sets in $X$.*

---

Received by the editors: Oct. 09, 2015, Accepted: Jan. 15, 2016.
2010 *Mathematics Subject Classification.* 54D05; 54A10; 54A05.
*Key words and phrases.* $Cl-Cl$ - separated set, $Cl-Cl$ - connected space, connected space.

It is obvious that every connected space is a $Cl-Cl$ - connected space but the converse need not hold in general.

**Example 2.** *Let $\Re$ be the real line with the usual topology on $\Re$. Let $X=(0,1)\cup(1,2)$. Consider $A=(0,1)\cap X$ and $B=(1,2)\cap X$, then $X$ is not connected set in $\Re$, since $X=A\cup B$, $A\cap Cl(B)=\emptyset=Cl(A)\cap B$. But the set $X$ is a $Cl-Cl$ - connected set.*

**Definition 3.** [3] *A subset $A$ of $X$ is $V-\theta$ - connected if it cannot be expressed as the union of nonempty subsets with disjoint closed neighbourhoods in $X$, i.e., if there are no disjoint nonempty sets $B_1$ and $B_2$ and no open sets $U$ and $V$ such that $A=B_1\cup B_2$, $B_1\subset U, B_2\subset V$ and $Cl(U)\cap Cl(V)=\emptyset$.*

**Theorem 2.** *Every $V-\theta$ - connected space is a $Cl-Cl$ - connected space.*

*Proof.* The proof is obvious from Definition 3. □

Following examples show that $T_0$ - space and $Cl-Cl$ - connected space are independent concept.

**Example 3.** *Let $X=\{1,2\}$, $\tau=\{\emptyset,\{1\},\{2\},\{1,2\}\}$. Then $X=\{1\}\cup\{2\}$ and $Cl(\{1\})\cap Cl(\{2\})=\emptyset$. The space is a $T_0$ - space but $X$ is not a $Cl-Cl$ - connected space.*

**Example 4.** *Let $X$ be a set such that $\mid X\mid\geq 2$. Let $\tau$ be the indiscrete topology on $X$. Then $(X,\tau)$ is not a $T_0$ - space but it is a $Cl-Cl$ - connected space.*

**Theorem 3.** *A space $X$ is $Cl-Cl$ - connected if and only if it cannot be expressed as the disjoint union of two nonempty clopen sets.*

*Proof.* Let $X$ be a $Cl-Cl$ - connected space. If possible suppose that $X=W_1\cup W_2$, where $W_1\cap W_2=\emptyset$, $W_1(\neq\emptyset)$ is a clopen set in $X$ and $W_2(\neq\emptyset)$ is a clopen set in $X$. Since $W_1$ and $W_2$ are clopen sets in $X$, then $Cl(W_1)\cap Cl(W_2)=\emptyset$. Therefore $X$ is not a $Cl-Cl$ - connected space. This is a contradiction.

Conversely suppose that $X\neq W_1\cup W_2$ and $W_1\cap W_2=\emptyset$, where $W_1$ is a nonempty clopen set and $W_2$ is a nonempty clopen set in $X$. We shall prove that $X$ is a $Cl-Cl$ - connected space.

If possible suppose that $X$ is not a $Cl-Cl$ - connected space, then there exist $Cl-Cl$ - separated sets $A$ and $B$ such that $X=A\cup B$. Then $X=Cl(A)\cup Cl(B)$ and $Cl(A)\cap Cl(B)=\emptyset$. Set $W_1=Cl(A)$ and $W_2=Cl(B)$. Then $W_1$ and $W_2$ are nonempty clopen sets.

Moreover, we have $W_1\cup W_2=X$ and $W_1\cap W_2=\emptyset$. This is a contradiction. So $X$ is a $Cl-Cl$ - connected space. □

**Theorem 4.** *Let $X$ be a space. If $A$ is a $Cl-Cl$ - connected subset of $X$ and $H$, $G$ are $Cl-Cl$ - separated subsets of $X$ with $A\subset H\cup G$, then either $A\subset H$ or $A\subset G$.*

*Proof.* Let $A$ be a $Cl-Cl$ - connected set. Let $A \subset H \cup G$. Since $A = (A \cap H) \cup (A \cap G)$, then $Cl(A \cap G) \cap Cl(A \cap H) \subset Cl(G) \cap Cl(H) = \emptyset$. Suppose $A \cap H$ and $A \cap G$ are nonempty. Then $A$ is not $Cl-Cl$ - connected. This is a contradiction. Thus, either $A \cap H = \emptyset$ or $A \cap G = \emptyset$. This implies that $A \subset H$ or $A \subset G$. □

**Theorem 5.** *If $A$ and $B$ are $Cl-Cl$ - connected sets of a space $X$ and $A$ and $B$ are not $Cl-Cl$ - separated, then $A \cup B$ is $Cl-Cl$ - connected.*

*Proof.* Let $A$ and $B$ be $Cl-Cl$ - connected sets in $X$. Suppose $A \cup B$ is not $Cl-Cl$ - connected. Then, there exist two nonempty disjoint $Cl-Cl$ - separated sets $G$ and $H$ such that $A \cup B = G \cup H$. Suppose that $Cl(G) \cap Cl(H) = \emptyset$. Since $A$ and $B$ are $Cl-Cl$ - connected, by Theorem 4, either $A \subset G$ and $B \subset H$ or $B \subset G$ and $A \subset H$.

Case (i). If $A \subset G$ and $B \subset H$, then $A \cap H = B \cap G = \emptyset$. Therefore, $(A \cup B) \cap G = (A \cap G) \cup (B \cap G) = (A \cap G) \cup \emptyset = A \cap G = A$. Also, $(A \cup B) \cap H = (A \cap H) \cup (B \cap H) = B \cap H = B$. Now, $Cl(A) \cap Cl(B) = Cl((A \cup B) \cap G) \cap Cl((A \cup B) \cap H) \subset Cl(H) \cap Cl(G) = \emptyset$. Thus, $A$ and $B$ are $Cl-Cl$ - separated, which is a contradiction. Hence, $A \cup B$ is $Cl-Cl$ - connected.

Case (ii). If $B \subset G$ and $A \subset H$, then $B \cap H = A \cap G = \emptyset$. Therefore $(A \cup B) \cap H = (A \cap H) \cup (B \cap H) = (A \cap H) \cup \emptyset = A$. Also, $(A \cup B) \cap G = (A \cap G) \cup (B \cap G) = B \cap G = B$. Now, $Cl(A) \cap Cl(B) = Cl((A \cup B) \cap H) \cap Cl((A \cup B) \cap G) \subset Cl(H) \cap Cl(G) = \emptyset$. Thus, $A$ and $B$ are $Cl-Cl$ - separated, which is a contradiction. Hence, $A \cup B$ is $Cl-Cl$ - connected. □

**Theorem 6.** *If $\{M_i : i \in I\}$ is a nonempty family of $Cl-Cl$ - connected sets of a space $X$, with $\cap_{i \in I} M_i \neq \emptyset$, then $\cup_{i \in I} M_i$ is $Cl-Cl$ - connected.*

*Proof.* Suppose $\cup_{i \in I} M_i$ is not $Cl-Cl$ - connected. Then we have $\cup_{i \in I} M_i = H \cup G$, where $H$ and $G$ are $Cl-Cl$ - separated sets in $X$. Since $\cap_{i \in I} M_i \neq \emptyset$, we have a point $x \in \cap_{i \in I} M_i$. Since $x \in \cup_{i \in I} M_i$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in M_i$ for each $i \in I$, then $M_i$ and $H$ intersect for each $i \in I$. By Theorem 4, $M_i \subset H$ or $M_i \subset G$. Since $H$ and $G$ are disjoint, $M_i \subset H$ for all $i \in I$ and hence $\cup_{i \in I} M_i \subset H$. This implies that $G$ is empty. This is a contradiction. Suppose that $x \in G$. By the similar way, we have that $H$ is empty. This is a contradiction. Thus, $\cup_{i \in I} M_i$ is $Cl-Cl$ - connected. □

**Theorem 7.** *Let $X$ be a space, $\{A_\alpha : \alpha \in \triangle\}$ be a family of $Cl-Cl$ - connected sets and $A$ be a $Cl-Cl$ - connected set. If $A \cap A_\alpha \neq \emptyset$ for every $\alpha \in \triangle$, then $A \cup (\cup_{\alpha \in \triangle} A_\alpha)$ is $Cl-Cl$ - connected.*

*Proof.* Since $A \cap A_\alpha \neq \emptyset$ for each $\alpha \in \triangle$, by Theorem 6, $A \cup A_\alpha$ is $Cl-Cl$ - connected for each $\alpha \in \triangle$. Moreover, $A \cup (\cup A_\alpha) = \cup (A \cup A_\alpha)$ and $\cap (A \cup A_\alpha) \supset A \neq \emptyset$. Thus by Theorem 6, $A \cup (\cup A_\alpha)$ is $Cl-Cl$ - connected. □

**Theorem 8.** *The continuous image of a $Cl-Cl$ - connected space is a $Cl-Cl$ - connected space.*

*Proof.* Let $f : X \to Y$ be a continuous map and $X$ be a $Cl - Cl$ - connected space. If possible suppose that $f(X)$ is not a $Cl - Cl$ - connected subset of $Y$. Then, there exist nonempty $Cl - Cl$ - separated sets $A$ and $B$ such that $f(X) = A \cup B$. Since $f$ is continuous and $Cl(A) \cap Cl(B) = \emptyset$, $Cl(f^{-1}(A)) \cap Cl(f^{-1}(B)) \subset f^{-1}(Cl(A)) \cap f^{-1}(Cl(B)) = f^{-1}(Cl(A) \cap Cl(B)) = \emptyset$. Since $A$ and $B$ are nonempty, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Therefore, $f^{-1}(A)$ and $f^{-1}(B)$ are $Cl - Cl$ - separated and $X = f^{-1}(A) \cup f^{-1}(B)$. This contradicts that $X$ is $Cl - Cl$ - connected. Therefore, $f(X)$ is $Cl - Cl$ - connected. □

**Theorem 9.** *Let $X$ be a space then following are equivalent conditions:*

*(1) $X$ is not $Cl - Cl$ - connected;*

*(2) $X = W_1 \cup W_2$, $W_1 \cap W_2 = \emptyset$, where $W_1 (\neq \emptyset)$ is clopen set in $X$ and $W_2 (\neq \emptyset)$ is a clopen set in $X$;*

*(3) there is a continuous map $f : X \to (Y, \sigma)$ such that $f(x) = 0$ if $x \in W_1$ and $f(x) = 1$ if $x \in W_2$, where $Y = \{0, 1\}$ and $\sigma$ is the discrete topology on $Y$.*

*Proof.* (1)$\Leftrightarrow$ (2) is obvious from Theorem 3.

(2) $\Rightarrow$ (3): Let $Y = \{0, 1\}$ and $\sigma$ is the discrete topology, then $(Y, \sigma)$ is a topological space. Let $f : X \to (Y, \sigma)$ be a function defined by $f(W_1) = 0$ and $f(W_2) = 1$. Then $f$ is a continuous surjection such that $f(x) = 0$ for each $x \in W_1$ and $f(x) = 1$ for each $x \in W_2$.

(3) $\Rightarrow$ (2): Here $W_1 = f^{-1}(0)$ is a clopen set of $X$ and $W_2 = f^{-1}(1)$ is a clopen set of $X$. And also $X$ is a disjoint union of nonempty sets $W_1$ and $W_2$. □

**Lemma 10.** [1] *Let $(X, \tau)$ be a topological space and $A \subset X$. Then the topologies $\tau$, $\tau_s$ and $\tau_\theta$ have same family of open and closed sets, i.e., $CO(\tau_\theta) = CO(\tau_s) = CO(\tau)$.*

**Corollary 11.** *The $Cl - Cl$ - connection of $\theta$ - topology, semi-regularization topology and original topology are same concept.*

## References


[1] Mršević, M., Andrijević, D.: On $\theta$ - connectedness and $\theta$ - closure, Topology and its Applications, 123, 157 - 166 (2002)

[2] Veličko, N. V., $H$ - closed topological spaces, Mat. Sb. 70, 98 - 112 (1966); Math. USSR Sb. 78, 103 - 118 (1969)

[3] Veličko, N. V., On the theory of $H$ - closed topological spaces, Sibirskii Math. Z. 8, 754 - 763 (1967); Siberian Math J. 8, 569 - 579 (1967)

[4] Willard, S., General Topology, Addison-Wesley Publ. Comp., (1970)



*Current address*: S. Modak, Department of Mathematics, University of Gour Banga P.O. Mokdumpur, Malda - 732103, India

*E-mail address*: `spmodak2000@yahoo.co.in`

*Current address*: T. Noiri, 2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumomoto-ken, 869-5142 JAPAN

*E-mail address*: `t.noiri@nifty.com`