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WEIGHTED APPROXIMATION PROPERTIES OF STANCU TYPE MODIFICATION OF q-SZÁSZ-DURRMEYER OPERATORS

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Abstract. In this paper, we are dealing with q-Szász-Mirakyan-Durrmeyer-Stancu operators. Firstly, we establish moments of these operators and estimate convergence results. We discuss a Voronovskaja type result for the operators. We shall give the weighted approximation properties of these operators. Furthermore, we study the weighted statistical convergence for the operators.

1. Introduction

Some researchers studied the well-known Szász-Mirakyan operators and estimated some approximation results. The most commonly used integral modifications of the Szász-Mirakyan operators are Kantorovich and Durrmeyer type operators. In 1954, R. S. Phillips [28] defined the well-known P_n positive operators. Some approximation properties of these operators were studied by Gupta and Srivastava [16] and by May [24]. Recently, Gupta [10] introduced and studied approximation properties of q-Durrmeyer operators. Gupta and Heping [15] introduced the q-Durrmeyer type operators and studied estimation of the rate of convergence for continuous functions in terms of modulus of continuity. Some other analogues of the Bernstein-Durrmeyer operators related to the q-Bernstein basis functions have been studied by Derriennic [4]. Also, many authors studied the q-analogue of operators in [3], [5], [7], [20], [22], [23], and [27]. The q-analogue and integral modifications of Szász-Mirakyan operators have been studied by researchers in [2], [11], [13], [14], [15], and [22]. In 1993, Gupta [12] filled the gaps and improved the results of [29]. To approximate Lebesgue integrable functions on the interval $[0,\infty)$, the Szász-Mirakyan-Baskakov operators are defined in [14] as

$$G_n(f;x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\, f(t)\, d_q t,$$

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where $x\in[0,\infty)$ and

$$s_{n,k}\left(x\right)=e^{-nx}\frac{\left(nx\right)^{k}}{k!},\ p_{n,k}\left(t\right)=\binom{n+k-1}{k}\frac{x^{k}}{\left(1+x\right)^{n+k}}.$$

Based on q-exponential function Mahmudov [21], introduced the following q-Szász-Mirakyan operators

$$\begin{aligned}S_{n,q}\left(f,x\right) &= \frac{1}{E_{q}\left(\left[n\right]x\right)}\sum_{k=0}^{\infty}\frac{\left(\left[n\right]x\right)^{k}}{\left[k\right]!}q^{k(k-1)/2}f\left(\frac{\left[k\right]}{q^{k-2}\left[n\right]}\right)\\ &= \sum_{k=0}^{\infty}s_{n,k}^{q}\left(x\right)f\left(\frac{\left[k\right]}{q^{k-2}\left[n\right]}\right),\end{aligned}$$

where

$$s_{n,k}^{q}\left(x\right)=\frac{\left(\left[n\right]x\right)^{k}}{\left[k\right]!}q^{k(k-1)/2}\frac{1}{E_{q}\left(\left[n\right]x\right)}. \tag{1.1}$$

He obtained the moments as

$$S_{n,q}\left(1,x\right)=1,\ S_{n,q}\left(t,x\right)=qx,\text{ and }S_{n,q}\left(t^{2},x\right)=qx^{2}+\frac{q^{2}x}{\left[n\right]}.$$

In [14], Gupta et al. introduced the q-analogue of the Szász-Mirakyan-Durrmeyer operators as

$$G_{n}^{q}\left(f;x\right)=\left[n-1\right]\sum_{k=0}^{\infty}q^{k}s_{n,k}^{q}\left(x\right)\int_{0}^{\infty/A}p_{n,k}^{q}\left(t\right)f\left(t\right)d_{q}t,$$

where $s_{n,k}^{q}\left(x\right)$ is given in (1.1) and

$$p_{n,k}^{q}\left(t\right)=\begin{bmatrix} n+k-1\\ k\end{bmatrix}q^{k(k-1)/2}\frac{t^{k}}{\left(1+t\right)_{q}^{n+k}}. \tag{1.2}$$

They obtained its moments as

$$\begin{aligned}G_{n}^{q}\left(1;x\right) &= 1,\ G_{n}^{q}\left(t;x\right)=\frac{\left[n\right]}{q^{2}\left[n-2\right]}x+\frac{1}{q\left[n-2\right]},\\ G_{n}^{q}\left(t^{2};x\right) &= \tfrac{\left[n\right]^{2}}{q^{6}\left[n-2\right]\left[n-3\right]}x^{2}+\tfrac{\left(1+q\right)^{2}\left[n\right]}{q^{5}\left[n-2\right]\left[n-3\right]}x+\tfrac{1+q}{q^{3}\left[n-2\right]\left[n-3\right]}.\end{aligned}$$

Also, Mahmudov and Kaffaoğlu [22] studied on q-Szász-Mirakyan-Durrmeyer operators but they defined different operator. They gave the operator as

$$D_{n,q}\left(f,x\right)=\left[n\right]\sum_{k=0}^{\infty}q^{k}s_{n,k}\left(q;x\right)\int_{0}^{\infty/(1-q)}s_{n,k}\left(q;t\right)f\left(t\right)d_{q}t, \tag{1.3}$$

where $s_{n,k}(q;x)$ is given by (1.1). They gave the moments as

$$D_{n,q}(1,x) = 1,\ D_{n,q}(t,x) = \frac{1}{q^2}x + \frac{1}{q[n]}, \tag{1.4}$$

$$D_{n,q}(t^2,x) = \frac{1}{q^6}x^2 + \frac{(1+q)^2}{q^5[n]}x + \frac{1+q}{q^3}\frac{1}{[n]^2}. \tag{1.5}$$

Gupta and Karslı [17] extended the $G_n^q(f;x)$ operators and introduced q-Szász-Mirakyan-Durrmeyer-Stancu operators as

$$G_{n,\alpha,\beta}^q(f;x) = [n-1]\sum_{k=0}^{\infty} s_{n,k}^q(x)\, q^k \int_0^{\infty/A} p_{n,k}^q(t)\, f\left(\frac{[n]t+\alpha}{[n]+\beta}\right) d_q t,$$

where $s_{n,k}^q(x)$ given in (1.1) and $p_{n,k}^q(t)$ given in (1.2). They gave the moments as

$$\begin{aligned} G_{n,\alpha,\beta}^q(1;x) &= 1,\ G_{n,\alpha,\beta}^q(t;x) = \frac{[n]^2}{q^2[n-2]([n]+\beta)}x + \frac{[n]+q\alpha[n-2]}{q[n-2]([n]+\beta)}, \\ G_{n,\alpha,\beta}^q(t^2;x) &= \left(\frac{[n]}{[n]+\beta}\right)^2\left(\tfrac{[n]^2x^2}{q^6[n-2][n-3]} + \tfrac{(1+q)^2[n]x}{q^5[n-2][n-3]} + \tfrac{1+q}{q^3[n-2][n-3]}\right) \\ &\quad + \frac{2\alpha[n]}{([n]+\beta)^2}\left(\frac{[n]x+q}{q^2[n-2]}\right) + \left(\frac{\alpha}{[n]+\beta}\right)^2. \end{aligned}$$

We now extend the studies and introduce for $0 \le \alpha \le \beta$, and every $n \in \mathbb{N}$, $q \in (0,1)$ the Stancu type generalization of (1.3) operator as

$$D_{n,q}^{\alpha,\beta}(f,x) = [n]\sum_{k=0}^{\infty} q^k s_{n,k}(q;x)\int_0^{\infty/(1-q)} s_{n,k}(q;t)\, f\left(\frac{[n]t+\alpha}{[n]+\beta}\right) d_q t, \tag{1.6}$$

where $f \in C[0,\infty)$ and $x \in [0,\infty)$.

We first mention some notations of q-calculus. Throughout the present article q is a real number satisfying the inequality $0 < q \le 1$. For $n \in \mathbb{N}$,

$$[n]_q = [n] := \begin{cases} (1-q^n)/(1-q), & q \ne 1 \\ n, & q = 1 \end{cases},$$

$$[n]_q! = [n]! := \begin{cases} [n][n-1]\ldots[1], & n \ge 1 \\ 1, & n = 0 \end{cases}$$

and

$$(1+x)_q^n := \begin{cases} \prod_{j=0}^{n-1}\left(1+q^j x\right), & n = 1,2,\ldots \\ 1, & n = 0. \end{cases}$$

For integers $0 \le k \le n$, the q-polynomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

The q-analogue of integration, discovered by Jackson [18] in the interval $[0,a]$, is defined by

$$\int_0^a f(x)\,d_qx := a(1-q)\sum_{n=0}^{\infty} f(aq^n)\,q^n,\ 0<q<1 \text{ and } a>0.$$

The q-improper integral used in the present paper is defined as

$$\int_0^{\infty/A} f(x)\,d_qx := (1-q)\sum_{n=0}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A},\ A>0,$$

provided the sum converges absolutely. The two q-Gamma functions are defined as

$$\Gamma_q(x) = \int_0^{1/1-q} t^{x-1}E_q(-qt)\,d_qt \text{ and } \gamma_q^A(x) = \int_0^{\infty/A(1-q)} t^{x-1}e_q(-t)\,d_qt.$$

There are two q-analogues of the exponential function e^x, see [19],

$$e_q(x) = \sum_{k=0}^{\infty}\frac{x^k}{[k]!} = \frac{1}{(1-(1-q)\,x)_q^{\infty}},\ |x| < \frac{1}{1-q}\quad |q|<1,$$

and

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{x^k}{[k]!} = (1+(1-q)\,x)_q^{\infty},\ |q|<1.$$

By Jackson [18], it was shown that the q-Beta function defined in the usual formula

$$B_q(t,s) = \frac{\Gamma_q(s)\,\Gamma_q(t)}{\Gamma_q(s+t)}$$

has the q-integral representation, which is a q-analogue of Euler's formula:

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1}\,d_qx,\ t,s>0.$$

2. Moments

In this section, we will calculate the moments of $D_{n,q}^{\alpha,\beta}\left(t^i,x\right)$ operators for $i = 0,1,2$. By the definition of q-Gamma function γ_q^A, we have

$$\int_0^{\infty/(1-q)} t^s s_{n,k}(q;t)\,d_qt = \frac{q^{k(k-1)/2}}{[n]^{s+1}[k]!}\frac{[k+s]!}{q^{(k+s)(k+s+1)/2}},\ s=0,1,2,...$$

Lemma 1. *We have*

$$\begin{aligned} D_{n,q}^{\alpha,\beta}(1,x) &= 1,\ D_{n,q}^{\alpha,\beta}(t,x)=\frac{[n]}{q^2([n]+\beta)}x+\left(\frac{1}{q}+\alpha\right)\frac{1}{[n]+\beta}, \\ D_{n,q}^{\alpha,\beta}\left(t^2,x\right) &= \frac{[n]^2}{q^6([n]+\beta)^2}x^2+\frac{[n]}{([n]+\beta)^2}\left(\frac{(1+q)^2}{q^5}+\frac{2\alpha}{q^2}\right)x \\ &\quad +\frac{1}{([n]+\beta)^2}\left(\frac{1+q}{q^3}+\frac{2\alpha}{q}+\alpha^2\right). \end{aligned}$$

Proof. We know moments of $D_{n,q}(f,x)$ from (1.4) and (1.5), see [22]. Using the these formulas, we get

$$D_{n,q}^{\alpha,\beta}(1,x)=D_{n,q}(1,x)=1,$$

and

$$\begin{aligned} D_{n,q}^{\alpha,\beta}(t,x) &= \frac{[n]}{[n]+\beta}D_{n,q}(t,x)+\frac{\alpha}{[n]+\beta}D_{n,q}(1,x) \\ &= \frac{[n]}{[n]+\beta}\left(\frac{1}{q^2}x+\frac{1}{q[n]}\right)+\frac{\alpha}{[n]+\beta} \\ &= \frac{[n]}{q^2([n]+\beta)}x+\left(\frac{1}{q}+\alpha\right)\frac{1}{[n]+\beta}. \end{aligned}$$

Finally, we have

$$\begin{aligned} D_{n,q}^{\alpha,\beta}\left(t^2,x\right) &= \left(\frac{[n]}{[n]+\beta}\right)^2 D_{n,q}\left(t^2,x\right)+\frac{2\alpha[n]}{([n]+\beta)^2}D_{n,q}(t,x)+\left(\frac{\alpha}{[n]+\beta}\right)^2 \\ &= \left(\frac{[n]}{[n]+\beta}\right)^2\left(\frac{1}{q^6}x^2+\frac{(1+q)^2}{q^5[n]}x+\frac{1+q}{q^3}\frac{1}{[n]^2}\right) \\ &\quad +\frac{2\alpha[n]}{([n]+\beta)^2}\left(\frac{1}{q^2}x+\frac{1}{q[n]}\right)+\left(\frac{\alpha}{[n]+\beta}\right)^2 \\ &= \frac{[n]^2}{q^6([n]+\beta)^2}x^2+\frac{[n]}{([n]+\beta)^2}\left(\frac{(1+q)^2}{q^5}+\frac{2\alpha}{q^2}\right)x \\ &\quad +\frac{1}{([n]+\beta)^2}\left(\frac{1+q}{q^3}+\frac{2\alpha}{q}+\alpha^2\right). \end{aligned}$$

□

Remark 1. *For $q\to 1^-$, $D_{n,q}^{\alpha,\beta}$ reduces to $S_{n,0}^{\alpha,\beta}$ operators which are given in* [13]. *Also, we have the central moments as*

$$\mu_{n,1}^{\alpha,\beta}(q,x):=D_{n,q}^{\alpha,\beta}(t-x,x)=\left(\frac{[n]}{q^2([n]+\beta)}-1\right)x+\left(\frac{1}{q}+\alpha\right)\frac{1}{[n]+\beta} \quad (2.1)$$

$$\begin{aligned}\mu_{n,2}^{\alpha,\beta}(q,x) \quad : \quad &= D_{n,q}^{\alpha,\beta}\left((t-x)^2, x\right) = \left(\frac{[n]^2}{q^6([n]+\beta)^2} - \frac{2[n]}{q^2([n]+\beta)} + 1\right)x^2 \\ &+ \left(\frac{[n]}{([n]+\beta)^2}\left(\frac{(1+q)^2}{q^5} + \frac{2\alpha}{q^2}\right) - \frac{2}{[n]+\beta}\left(\frac{1}{q}+\alpha\right)\right)x \\ &+ \frac{1}{([n]+\beta)^2}\left(\frac{1+q}{q^3} + \frac{2\alpha}{q} + \alpha^2\right). \qquad (2.2)\end{aligned}$$

3. Local Approximation

Let $C_B[0,\infty)$ be the set of all real-valued continuous bounded functions f on $[0,\infty)$, endowed with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. The Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf\{\|f-g\| + \delta\|g''\| : g \in C_B^2[0,\infty)\},$$

where $C_B^2[0,\infty) := \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. There exists a positive constant $C > 0$ such that

$$K_2(f,\delta) \le C\omega_2\left(f, \sqrt{\delta}\right), \qquad (3.1)$$

where $\delta > 0$ and the second order modulus of smoothness, for $f \in C_B[0,\infty)$, is defined as

$$\omega_2\left(f; \sqrt{\delta}\right) = \sup_{0<h\le\delta}\sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

We denote the usual modulus of continuity for $f \in C_B[0,\infty)$ as

$$\omega(f;\delta) = \sup_{0<h\le\delta}\sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

Now we state our next main result.

Lemma 2. *Let $f \in C_B[0,\infty)$. Then, for all $g \in C_B^2[0,\infty)$, we have*

$$\left|{}^*D_{n,q}^{\alpha,\beta}(g,x) - g(x)\right| \le \left(\mu_{n,2}^{\alpha,\beta}(q,x) + \left(\mu_{n,1}^{\alpha,\beta}(q,x)\right)^2\right)\|g''\|,$$

where

$${}^*D_{n,q}^{\alpha,\beta}(f,x) = D_{n,q}^{\alpha,\beta}(f,x) + f(x) - f\left(\frac{[n]x}{q^2([n]+\beta)} + \left(\frac{1}{q}+\alpha\right)\frac{1}{[n]+\beta}\right). \qquad (3.2)$$

Proof. From (3.2), we have

$${}^*D_{n,q}^{\alpha,\beta}(t-x,x) = D_{n,q}^{\alpha,\beta}(t-x,x) - \mu_{n,1}^{\alpha,\beta}(q,x) = 0. \qquad (3.3)$$

Using the Taylor's formula

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du,$$

we can write by (3.3) that

$$\begin{aligned} {}^{*}D_{n,q}^{\alpha,\beta}\left(g,x\right)-g\left(x\right) &= {}^{*}D_{n,q}^{\alpha,\beta}\left(t-x,x\right)g'\left(x\right)+{}^{*}D_{n,q}^{\alpha,\beta}\left(\int_{x}^{t}\left(t-u\right)g''\left(u\right)du,x\right) \\ &= D_{n,q}^{\alpha,\beta}\left(\int_{x}^{t}\left(t-u\right)g''\left(u\right)du,x\right) \\ &\quad -\int_{x}^{\mu_{n,1}^{\alpha,\beta}(q,x)+x}\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)+x-u\right)g''\left(u\right)du. \end{aligned}$$

On the other hand, since

$$\left|\int_{x}^{t}\left(t-u\right)g''\left(u\right)du\right|\leq\left|\int_{x}^{t}\left|t-u\right|\left|g''\left(u\right)\right|du\right|\leq\left(t-x\right)^{2}\left\|g''\right\|$$

and

$$\left|\int_{x}^{\mu_{n,1}^{\alpha,\beta}(q,x)+x}\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)+x-u\right)g''\left(u\right)du\right|\leq\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right)^{2}\left\|g''\right\|,$$

we conclude that

$$\left|{}^{*}D_{n,q}^{\alpha,\beta}\left(g,x\right)-g\left(x\right)\right|\leq\left(\mu_{n,2}^{\alpha,\beta}\left(q,x\right)+\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right)^{2}\right)\left\|g''\right\|.$$

Here we should say that $\mu_{n,1}^{\alpha,\beta}\left(q,x\right)$ and $\mu_{n,2}^{\alpha,\beta}\left(q,x\right)$ are given by (2.1) and (2.2), respectively. □

Theorem 1. *Let $f\in C_{B}[0,\infty)$. Then for every $x\in[0,\infty)$, there exists a constant $L>0$ such that*

$$\begin{aligned} \left|D_{n,q}^{\alpha,\beta}\left(f,x\right)-f\left(x\right)\right| &\leq L\omega_{2}\left(f,\sqrt{\mu_{n,2}^{\alpha,\beta}\left(q,x\right)+\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right)^{2}}\right) \\ &\quad +\omega\left(f,\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right). \end{aligned}$$

Proof. From (3.2), we can write that

$$\begin{aligned} \left|D_{n,q}^{\alpha,\beta}\left(f,x\right)-f\left(x\right)\right| &\leq \left|{}^{*}D_{n,q}^{\alpha,\beta}\left(f,x\right)-f\left(x\right)\right|+\left|f\left(x\right)-f\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)+x\right)\right| \\ &\leq \left|{}^{*}D_{n,q}^{\alpha,\beta}\left(f-g,x\right)-\left(f-g\right)\left(x\right)\right| \\ &\quad +\left|f\left(x\right)-f\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)+x\right)\right|+\left|{}^{*}D_{n,q}^{\alpha,\beta}\left(g,x\right)-g\left(x\right)\right|. \end{aligned}$$

Now, taking into account the boundedness of ${}^{*}D_{n,q}^{\alpha,\beta}$ and using Lemma 3, we get

$$\begin{aligned}\left|D_{n,q}^{\alpha,\beta}\left(f,x\right)-f\left(x\right)\right| &\leq 4\left\|f-g\right\|+\left|f\left(x\right)-f\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)+x\right)\right| \\ &\quad+\left(\mu_{n,2}^{\alpha,\beta}\left(q,x\right)+\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right)^{2}\right)\left\|g''\right\| \\ &\leq 4\left\|f-g\right\|+\left(\mu_{n,2}^{\alpha,\beta}\left(q,x\right)+\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right)^{2}\right)\left\|g''\right\| \\ &\quad+\omega\left(f,\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right).\end{aligned}$$

Now, taking infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and using (3.1), we get

$$\begin{aligned}\left|D_{n,q}^{\alpha,\beta}\left(f,x\right)-f\left(x\right)\right| &\leq 4K_2\left(f,\left(\mu_{n,2}^{\alpha,\beta}\left(q,x\right)+\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right)^{2}\right)\right) \\ &\quad+\omega\left(f,\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right) \\ &\leq L\omega_2\left(f,\sqrt{\mu_{n,2}^{\alpha,\beta}\left(q,x\right)+\left(\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right)^{2}}\right) \\ &\quad+\omega\left(f,\mu_{n,1}^{\alpha,\beta}\left(q,x\right)\right),\end{aligned}$$

where $L = 4C > 0$. □

Theorem 2. *Let $0 < \alpha \leq 1$ and E be any bounded subset of the interval $[0,\infty)$. Then, if $f \in C_B[0,\infty)$ is locally $Lip_M(\alpha)$, i.e. the condition*

$$\left|f\left(y\right)-f\left(x\right)\right| \leq M\left|y-x\right|^{\alpha},\ y\in E \text{ and } x\in[0,\infty),$$

holds, then, for each $x \in [0,\infty)$, we have

$$\left|D_{n,q}^{\alpha,\beta}\left(f,x\right)-f\left(x\right)\right| \leq M\left[\left(\mu_{n,2}^{\alpha,\beta}\left(q,x\right)\right)^{\alpha/2}+2d\left(x,E\right)^{\alpha}\right].$$

Here, M is a constant depending on α and f, and $d(x,E)$ is the distance between x and E defined as

$$d\left(x,E\right)=\inf\{|y-x| : y\in E\}.$$

Proof. Let $\bar{E}$ denotes the closure of E in $[0,\infty)$. Then, there exists a point $x_0 \in \bar{E}$ such that $|x-x_0| = d(x,E)$. Using the triangle inequality

$$\left|f\left(y\right)-f\left(x\right)\right| \leq \left|f\left(y\right)-f\left(x_0\right)\right|+\left|f\left(x\right)-f\left(x_0\right)\right|$$

we get, by the definition of $Lip_M(\alpha)$

$$\begin{aligned}\left|D_{n,q}^{\alpha,\beta}(f,x)-f(x)\right| &\leq D_{n,q}^{\alpha,\beta}(|f(y)-f(x)|,x)\\ &\leq D_{n,q}^{\alpha,\beta}(|f(y)-f(x_0)|,x)+D_{n,q}^{\alpha,\beta}(|f(x)-f(x_0)|,x)\\ &\leq M\{D_{n,q}^{\alpha,\beta}(|y-x_0|^\alpha,x)+|x-x_0|^\alpha\}\\ &\leq M\{D_{n,q}^{\alpha,\beta}(|y-x|^\alpha+|x-x_0|^\alpha,x)+|x-x_0|^\alpha\}\\ &= M\{D_{n,q}^{\alpha,\beta}(|y-x_0|^\alpha,x)+2|x-x_0|^\alpha\}.\end{aligned}$$

Using the Hölder inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, we find that

$$\begin{aligned}\left|D_{n,q}^{\alpha,\beta}(f,x)-f(x)\right| &\leq M\{\left(D_{n,q}^{\alpha,\beta}\left((y-x_0)^2,x\right)\right)^{\alpha/2}+2d(x,E)^\alpha\}\\ &= M\{\left(\mu_{n,2}^{\alpha,\beta}(q,x)\right)^{\alpha/2}+2d(x,E)^\alpha\}.\end{aligned}$$

Thus, we have the desired result. □

4. Voronovskaja Type Theorem

In this section we give Voronovskaja type result for $D_{n,q}^{\alpha,\beta}$ operators.

Lemma 3. *Let $q\in(0,1)$. We have*

$$\begin{aligned}D_{n,q}^{\alpha,\beta}\left(t^3,x\right) &= \left(\frac{[n]}{q^4([n]+\beta)}\right)^3x^3+\frac{[n]^2}{([n]+\beta)^3}\left(\frac{[3]^2}{q^{11}}+\frac{3\alpha}{q^6}\right)x^2\\ &\quad+\frac{[n]}{([n]+\beta)^3}\left(\frac{[2][3]^2}{q^9}+\frac{3\alpha[2]^2}{q^5}+\frac{3\alpha^2}{q^2}\right)x\\ &\quad+\frac{1}{([n]+\beta)^3}\left(\frac{[2][3]}{q^6}+\frac{3\alpha[2]}{q^3}+\frac{3\alpha^2}{q}+\alpha^3\right),\\ D_{n,q}^{\alpha,\beta}\left(t^4,x\right) &= \left(\frac{[n]}{q^5([n]+\beta)}\right)^4x^4+\frac{[n]^3}{([n]+\beta)^4}\left(\frac{[4]^2}{q^{19}}+\frac{4\alpha}{q^{12}}\right)x^3\\ &\quad+\frac{[n]^2}{([n]+\beta)^4}\left(\frac{[3]^2[4]^2}{q^{17}}+\frac{4\alpha[3]^2}{q^{11}}+\frac{6\alpha^2}{q^6}\right)x^2\\ &\quad+\frac{[n]}{([n]+\beta)^4}\left(\frac{[2][3][4]^2}{q^{14}}+\frac{4\alpha[2][3]^2}{q^9}+\frac{6\alpha^2[n]^2}{q^5}+\frac{4\alpha^3}{q^2}\right)x\\ &\quad+\frac{1}{([n]+\beta)^4}\left(\frac{[2][3][4]}{q^{10}}+\frac{4\alpha[2][3]}{q^6}+\frac{6\alpha^2[2]}{q^3}+\frac{4\alpha^3}{q}+\alpha^4\right),\end{aligned}$$

and

$$
\begin{aligned}
\mu_{n,4}^{\alpha,\beta}(q,x) &= D_{n,q}^{\alpha,\beta}\left((t-x)^4, x\right) \\
&= x^4\left\{\frac{[n]^4}{q^{20}([n]+\beta)^4} - \frac{4[n]^3}{q^{12}([n]+\beta)^3} + \frac{6[n]^2}{q^6([n]+\beta)^2}\right. \\
&\quad \left. - \frac{4[n]}{q^2([n]+\beta)} + 1\right\} \\
&\quad + x^3\left\{\frac{[n]^3}{([n]+\beta)^4}\left(\frac{[4]^2}{q^{19}} + \frac{4\alpha}{q^{12}}\right) - \frac{4[n]^2}{([n]+\beta)^3}\left(\frac{[3]^2}{q^{11}} + \frac{3\alpha}{q^6}\right)\right. \\
&\quad \left. + \frac{6[n]}{([n]+\beta)^2}\left(\frac{[2]^2}{q^5} + \frac{2\alpha}{q^2}\right) - \frac{4}{[n]+\beta}\left(\frac{1}{q} + \alpha\right)\right\} \\
&\quad + x^2\left\{\frac{[n]^2}{([n]+\beta)^4}\left(\frac{[3]^2[4]^2}{q^{17}} + \frac{4\alpha[3]^2}{q^{11}} + \frac{6\alpha^2}{q^6}\right)\right. \\
&\quad - \frac{4[n]}{([n]+\beta)^3}\left(\frac{[2][3]^2}{q^9} + \frac{3\alpha[2]^2}{q^5} + \frac{3\alpha^2}{q^2}\right) \\
&\quad \left. + \frac{6}{([n]+\beta)^2}\left(\frac{[2]}{q^3} + \frac{2\alpha}{q} + \alpha^2\right)\right\} \\
&\quad + x\left\{\frac{[n]}{([n]+\beta)^4}\left(\frac{[2][3][4]^2}{q^{14}} + \frac{4\alpha[2][3]^2}{q^9} + \frac{6\alpha^2[2]^2}{q^5} + \frac{4\alpha^3}{q^2}\right)\right. \\
&\quad \left. - \frac{4}{([n]+\beta)^3}\left(\frac{[2][3]}{q^6} + \frac{3\alpha[2]}{q^3} + \frac{3\alpha^2}{q} + \alpha^3\right)\right\} \\
&\quad + \frac{1}{([n]+\beta)^4}\left(\tfrac{[2][3][4]}{q^{10}} + \tfrac{4\alpha[2][3]}{q^6} + \tfrac{6\alpha^2[2]}{q^3} + \tfrac{4\alpha^3}{q} + \alpha^4\right).
\end{aligned}
$$

Theorem 3. *Let f be bounded and integrable on the interval , second derivative of f exists at a fixed point $x \in [0,\infty)$ and $q_n \in (0,1)$ such that $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. Then, the following equality holds*

$$
\begin{aligned}
\lim_{n\to\infty}[n]_{q_n}\left(D_{n,q_n}^{\alpha,\beta}(f,x) - f(x)\right) &= ((2-2a-\beta)x + 1 + \alpha) f'(x) \\
&\quad + \left((1-a)x^2 + x\right) f''(x).
\end{aligned}
$$

Proof. By the Taylor's formula we can write

$$
f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{4.1}
$$

where $r(t,x)$ is the Peano form of the remainder, $r(.,x)$ is bounded and $\lim_{t\to x} r(t,x) = 0$. By applying the operator $D_{n,q}^{\alpha,\beta}$ of (4.1) relation, we obtain

$$\begin{aligned}[n]_{q_n}\left(D_{n,q_n}^{\alpha,\beta}(f,x) - f(x)\right) &= f'(x)\,[n]_{q_n}\left(D_{n,q_n}^{\alpha,\beta}(t-x,x)\right) \\ &\quad + \frac{f''(x)}{2}[n]_{q_n}\left(D_{n,q_n}^{\alpha,\beta}\left((t-x)^2,x\right)\right) \\ &\quad + [n]_{q_n}\left(D_{n,q_n}^{\alpha,\beta}\left(r(t,x)(t-x)^2,x\right)\right).\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$[n]_{q_n}\left(D_{n,q_n}^{\alpha,\beta}\left(r(t,x)(t-x)^2,x\right)\right) \le \sqrt{D_{n,q_n}^{\alpha,\beta}\left(r^2(t,x),x\right)}\sqrt{[n]_{q_n}^2 D_{n,q_n}^{\alpha,\beta}\left((t-x)^4,x\right)}.$$

Observe that $r^2(x,x) = 0$ and $r^2(.,x)$ is bounded. The sequence $\{D_{n,q_n}^{\alpha,\beta}\}$ converges to f uniformly on $[0,A] \subset [0,\infty)$, for each f which is bounded, integrable and has second derivative existing at a fixed point $x \in [0,\infty)$, $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} q_n^n = a$. Then, it follows that

$$\lim_{n\to\infty} D_{n,q_n}^{\alpha,\beta}\left(r^2(t,x),x\right) = r^2(x,x) = 0$$

uniformly with respect to $x \in [0,A]$. So, we get

$$\lim_{n\to\infty}[n]_{q_n}\left(D_{n,q_n}^{\alpha,\beta}\left(r(t,x)(t-x)^2,x\right)\right) = 0.$$

Using Remark 1, we have the following equality as

$$\begin{aligned}\lim_{n\to\infty}[n]_{q_n}\left(D_{n,q_n}^{\alpha,\beta}(f,x) - f(x)\right) &= \left((2-2a-\beta)x + 1 + \alpha\right)f'(x) \\ &\quad + \left((1-a)x^2 + x\right)f''(x).\end{aligned}$$

□

5. Weighted Approximation

Now we give the weighted approximation theorem. Let us give some definitions to be considered here. Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \le M_f\left(1+x^2\right)$, where M_f is a constant depending only on f. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$. Also, let $C_{x^2}^*[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C_{x^2}^*[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$.

Theorem 4. *Let $q = q_n$ satisfies $q_n \in (0,1)$ and let $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} q_n^n = a$. Then, for each $f \in C_{x^2}^*[0,\infty)$, we have*

$$\lim_{n\to\infty}\left\|D_{n,q_n}^{\alpha,\beta}(f,x) - f(x)\right\|_{x^2} = 0.$$

Proof. Using the Theorem presented in [8] we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \left\| D_{n,q_n}^{\alpha,\beta}\left(t^{\nu},x\right)-x^{\nu}\right\|_{x^2}=0,\ \nu=0,1,2.$$

Since $D_{n,q}^{\alpha,\beta}(1,x)=1$, it is sufficient to show that $\lim_{n\to\infty}\left\| D_{n,q_n}^{\alpha,\beta}\left(t^{\nu},x\right)-x^{\nu}\right\|_{x^2}=0$, $\nu=1,2$.

We can write from Remark 1,

$$\begin{aligned}
\left\| D_{n,q_n}^{\alpha,\beta}(t,x)-x\right\|_{x^2} &= \sup_{x\in[0,\infty)}\left|\frac{\left([n]_{q_n}-q_n^2\left([n]_{q_n}+\beta\right)\right)x+q_n+q_n^2\alpha}{q_n^2\left([n]_{q_n}+\beta\right)}\right|\frac{1}{1+x^2}\\
&\le \left(1-\frac{[n]_{q_n}}{q_n^2\left([n]_{q_n}+\beta\right)}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\
&\quad+\frac{q_n+q_n^2\alpha}{q_n^2\left([n]_{q_n}+\beta\right)}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}\\
&= \frac{1}{2}\left(1-\frac{[n]_{q_n}}{q_n^2\left([n]_{q_n}+\beta\right)}\right)+\frac{q_n+q_n^2\alpha}{q_n^2\left([n]_{q_n}+\beta\right)},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\left\| D_{n,q_n}^{\alpha,\beta}(t,x)-x\right\|_{x^2}=0.$$

Finally

$$\begin{aligned}
\left\| D_{n,q_n}^{\alpha,\beta}\left(t^2,x\right)-x^2\right\|_{x^2} &\le \left|\frac{[n]^2}{q^6\left([n]+\beta\right)^2}-1\right|\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\\
&\quad+\frac{[n]}{\left([n]+\beta\right)^2}\left(\frac{(1+q)^2}{q^5}+\frac{2\alpha}{q^2}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\
&\quad+\frac{1}{\left([n]+\beta\right)^2}\left(\frac{1+q}{q^3}+\frac{2\alpha}{q}+\alpha^2\right)\sup_{x\in[0,\infty)}\frac{1}{1+x^2},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\left\| D_{n,q_n}^{\alpha,\beta}\left(t^2,x\right)-x^2\right\|_{x^2}=0.$$

Thus the proof is completed. □

6. Statistical Convergence

A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be statistically convergent to a number L, denoted by $st-\lim_n x_n=L$ if, for every $\varepsilon>0$,

$$\delta\{n\in\mathbb{N}:|x_n-L|\ge\varepsilon\}=0,$$

where

$$\delta\left(K\right)=\lim_{n}\frac{1}{n}\sum_{j=1}^{n}\chi_{K}\left(j\right)$$

is the natural density of set $K\subseteq\mathbb{N}$ and χ_K is the characteristic function of K. For instant

$$x_n=\begin{cases}\log n & n\in\{10^k,\ k\in\mathbb{N}\}\\ 1 & \text{otherwise}\end{cases},$$

series $\left(x_n\right)_{n\in\mathbb{N}}$ converges statistically, but $\lim\limits_{n}x_n$ does not exist. We note that convergence of a sequence implies statistical convergence, but converse need not be true (details can be found in [1, 6, 9, 25, 26]).

A useful Korovkin type theorem for statistical convergence on continuous function space has been proved by Gadjiev and Orhan [9].

Since useful Korovkin theorem doesn't work on infinitive intervals, a weighted Korovkin type theorem is given by Gadjiev [8] in order to obtain approximation properties on infinite intervals.

Agratini and Doğru obtained the weighted statistical approximation by q-Szász type operators in [1]. There are many weighted statistical convergence works for q-Szász-Mirakyan operators (for instance see [25, 26]). The main purpose of this part is to obtain weighted statistical approximation properties of the operators defined in (1.6).

Theorem 5. *Let* $\left(q_n\right)_{n\in\mathbb{N}}$ *be a sequence satisfying*

$$st-\lim_{n}q_n=1 \text{ and } st-\lim_{n}q_n^n=a\ \ (a<1) \tag{6.1}$$

then for each function $C[0,\infty)$, theoperator $D_{n,q_n}^{\alpha,\beta}f$ weighted statistically converges to f, that is

$$st-\lim_{n}\left\|D_{n,q_n}^{\alpha,\beta}f-f\right\|_{x^2}=0.$$

Proof. It is clear that

$$st-\lim_{n}\left\|D_{n,q_n}^{\alpha,\beta}\left(1,x\right)-1\right\|_{x^2}=0. \tag{6.2}$$

Based on Lemma 1, we have

$$\begin{aligned}\sup_{x\in[0,\infty)}\left|D_{n,q_n}^{\alpha,\beta}\left(t,x\right)-x\right| &= \sup_{x\in[0,\infty)}\left|\begin{array}{c}\left(\frac{[n]_{q_n}}{q_n^2\left([n]_{q_n}+\beta\right)}-1\right)x\\ +\left(\frac{1}{q_n}+\alpha\right)\frac{1}{[n]_{q_n}+\beta}\end{array}\right|\frac{1}{1+x^2}\\ &\leq \left(\frac{[n]_{q_n}}{q_n^2\left([n]_{q_n}+\beta\right)}-1\right)\frac{1}{2}+\left(\frac{1}{q_n}+\alpha\right)\frac{1}{[n]_{q_n}+\beta}.\end{aligned}$$

Using the conditions (6.1), we get

$$st-\lim_n\left(\frac{[n]_{q_n}}{q_n^2\left([n]_{q_n}+\beta\right)}-1\right)=0 \text{ and } st-\lim_n\left(\frac{1}{q_n}+\alpha\right)\frac{1}{[n]_{q_n}+\beta}=0.$$

For each $\varepsilon>0$, we define the following sets:

$$\begin{aligned}
D &:= \left\{n\in\mathbb{N}:\left\|D_{n,q_n}^{\alpha,\beta}(t,x)-x\right\|_{x^2}\geq\varepsilon\right\},\\
D_1 &:= \left\{n\in\mathbb{N}:\frac{1}{2}\left(\frac{[n]_{q_n}}{q_n^2\left([n]_{q_n}+\beta\right)}-1\right)\geq\frac{\varepsilon}{2}\right\},\\
D_2 &:= \left\{n\in\mathbb{N}:\left(\frac{1}{q_n}+\alpha\right)\frac{1}{[n]_{q_n}+\beta}\geq\frac{\varepsilon}{2}\right\}.
\end{aligned}$$

Thus, we obtain $D\subseteq D_1\cup D_2$, i.e., $\delta(D)\leq\delta(D_1)+\delta(D_2)=0$. Therefore,

$$st-\lim_n\left\|D_{n,q_n}^{\alpha,\beta}(t,x)-x\right\|_{x^2}=0. \tag{6.3}$$

A similar calculation reveals that

$$\begin{aligned}
\sup_{x\in[0,\infty)}\left|D_{n,q_n}^{\alpha,\beta}\left(t^2,x\right)-x^2\right| &= \sup_{x\in[0,\infty)}\Big|\Big(\frac{[n]_{q_n}^2}{q_n^6\left([n]_{q_n}+\beta\right)^2}-1\Big)x^2\\
&\quad+\frac{[n]_{q_n}}{\left([n]_{q_n}+\beta\right)^2}\left(\frac{(1+q_n)^2}{q_n^5}+\frac{2\alpha}{q_n^2}\right)x\\
&\quad+\frac{1}{\left([n]_{q_n}+\beta\right)^2}\left(\frac{1+q_n}{q_n^3}+\frac{2\alpha}{q_n}+\alpha^2\right)\Big|\frac{1}{1+x^2}\\
&= \left(\frac{[n]_{q_n}^2}{q_n^6\left([n]_{q_n}+\beta\right)^2}-1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\\
&\quad+\frac{[n]_{q_n}}{\left([n]_{q_n}+\beta\right)^2}\left(\frac{(1+q_n)^2}{q_n^5}+\frac{2\alpha}{q_n^2}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\
&\quad+\frac{1}{\left([n]_{q_n}+\beta\right)^2}\left(\frac{1+q_n}{q_n^3}+\frac{2\alpha}{q_n}+\alpha^2\right)\sup_{x\in[0,\infty)}\frac{1}{1+x^2}.
\end{aligned}$$

Using the conditions (6.1), we get

$$\begin{aligned} st-\lim_n\left(\frac{[n]_{q_n}^2}{q_n^6\left([n]_{q_n}+\beta\right)^2}-1\right) &= 0, \\ st-\lim_n\frac{[n]_{q_n}}{\left([n]_{q_n}+\beta\right)^2}\left(\frac{(1+q_n)^2}{q_n^5}+\frac{2\alpha}{q_n^2}\right) &= 0, \end{aligned}$$

and

$$st-\lim_n\frac{1}{\left([n]_{q_n}+\beta\right)^2}\left(\frac{1+q_n}{q_n^3}+\frac{2\alpha}{q_n}+\alpha^2\right)=0.$$

For each $\varepsilon>0$, we define the following sets:

$$\begin{aligned} B \quad &: \quad =\left\{n\in\mathbb{N}:\left\|D_{n,q_n}^{\alpha,\beta}\left(t^2,x\right)-x^2\right\|_{x^2}\geq\varepsilon\right\}, \\ B_1 \quad &: \quad =\left\{n\in\mathbb{N}:\frac{[n]_{q_n}^2}{q_n^6\left([n]_{q_n}+\beta\right)^2}-1\geq\frac{\varepsilon}{3}\right\}, \\ B_2 \quad &: \quad =\left\{n\in\mathbb{N}:\frac{[n]_{q_n}}{2\left([n]_{q_n}+\beta\right)^2}\left(\frac{(1+q_n)^2}{q_n^5}+\frac{2\alpha}{q_n^2}\right)\geq\frac{\varepsilon}{3}\right\}, \\ B_3 \quad &: \quad =\left\{n\in\mathbb{N}:\frac{1}{\left([n]_{q_n}+\beta\right)^2}\left(\frac{1+q_n}{q_n^3}+\frac{2\alpha}{q_n}+\alpha^2\right)\geq\frac{\varepsilon}{3}\right\}. \end{aligned}$$

Thus, we obtain $B\subseteq B_1\cup B_2\cup B_3$, i.e., $\delta(B)\leq\delta(B_1)+\delta(B_2)+\delta(B_3)=0$. Therefore,

$$st-\lim_n\left\|D_{n,q_n}^{\alpha,\beta}\left(t^2,x\right)-x^2\right\|_{x^2}=0. \tag{6.4}$$

Thus, by using equations (6.2), (6.3) and (6.4), we get the result. □

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