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BRÜCK CONJECTURE-A DIFFERENT PERSPECTIVE

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Abstract. The purpose of the paper is to obtain some sufficient conditions for which two differential polynomials sharing a small function satisfies conclusions of Brück [3] conjecture. The result present in the paper will unify, improve and generalize several existing results. We have exhibited a number of examples to show that some conditions used in the paper are essential. In the concluding part of the paper we propose two open problems for further investigations.

1. Introduction Definitions and Results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities) and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)) \qquad (r \longrightarrow \infty, r \notin E).$$

A meromorphic function $a(\not\equiv \infty)$ is called a small function with respect to f provided that $T(r, a) = S(r, f)$ as $(r \longrightarrow \infty, r \notin E)$. If $a = a(z)$ is a small function we define that f and g share a IM or a CM according as $f - a$ and $g - a$ share 0 CM or 0 IM respectively.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.
Also it is known to us that the hyper order of $f(z)$, denoted by $\rho_2(f)$, is defined by

$$\rho_2(f) = \limsup_{r \longrightarrow \infty} \frac{log\ log\ T(r, f)}{log\ r}.$$

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The uniqueness problem of entire and meromorphic functions sharing values with their derivatives is a special case of the uniqueness theory with distinguishable entity. The research on this problem was initiated by Rubel and Yang [17]. Analogous to the Nevanlinna 5 value theorem they first showed that for the uniqueness of entire functions and their derivatives one usually needs sharing of only two values CM. In 1979, analogous result corresponding to IM sharing was obtained by E. Mues and N. Steinmetz [16] in the following manner.

Theorem A. [16] *Let f be a non-constant entire function. If f and $f^{'}$ share two distinct values a, b IM then $f^{'} \equiv f$.*

Subsequently, similar considerations have been made with respect to higher derivatives and more general (linear) differential expressions as well.

Above results motivated researchers to study the relation between an entire function and its derivative counterpart for one CM shared value. In 1996, in this direction the following famous conjecture was proposed by R. Brück [3].

Conjecture: *Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and $f^{'}$ share a finite value a CM, then $\frac{f^{'}-a}{f-a} = c$, where c is a non zero constant.*

Brück himself proved the conjecture for $a = 0$ where as for $a \neq 0$, Brück [3] verified the conjecture under the assumption $N(r, 0; f^{'}) = S(r, f)$ without any growth condition. Following example shows the fact that one can not simply replace the value 1 by a small function $a(z)(\not\equiv 0, \infty)$.

Example 1.1. *Let $f = 1 + e^{e^z}$ and $a(z) = \frac{1}{1-e^{-z}}$.*

By *Lemma 2.6* of [7] [p. 50] we know that a is a small function of f. Also it can be easily seen that f and $f^{'}$ share a CM and $N(r, 0; f^{'}) = 0$ but $f - a \neq c\,(f^{'} - a)$ for every nonzero constant c. We note that $f - a = e^{-z}\,(f^{'} - a)$. So in this case additional suppositions are required.

In 1998, Gundersen and Yang [6] removed the supposition $N(r, 0; f^{'}) = 0$ in [3] for entire function of finite order and thus establishes the Brück conjecture in the following manner.

Theorem B. [6] *Let f be a non-constant entire function of finite order. If f, $f^{(1)}$ share one finite non-zero value a CM, then $\frac{f^{(1)}-a}{f-a} = c$ where c is a nonzero constant.*

Following example exhibited by Gundersen and Yang [6] shows that the corresponding conjecture for meromorphic functions fails in general.

Example 1.2. $f(z) = \frac{2e^z+z+1}{e^z+1}$. *Clearly f and $f^{'}$ share 1 CM and f is of finite order but for a non zero constant c, $\frac{f^{'}-1}{f-1} \neq c$.*

In the next year, Yang [18] further extended *Theorem B* to higher order derivatives and obtained the following result.

Theorem C. [18] *Let f be a non-constant entire function of finite order and let $a(\neq 0)$ be a finite constant. If f, $f^{(k)}$ share the value a CM then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant, where $k(\geq 1)$ is an integer.*

Zhang [20] studied the conjecture for meromorphic function corresponding to CM value sharing of a meromorphic function with its k-th derivative.

Meanwhile a new notion of scalings between CM and IM known as weighted sharing ([8]-[9]), appeared in the uniqueness literature.

Definition 1.1. [8, 9] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively. We now require the following definition.

Definition 1.2. [19] *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.*

Using weighted sharing method, in 2005, Zhang [21] further extended the results of Lahiri-Sarkar [12] and that of Zhang [20] to a small function and proved the following result.

Theorem D. [21] *Let f be a non-constant meromorphic function and $k(\geq 1)$, $l(\geq 0)$ be integers. Also let $a \equiv a(z)$ $(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l(\geq 2)$ and*

$$2\overline{N}(r, \infty; f) + N_2\left(r, 0; f^{(k)}\right) + N_2\left(r, 0; (f/a)^{'}\right) < (\lambda + o(1))\, T\left(r, f^{(k)}\right) \tag{1.1}$$

or $l = 1$ and

$$2\overline{N}(r, \infty; f) + N_2\left(r, 0; f^{(k)}\right) + 2\overline{N}\left(r, 0; (f/a)^{'}\right) < (\lambda + o(1))\, T\left(r, f^{(k)}\right) \tag{1.2}$$

for $r \in I$, where $0 < \lambda < 1$ then $\frac{f^{(k)}-a}{f-a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

In 2008, Zhang and Lü [22] further investigated the analogous problem of Brück conjecture in a different way than that was studied earlier. Zhang and Lü [22] obtained the following theorem.

Theorem E. [22] *Let f be a non-constant meromorphic function and $k(\geq 1)$, $n(\geq 1)$ and $l(\geq 0)$ be integers. Also let $a \equiv a(z)$ ($\not\equiv 0, \infty$) be a meromorphic small function. Suppose that $f^n - a$ and $f^{(k)} - a$ share $(0, l)$. If $l = \infty$ and*

$$2\overline{N}(r,\infty;f) + N_2\left(r,0;f^{(k)}\right) + \overline{N}\left(r,0;(f^n/a)^{'}\right) < (\lambda + o(1))\, T\left(r, f^{(k)}\right) \quad (1.3)$$

for $r \in I$, where $0 < \lambda < 1$ then $\frac{f^{(k)}-a}{f^n-a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

At the end of [22] the following question was raised by Zhang and Lü [22]. What will happen if f^n and $[f^{(k)}]^m$ share a small function ?

In the direction of the above question, Liu [13] proved the following result.

Theorem F. [13] *Let f be a non-constant meromorphic function and $k(\geq 1)$, $n(\geq 1)$, $m(\geq 2)$ and $l(\geq 0)$ be integers. Also let $a \equiv a(z)$ ($\not\equiv 0, \infty$) be a meromorphic small function. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$. If $l = \infty$ and*

$$\frac{2}{m}\overline{N}(r,\infty;f) + \frac{2}{m}\overline{N}\left(r,0;f^{(k)}\right) + \frac{1}{m}\overline{N}\left(r,0;(f^n/a)^{'}\right) < (\lambda + o(1))\, T\left(r, f^{(k)}\right) \quad (1.4)$$

for $r \in I$, where $0 < \lambda < 1$ then $\frac{(f^{(k)})^m-a}{f^n-a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

Next we recall the following definition.

Definition 1.3. *Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be non negative integers.*
The expression $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}} \ldots (f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d(M_j) = \sum\limits_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum\limits_{i=0}^{k}(i+1)n_{ij}$.

The sum $P[f] = \sum\limits_{j=1}^{t} b_j M_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P) = max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_P = max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, \ldots, t$.

The numbers $\underline{d}(P) = min\{d(M_j) : 1 \leq j \leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\overline{d}(P)$=$\underline{d}(P)$.

$P[f]$ is called a Linear Differential Polynomial generated by f if $\overline{d}(P) = 1$. Otherwise $P[f]$ is called Non-linear Differential Polynomial. We also denote by $\mu = max\ \{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = max\ \{n_{1j} + 2n_{2j} + \ldots + kn_{kj} : 1 \leq j \leq t\}$.

So we see from the above discussion that the research have gradually been shifted towards finding the relation between a power of a function together with the differential monomial of that function. As a result it is quite natural to expect the extensions of *Theorems D-H* up to differential polynomial generated by f. In this direction, in 2010, in an attempt to improve *Theorem D*, Li and Yang [14] obtained the following.

Theorem G. [14] *Let f be a non-constant meromorphic function $P[f]$ be a differential polynomial generated by f. Also let $a \equiv a(z)$ ($\not\equiv 0, \infty$) be a small meromorphic* *function. Suppose that* $f-a$ *and* $P[f]-a$ *share* $(0,l)$ *and* $(t-1)\overline{d}(P) \leq \sum\limits_{j=1}^{t} d(M_j)$. *If* $l(\geq 2)$ *and*

$$2\overline{N}(r,\infty;f)+N_2\left(r,0;P[f]\right)+N_2\left(r,0;(f/a)^{'}\right)<(\lambda+o(1))\,T\left(r,P[f]\right) \quad (1.5)$$

or $l=1$ *and*

$$2\overline{N}(r,\infty;f)+N_2\left(r,0;P[f]\right)+2\overline{N}\left(r,0;(f/a)^{'}\right)<(\lambda+o(1))\,T\left(r,P[f]\right) \quad (1.6)$$

for $r\in I$, *where* $0<\lambda<1$ *then* $\frac{P[f]-a}{f-a}=c$ *for some constant* $c\in\mathbb{C}/\{0\}$.

Natural question would be whether *Theorem G* is true for any differential polynomial without the supposition taken over its degree $\overline{d}(P)$? This is one among the motivations of writing the paper. Next question is that whether the two settings of sharing functions in the above theorems can both be extended up to differential polynomials ? The main intention of the paper is to obtain the possible answers of the above questions in such a way that all the *Theorems D-G* can be brought under a single theorem which improves all of them. Henceforth by b_j, $j=1,2,\ldots,t$ and c_i $i=1,2,\ldots,u$ we denote small functions in f and we also suppose that $P[f]=\sum\limits_{j=1}^{t} b_jM_j[f]$ and $Q[f]=\sum\limits_{i=1}^{u} c_iM_i[f]$ be two differential polynomial generated by f. Following theorem is the main result of the paper.

Theorem 1.1. *Let* f *be a non-constant meromorphic function,* $m(\geq 1)$ *be a positive integer or infinity and* $a\equiv a(z)$ $(\not\equiv 0,\infty)$ *be a small meromorphic function. Suppose that* $P[f]$ *and* $Q[f]$ *be two differential polynomial generated by* f *such that* $Q[f]$ *contains at least one derivative. Suppose that* $P[f]-a$ *and* $Q[f]-a$ *share* $(0,l)$. *If* $l=\infty$ *and*

$$2\overline{N}(r,\infty;f)+N_2\left(r,0;Q[f]\right)+\overline{N}\left(r,0;(P[f]/a)^{'}\right)<(\lambda+o(1))\,T\left(r,Q[f]\right) \quad (1.7)$$

or $2\leq l<\infty$ *and*

$$2\overline{N}(r,\infty;f)+N_2\left(r,0;Q[f]\right)+N_2\left(r,0;(P[f]/a)^{'}\right)<(\lambda+o(1))\,T\left(r,Q[f]\right) \quad (1.8)$$

or $l=1$ *and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f)+N_2\left(r,0;Q[f]\right)+\overline{N}\left(r,0;(P[f]/a)^{'}\right) \\ & +\overline{N}\left(r,0;(P[f]/a)^{'}\mid (P[f]/a)\neq 0\right) \\ < \;& (\lambda+o(1))\,T\left(r,Q[f]\right) \end{aligned} \quad (1.9)$$

for $r\in I$, *where* $0<\lambda<1$ *then either* **a**) $\frac{Q[f]-a}{P[f]-a}=c$, *for some constant* $c\in\mathbb{C}/\{0\}$ *or* **b**) $P[f]Q[f]-aQ[f](1+d)\equiv -da^2$, *for a non-zero constant* $d\in\mathbb{C}$.
In particular,
if i) $P[f]=b_1f^n+b_2f^{n-1}+b_3f^{n-2}+\ldots+b_{t-1}f$ *or if*

ii) $\underline{d}(Q) > 2\overline{d}(P) - \underline{d}(P)$ *and each monomial of* $Q[f]$ *contains a term involving a power of* f*, then the conclusion (b) does not hold.*

Following four examples show that (1.7)-(1.9) are not necessary when (i) or (ii) of Theorem 1.1 occurs.

Example 1.3. *Let* $f(z) = \frac{ie^z}{e^z-1}$. $P[f] = f^2$, $Q[f] = if - if^{'}$. *Then clearly* $P[f]$ *and* $Q[f]$ *share any non-zero complex number* a *CM and* $\frac{Q[f]-a}{P[f]-a} = 1$, *but (1.7)-(1.9) are not satisfied.*

Example 1.4. *Let* $f(z) = \frac{i}{e^z+1}$. $P[f] = -f^2 - if^3$, $Q[f] = ff^{'}$. *Then clearly* $P[f]$ *and* $Q[f]$ *share any non-zero complex number* a *CM and* $\frac{Q[f]-a}{P[f]-a} = 1$, *but (1.7)-(1.9) are not satisfied. Here we note that* $3 = \underline{d}(Q) > 2\overline{d}(P) - \underline{d}(P) = 2$.

Example 1.5. *Let* $f(z) = \frac{i}{e^z-1}$. $P[f] = \frac{1}{2}[f^{'} + f^{''}]$, $Q[f] = -2f^2f^{'} - iff^{''}$. *Then clearly* $P[f] = Q[f] = \frac{ie^z}{(e^z-1)^3}$ *share any non-zero complex number* a *CM and* $\frac{Q[f]-a}{P[f]-a} = 1$, *but (1.7)-(1.9) are not satisfied. Here we see that* $2 = \underline{d}(Q) > 2\overline{d}(P) - \underline{d}(P) = 1$.

Example 1.6. *Let* $f(z) = \frac{i}{e^z+1}$. $P[f] = {f^{'}}^2 - ff^{''}$, $Q[f] = 2f^3f^{'} - if^2f^{''}$. *Then clearly* $P[f] = Q[f] = \frac{-e^z}{(e^z+1)^4}$ *share any non-zero complex number* a *CM and* $\frac{Q[f]-a}{P[f]-a} = 1$, *but (1.7)-(1.9) are not satisfied. Here we see that* $3 = \underline{d}(Q) > 2\overline{d}(P) - \underline{d}(P) = 2$.

We now give the next four examples the first two of which show that both the conditions stated in (ii) are essential in order to obtain conclusion (a) in Theorem 1.1 for homogeneous differential polynomials $P[f]$ where as the rest two substantiate the same for non homogeneous differential polynomials.

Example 1.7. *Let* $f(z) = e^z - e^{-z}$. $P[f] = \frac{1}{2}[f^{'} + f^{''}]$, $Q[f] = \frac{1}{2}[-f + f^{'}]$. *Then clearly* $P[f] = e^z$ *and* $Q[f] = e^{-z}$ *share 1 CM. Here (1.7)-(1.9) are satisfied, but* $\frac{Q[f]-1}{P[f]-1} = -e^{-z}$, *rather* $P[f]Q[f] = 1$. *Here* $1 = \underline{d}(Q) \not> 2\overline{d}(P) - \underline{d}(P) = 1$.

Example 1.8. *Let* $f(z) = e^z - e^{-z}$. $P[f] = -f^2 + f^{'}f^{''}$, $Q[f] = \frac{1}{4}[f^2 + {f^{'}}^2] + \frac{1}{2}ff^{'}$. *Then clearly* $P[f] = 2 - 2e^{-2z}$ *and* $Q[f] = e^{2z}$ *share both* $1+i$ *and* $1-i$ *CM. Here (1.7)-(1.9) are satisfied and* $P[f]Q[f] - 2Q[f] + 2 = 0$. *When we consider* $1+i$ *as the shared value then* $\frac{Q[f]-(1+i)}{P[f]-(1+i)} = \frac{e^{2z}}{1-i}$, *on the other hand when we consider* $1-i$ *as the shared value then* $\frac{Q[f]-(1-i)}{P[f]-(1-i)} = \frac{e^{2z}}{1+i}$. *Here* $2 = \underline{d}(Q) \not> 2\overline{d}(P) - \underline{d}(P) = 2$.

Example 1.9. *Let* $f(z) = e^z + e^{-z}$. $P[f] = \frac{1}{2}[f + f^{'} + {f^{'}}^2 - {f^{''}}^2]$, $Q[f] = \frac{1}{2}[-f^{'} + f^{''}]$. *Then clearly* $P[f] = e^z - 2$ *and* $Q[f] = e^{-z}$ *share both* $-1+\sqrt{2}$, $-1-\sqrt{2}$ *CM. Here (1.7)-(1.9) are satisfied and* $P[f]Q[f] + 2Q[f] - 1 = 0$. *When we consider* $-1+\sqrt{2}$

as the shared value then $\frac{Q[f]-(-1+\sqrt{2})}{P[f]-(-1+\sqrt{2})}=\frac{1-\sqrt{2}}{e^z}$*, on the other hand when we consider* $-1-\sqrt{2}$ *as the shared value then* $\frac{Q[f]-(-1-\sqrt{2})}{P[f]-(-1-\sqrt{2})}=\frac{1+\sqrt{2}}{e^z}$*. We also note that here* $\overline{d}(P)\neq\underline{d}(P)$, $1=\underline{d}(Q)\not>2\overline{d}(P)-\underline{d}(P)=2$.

Example 1.10. *Let* $f(z)=cosz$. $P[f]=-f-if^{'}+(1+i)f^{'2}+(1+i)f^{''2}$, $Q[f]=if-f^{'''}$. *Then clearly* $P[f]=1+i-e^{-iz}$ *and* $Q[f]=ie^{iz}$ *share both* i *and* 1 *CM. Here (1.7)-(1.9) are satisfied and* $P[f]Q[f]-(1+i)Q[f]+i=0$. *When we consider* i *as the shared value then* $\frac{Q[f]-i}{P[f]-i}=ie^{iz}$*, on the other hand when we consider* 1 *as the shared value then* $\frac{Q[f]-1}{P[f]-1}=e^{iz}$*. We also note that here* $\overline{d}(P)\neq\underline{d}(P)$, $1=\underline{d}(Q)\not>2\overline{d}(P)-\underline{d}(P)=3$.

The following two examples show that in order to obtain conclusions (a) or (b) of *Theorem 1.1*, (1.7)-(1.9) are essential.

Example 1.11. *Let* $f(z)=sinz$. $P[f]=f^2+f^{'2}+f^{'}+if^{''}-[f^{'}-if]^2$, $Q[f]=if+f^{'}$. *Then clearly* $P[f]=1+e^{-iz}-e^{-2iz}$ *and* $Q[f]=e^{iz}$ *share* 1 *CM. Since* $\frac{Q[f]-1}{P[f]-1}=e^{2iz}$ *and* $P[f]Q[f]-Q[f]+\frac{1}{Q}-1=0$*, neither of the conclusions of* Theorem 1.1 *is satisfied, nor any one of (1.7)-(1.9) is satisfied. Here we note that* $1=\underline{d}(Q)\not>2\overline{d}(P)-\underline{d}(P)=3$.

Example 1.12. *Let* $f(z)=cosz$. $P[f]=f+if^{'}$, $Q[f]=f^2+f^{'2}-(f+if^{'})(-if^{'}-f^{''})^2+if^{'2}+f^{''2}$. *Then clearly* $P[f]=e^{-iz}$ *and* $Q[f]=e^{2iz}-e^{iz}+1$ *share* 1 *CM. Since* $\frac{Q[f]-1}{P[f]-1}=-e^{2iz}$ *and* $P[f]Q[f]-(e^{iz}+e^{-iz})+1=0$*, neither of the conclusions of* Theorem 1.1 *is satisfied, nor any one of (1.7)-(1.9) is satisfied. Here we note that* $2=\underline{d}(Q)>2\overline{d}(P)-\underline{d}(P)=1$.

Though we use the standard notations and definitions of the value distribution theory available in [7], we explain some definitions and notations which are used in the paper.

Definition 1.4. [12]*Let* p *be a positive integer and* $a\in\mathbb{C}\cup\{\infty\}$.

(i) $N(r,a;f\mid\geq p)$ $(\overline{N}(r,a;f\mid\geq p))$*denotes the counting function (reduced counting function) of those a-points of* f *whose multiplicities are not less than* p.

(ii) $N(r,a;f\mid\leq p)$ $(\overline{N}(r,a;f\mid\leq p))$*denotes the counting function (reduced counting function) of those a-points of* f *whose multiplicities are not greater than* p.

Definition 1.5. [10] *Let* $a,b\in\mathbb{C}\cup\{\infty\}$*. We denote by* $N(r,a;f\mid g\neq b)$ *the counting function of those a-points of* f*, counted according to multiplicity, which are not the b-points of* g.

Definition 1.6. *{cf.[1], 2} Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. Let z_0 be a a-point of f with multiplicity p, a a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p>q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p=q=1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p=q\geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$.*

Definition 1.7. [8, 9] *Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H=\left(\frac{F^{''}}{F^{'}}-\frac{2F^{'}}{F-1}\right)-\left(\frac{G^{''}}{G^{'}}-\frac{2G^{'}}{G-1}\right). \tag{2.1}$$

Lemma 2.1. [21] *Let f be a non-constant meromorphic function and k be a positive integer, then*

$$N_p(r,0;f^{(k)})\leq N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.2. [11] *If $N(r,0;f^{(k)}\mid f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}\mid f\neq 0)\leq k\overline{N}(r,\infty;f)+N(r,0;f\mid<k)+k\overline{N}(r,0;f\mid\geq k)+S(r,f).$$

Lemma 2.3. [15] *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n}a_kf^k}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n\neq 0$ and $b_m\neq 0$. Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.4. [4] *Let f be a meromorphic function and $P[f]$ be a differential polynomial. Then*

$$m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + S(r, f).$$

Lemma 2.5. *Let f be a meromorphic function and $P[f]$ be a differential polynomial. Then we have*

$$\begin{aligned} N\left(r, \infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) &\leq (\Gamma_P - \overline{d}(P))\, \overline{N}(r, \infty; f) + (\overline{d}(P) - \underline{d}(P))\, N(r, 0; f \mid\geq k+1) \\ &\quad + \mu \overline{N}(r, 0; f \mid\geq k+1) + \overline{d}(P) N(r, 0; f \mid\leq k) + S(r, f). \end{aligned}$$

Proof. Let z_0 be a pole of f of order r, such that $b_j(z_0) \neq 0, \infty; 1 \leq j \leq t$. Then it would be a pole of $P[f]$ of order at most $r\overline{d}(P) + \Gamma_P - \overline{d}(P)$. Since z_0 is a pole of $f^{\overline{d}(P)}$ of order $r\overline{d}(P)$, it follows that z_0 would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order at most $\Gamma_P - \overline{d}(P)$. Next suppose z_1 is a zero of f of order $s(> k)$, such that $b_j(z_1) \neq 0, \infty; 1 \leq j \leq t$. Clearly it would be a zero of $M_j(f)$ of order $s.n_{0j} + (s-1)n_{1j} + \ldots + (s-k)n_{kj} = s.d(M_j) - (\Gamma_{M_j} - d(M_j))$. Hence z_1 be a pole of $\frac{M_j[f]}{f^{\overline{d}(P)}}$ of order

$$s.\overline{d}(P) - s.d(M_j) + (\Gamma_{M_j} - d(M_j)) = s(\overline{d}(P) - d(M_j)) + (\Gamma_{M_j} - d(M_j)).$$

So z_1 would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order at most

$$\max\{s(\overline{d}(P) - d(M_j)) + (\Gamma_{M_j} - d(M_j)) : 1 \leq j \leq t)\} = s(\overline{d}(P) - \underline{d}(P)) + \mu.$$

If z_1 is a zero of f of order $s \leq k$, such that $b_j(z_1) \neq 0, \infty : 1 \leq j \leq t$ then it would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order $s\overline{d}(P)$. Since the poles of $\frac{P[f]}{f^{\overline{d}(P)}}$ comes from the poles or zeros of f and poles or zeros of $b_j(z)$'s only, it follows that

$$\begin{aligned} N\left(r, \infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) &\leq (\Gamma_P - \overline{d}(P))\, \overline{N}(r, \infty; f) + (\overline{d}(P) - \underline{d}(P))\, N(r, 0; f \mid\geq k+1) \\ &\quad + \mu\, \overline{N}(r, 0; f \mid\geq k+1) + \overline{d}(P) N(r, 0; f \mid\leq k) + S(r, f). \end{aligned}$$

□

Lemma 2.6. [5] *Let $P[f]$ be a differential polynomial. Then*

$$T(r, P[f]) \leq \Gamma_P T(r, f) + S(r, f).$$

Lemma 2.7. *Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial. Then $S(r, P[f])$ can be replaced by $S(r, f)$.*

Proof. From *Lemma 2.6* it is clear that $T(r, P[f]) = O(T(r, f))$ and so the lemma follows. □

Lemma 2.8. *Let f be a non-constant meromorphic function and $P[f]$, $Q[f]$ be two differential polynomials. Then*

$$N(r,0;P[f]) \leq \frac{\overline{d}(P)-\underline{d}(P)}{\underline{d}(Q)} m\left(r,\frac{1}{Q[f]}\right) + (\Gamma_P - \overline{d}(P))\ \overline{N}(r,\infty;f)$$
$$+(\overline{d}(P)-\underline{d}(P))\ N(r,0;f\mid\geq k+1)+\mu\overline{N}(r,0;f\mid\geq k+1)$$
$$+\overline{d}(P)N(r,0;f\mid\leq k)+\overline{d}(P)\ N(r,0;f)+S(r,f).$$

Proof. For a fixed value of r, let $E_1 = \{\theta \in [0,2\pi] : \left|f(re^{i\theta})\right| \leq 1\}$ and E_2 be its complement. Since by definition

$$\sum_{i=0}^{k} n_{ij} \geq \underline{d}(Q),$$

for every $j = 1,2,\ldots,u$, it follows that on E_1

$$\left|\frac{Q[f]}{f^{\underline{d}(Q)}}\right| \leq \sum_{j=1}^{u}|c_j(z)|\prod_{i=1}^{k}\left|\frac{f^{(i)}}{f}\right|^{n_{ij}}|f|^{\sum\limits_{i=0}^{k} n_{ij}-\underline{d}(Q)} \leq \sum_{j=1}^{u}|c_j(z)|\prod_{i=1}^{k}\left|\frac{f^{(i)}}{f}\right|^{n_{ij}}.$$

Also we note that

$$\frac{1}{f^{\underline{d}(Q)}} = \frac{Q[f]}{f^{\underline{d}(Q)}}\ \frac{1}{Q[f]}.$$

Since on E_2, $\frac{1}{|f(z)|} < 1$, we have

$$\underline{d}(Q)m\left(r,\frac{1}{f}\right)$$
$$\begin{aligned}
&= \frac{1}{2\pi}\int\limits_{E_1}\log^+\frac{1}{|f(re^{i\theta})|^{\underline{d}(Q)}}d\theta + \frac{1}{2\pi}\int\limits_{E_2}\log^+\frac{1}{|f(re^{i\theta})|^{\underline{d}(Q)}}d\theta \\
&\leq \frac{1}{2\pi}\sum_{j=1}^{u}\left[\int\limits_{E_1}\log^+|c_j(z)|\,d\theta + \sum_{i=1}^{k}\int\limits_{E_1}\log^+\left|\frac{f^{(i)}}{f}\right|^{n_{ij}}d\theta\right] \\
&\quad + \frac{1}{2\pi}\int\limits_{E_1}\log^+\left|\frac{1}{Q[f(re^{i\theta})]}\right|d\theta \\
&\leq \frac{1}{2\pi}\int\limits_{0}^{2\pi}\log^+\left|\frac{1}{Q[f(re^{i\theta})]}\right|d\theta + S(r,f) = m\left(r,\frac{1}{Q[f]}\right) + S(r,f).
\end{aligned}$$

So using *Lemmas 2.4*, *2.5* and the first fundamental theorem we get

$$
\begin{aligned}
&N(r,0;P[f])\\
\leq\;& N\left(r,\infty;\frac{f^{\overline{d}(P)}}{P[f]}\right)+\overline{d}(P)N(r,0;f)\\
\leq\;& m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+N\left(r,\infty;\frac{P[f]}{f^{\overline{d}(P)}}\right)+\overline{d}(P)N(r,0;f)+S(r,f)\\
\leq\;& (\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)+(\Gamma_P-\overline{d}(P))\,\overline{N}(r,\infty;f)\\
&\qquad +(\overline{d}(P)-\underline{d}(P))\,N(r,0;f\mid\geq k+1)+\mu\overline{N}(r,0;f\mid\geq k+1)\\
&\qquad\quad +\overline{d}(P)N(r,0;f\mid\leq k)+\overline{d}(P)N(r,0;f)+S(r,f)\\
\leq\;& \frac{(\overline{d}(P)-\underline{d}(P))}{\underline{d}(Q)}m\left(r,\frac{1}{Q[f]}\right)+(\Gamma_P-\overline{d}(P))\,\overline{N}(r,\infty;f)\\
&+(\overline{d}(P)-\underline{d}(P))\,N(r,0;f\mid\geq k+1)\\
&+\mu\overline{N}(r,0;f\mid\geq k+1)+\overline{d}(P)N(r,0;f\mid\leq k)+\overline{d}(P)N(r,0;f)+S(r,f).
\end{aligned}
$$

□

3. Proof of the theorem

Proof of Theorem 1.1. Let $F=\frac{P[f]}{a}$ and $G=\frac{Q[f]}{a}$. Then $F-1=\frac{P[f]-a}{a}$, $G-1=\frac{Q[f]-a}{a}$. Since $P[f]-a$ and $Q[f]-a$ share $(0,l)$ it follows that F, G share $(1,l)$ except the zeros and poles of $a(z)$. We consider two cases the second of which is being split into several subcases.

Case 1 Let $H\not\equiv 0$.

From (2.1) we get

$$
\begin{aligned}
&N(r,\infty;H) && (3.1)\\
\leq\;& \overline{N}(r,\infty;F)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_0(r,0;F^{'})\\
&+\overline{N}_0(r,0;G^{'})+\overline{N}(r,0;a)+\overline{N}(r,\infty;a)+S(r,f),
\end{aligned}
$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined. Let z_0 be a simple zero of $F-1$. Then by a simple calculation we see that z_0 is a zero of H and hence

$$N_E^{1)}(r,1;F)=N(r,1;F\mid=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,F) \tag{3.2}$$

By the second fundamental theorem, *Lemma 2.7*, (3.1) and noting that $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)+S(r,f)$, we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,1;G)-N_0(r,0;G^{'})+S(r,G) \quad (3.3)\\ &\leq 2\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,0;F\mid\geq 2)\\ &\quad +\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})+S(r,f). \end{aligned}$$

While $l=\infty$, $\overline{N}_*(r,1;F,G)=0$. So

$$\begin{aligned} &\overline{N}(r,0;F\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'}) \quad (3.4)\\ \leq\ &\overline{N}(r,0;F^{'}). \end{aligned}$$

So

$$T(r,G)\leq 2\,\overline{N}(r,\infty;F)+N_2(r,0;G)+\overline{N}(r,0;F^{'})+S(r,f)$$

that is

$$T\left(r,Q[f]\right)\leq 2\,\overline{N}(r,\infty;f)+N_2(r,0;Q[f])+\overline{N}\left(r,0;(P[f]/a)^{'}\right)+S(r,f),$$

which contradicts (1.7)

While $l\geq 2$, (3.4) changes to

$$\begin{aligned} &\overline{N}(r,0;F\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'}) \quad (3.5)\\ \leq\ &\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,1;F\mid\geq l+1)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})\\ \leq\ &N_2(r,0;F^{'}). \end{aligned}$$

Hence

$$T(r,G)\leq 2\,\overline{N}(r,\infty;F)+N_2(r,0;G)+N_2(r,0;F^{'})+S(r,f)$$

that is

$$T\left(r,Q[f]\right)\leq 2\overline{N}(r,\infty;f)+N_2(r,0;Q[f])+N_2\left(r,0;(P[f]/a)^{'}\right)+S(r,f),$$

which contradicts (1.8).

While $l=1$ (3.4) changes to

$$\begin{aligned} &\overline{N}(r,0;F\mid\geq 2)+2\,\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})\\ \leq\ &\overline{N}(r,0;F^{'})+\overline{N}(r,0;F^{'}\mid F\neq 0). \end{aligned}$$

Similarly as above we have

$$\begin{aligned} &T\left(r,Q[f]\right)\leq 2\,\overline{N}(r,\infty;f)+N_2\left(r,0;Q[f]\right)+\overline{N}\left(r,0;(P[f]/a)^{'}\right)\\ &+\overline{N}\left(r,0;(P[f]/a)^{'}\mid(P[f]/a)\neq 0\right)+S(r,f), \end{aligned}$$

which contradicts (1.9).

Case 2 Let $H\equiv 0$.

On integration we get from

$$\frac{1}{F-1}\equiv\frac{C}{G-1}+D, \quad (3.6)$$

where C, D are constants and $C \neq 0$. From (3.6) it is clear that F and G share 1 CM. We first assume that $D \neq 0$. Then by (3.6) we get

$$\overline{N}(r,\infty;f) = S(r,f). \tag{3.7}$$

Clearly $\overline{N}(r,\infty;G) = \overline{N}(r,\infty;f) + S(r,f) = S(r,f)$.
From (3.6) we get

$$\frac{1}{F-1} = \frac{D\left(G-1+\frac{C}{D}\right)}{G-1} \tag{3.8}$$

Clearly from (3.8) we have

$$\overline{N}\left(r,1-\frac{C}{D};G\right) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;G) = S(r,f). \tag{3.9}$$

If $\frac{C}{D} \neq 1$, by the second fundamental theorem, *Lemma 2.7* and (3.9) we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\left(r,1-\frac{C}{D};G\right) + S(r,G) \\ &\leq \overline{N}(r,0;G) + S(r,f) \leq N_2(r,0;G) + S(r,f) \\ &\leq T(r,G) + S(r,f). \end{aligned}$$

So $T(r,G) = N_2(r,0;G) + S(r,f)$ that is, $T\left(r,Q[f]\right) = N_2\left(r,0;Q[f]\right) + S(r,f)$, which contradicts (1.7)-(1.9).
If $\frac{C}{D} = 1$ we get from (3.6)

$$\left(F-1-\frac{1}{C}\right)G \equiv -\frac{1}{C}. \tag{3.10}$$

i.e.,

$$P[f]Q[f] - aQ(1+d) \equiv -da^2,$$

for a non zero constant $d = \frac{1}{C} \in \mathbb{C}$. From (3.10) it follows that

$$N(r,0;f\mid\geq k+1) \leq N(r,0;Q[f]) \leq N(r,0;G) \leq N(r,0;a) = S(r,f). \tag{3.11}$$

When $P[f] = b_1f^n + b_2f^{n-1} + b_3f^{n-2} + \ldots + b_{t-1}f$, we see from (3.10) that

$$\frac{1}{f^{\overline{d}(Q)}\left(P[f]-(1+1/C)a\right)} \equiv -\frac{C}{a^2}\;\frac{Q[f]}{f^{\overline{d}(Q)}}.$$

Hence by the first fundamental theorem, (3.7), (3.11), *Lemmas 2.3*, *2.4* and *2.5* we get that

$$\begin{aligned}
& (n+\overline{d}(Q))T(r,f) \qquad\qquad (3.12)\\
= \;& T\left(r, f^{\overline{d}(Q)}(P[f]-(1+\frac{1}{C})a)\right)+S(r,f)\\
= \;& T\left(r, \frac{1}{f^{\overline{d}(Q)}(P[f]-(1+\frac{1}{C})a)}\right)+S(r,f)\\
= \;& T\left(r, \frac{Q[f]}{f^{\overline{d}(Q)}}\right)+S(r,f)\\
\leq \;& m\left(r, \frac{Q[f]}{f^{\overline{d}(Q)}}\right)+N\left(r, \frac{Q[f]}{f^{\overline{d}(Q)}}\right)+S(r,f)\\
\leq \;& (\overline{d}(Q)-\underline{d}(Q))\left[T(r,f)-\{N(r,0;f\mid\leq k)+N(r,0;f\mid\geq k+1)\}\right]+(\overline{d}(Q)-\underline{d}(Q))\\
& N(r,0;f\mid\geq k+1)+\mu\,\overline{N}(r,0;f\mid\geq k+1)+\overline{d}(Q)N(r,0;f\mid\leq k)+S(r,f)\\
\leq \;& (\overline{d}(Q)-\underline{d}(Q))T(r,f)+\underline{d}(Q)N(r,0;f\mid\leq k)+S(r,f).
\end{aligned}$$

From (3.12) it follows that

$$nT(r,f)\leq S(r,f),$$

which is absurd.

If $P[f]$ is a differential polynomial then we consider the following two subcases.

Subcase 2.1.

If $C=-1$ then from (3.6) we get $FG\equiv 1$, i.e., $P[f]Q[f]\equiv a^2$. It is clear that $\overline{N}(r,\infty;P[f])=\overline{N}(r,\infty;Q[f])=S(r,f)$.

First we observe that since each monomial of $Q[f]$ contains a term involving a power of f, we have $N(r,0;f)=S(r,f)$. So from the first fundamental theorem, *Lemma 2.4* and noting that $m\left(r,\frac{1}{f}\right)\leq\frac{1}{\underline{d}(Q)}m(r,\frac{1}{Q[f]}))$ we have

$$\begin{aligned}
T(r,Q[f]) \;&\leq\; T(r,P[f])+S(r,f)\\
&\leq\; m(r,\frac{P[f]}{f^{\overline{d}(P)}})+\overline{d}(P)m(r,f)+S(r,f)\\
&\leq\; (\overline{d}(P)-\underline{d}(P))m(r,\frac{1}{f})+\overline{d}(P)m(r,f)+S(r,f)\\
&\leq\; \frac{(\overline{d}(P)-\underline{d}(P))}{\underline{d}(Q)}m(r,\frac{1}{Q[f]})+\overline{d}(P)\{m(r,\frac{1}{f})+N(r,0;f)\}+S(r,f)\\
&\leq\; \frac{(\overline{d}(P)-\underline{d}(P))}{\underline{d}(Q)}m(r,\frac{1}{Q[f]})+\frac{\overline{d}(P)}{\underline{d}(Q)}m(r,\frac{1}{Q[f]})+S(r,f),
\end{aligned}$$

which is a contradiction as $\underline{d}(Q)>2\overline{d}(P)-\underline{d}(P)$.

Subcase 2.2.

Next we assume $C\neq -1$.

Then from (3.10) we have

$$\overline{N}(r,1+\frac{1}{C};F)=\overline{N}(r,\infty;G)=S(r,f).$$

So again noticing the fact that each monomial of $Q[f]$ contains a term involving a power of f, by the second fundamental theorem, *Lemma 2.8* we get

$$\begin{aligned} & T(r,P[f]) && (3.13)\\ \leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1+\frac{1}{C};F)+S(r,f)\\ \leq\ & N(r,0;P[f])+S(r,f)\\ \leq\ & \frac{\overline{d}(P)-\underline{d}(P)}{\underline{d}(Q)}T(r,P[f])+S(r,f), \end{aligned}$$

i.e.,

$$\frac{\underline{d}(Q)+\underline{d}(P)-\overline{d}(P)}{\underline{d}(Q)}T(r,P[f])\leq S(r,f). \tag{3.14}$$

Since by the given condition $\underline{d}(Q)>2\overline{d}(P)-\underline{d}(P)>\overline{d}(P)-\underline{d}(P)$ (3.14) leads to a contradiction.
Hence $D=0$ and so $\frac{G-1}{F-1}=C$ or $\frac{Q[f]-a}{P[f]-a}=C$. This proves the theorem. □

4. Concluding Remark and an Open Question

We see from the statement of *Theorem 1.1* that when (ii) occurs the conclusion of Brück conjecture can not be derived as a special case. Also (1.7) is better than the condition (3) in *Theorem 2* used in [20] for CM sharing and in fact (1.7) is the weakest inequality ever obtained when (i) of *Theorem 1.1* is satisfied. So natural question would be
i) Whether in any way (1.7) can further be relaxed and
ii) Can conclusion (a) of *Theorem 1.1* be obtained for two arbitrary differential polynomials $P[f]$ and $Q[f]$ sharing a small function $a\equiv a(z)$ ($\not\equiv 0,\infty$) CM or even under non zero finite weight without the help of (ii)?

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