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ON SEVEN DIMENSIONAL 3-SASAKIAN MANIFOLDS

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Abstract. 3-Sasakian manifolds in dimension seven have cocalibrated and nearly parallel G_2-structures. In this work, cocalibrated G_2-structure is deformed by one of the characteristic vector fields of the 3-Sasakian structure and a new G_2 structure is obtained whose metric has negative scalar curvature. In addition, the new G_2 structure has a nonzero Killing vector field. Then, by using this deformation, new covariant derivative on the spinor bundle is obtained and the new Dirac operator is written in terms of the Dirac operator before deformation.

1. INTRODUCTION

There exist several deformations of G_2 structures to obtain new G_2 structures. Some of them are conformal deformations which are extensively studied in [1, 2]. Other types of deformations use vector fields to get new G_2 structures [2]. 3-Sasakian manifolds are Einstein spaces of positive scalar curvature which have three compatible orthogonal Sasakian structures [3, 4]. Relations between the spectral properties of Dirac operator and seven dimensional 3-Sasakian manifolds are investigated by [5, 6, 7]. It is shown that seven dimensional 3-Sasakian manifolds have a coclosed and nearly parallel G_2-structures in [8]. In this work, to obtain a new G_2-structure from a fixed G_2-structure, one of the characteristic vector fields of the 3-Sasakian structure is used for changing the fundamental 3-form.

2. PRELIMINARIES

Let us consider $\mathbb{R}^7$ with the standard basis $\{e_1, ..., e_7\}$ and dual basis $\{e^1, ..., e^7\}$. The fundamental 3-form on $\mathbb{R}^7$ is defined as

$$\omega = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{256},$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$. The group G_2 is

$$G_2 = \{g \in GL(\mathbb{R}^7) | g^* \omega = \omega\}.$$

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Also the group G_2 is the automorphism group of octonions. The group G_2 is a compact, connected, simply connected and simple Lie subgroup of $SO(7)$ of dimension 14. A G_2-structure on a 7-dimensional manifold M is a reduction of the structure group of the frame bundle of M from $SO(7)$ to G_2. Let M be a Riemannian manifold with structure group G_2. The classification of such manifolds are done by Fernández and Gray by decomposing $\nabla\varphi$ into G_2-irreducible components. It turned out that there are 16 such classes [1].

Sasakian and 3-Sasakian manifolds are Riemannian manifolds with some additional conditions. These manifolds are studied by [3, 4]. If a $(2n+1)$-dimensional Riemannian manifold is equipped with a 1-form η, its dual vector field ξ and an endomorphism $\varphi : TM^{2n+1} \to TM^{2n+1}$ such that the conditions

$$\eta \wedge (d\eta)^3 \neq 0, \quad \eta(\xi) = 1, \quad \varphi^2 = -Id + \eta \otimes \xi,$$

$$g(\varphi(X), \varphi(Y)) = g(X,Y) - \eta(X) \cdot \eta(Y), \quad \nabla_X \xi = -\varphi(X),$$

$$(\nabla_X \varphi)(Y) = g(X,Y) \cdot \xi - \eta(Y) \cdot X$$

are satisfied, this manifold called a Sasakian manifold. In addition, if a $(4n+3)$-dimensional Riemannian manifold (M^{4n+3}, g) is equipped with three Sasakian structures $(\xi_i, \eta_i, \varphi_i)$, $i = 1, 2, 3$, such that

$$[\xi_1, \xi_2] = 2\xi_3, \quad [\xi_1, \xi_2] = 2\xi_3, \quad [\xi_1, \xi_2] = 2\xi_3$$

and

$$\begin{aligned}
\varphi_3 \circ \varphi_2 &= -\varphi_1 + \eta_2 \otimes \xi_3, \quad & \varphi_2 \circ \varphi_3 &= \varphi_1 + \eta_3 \otimes \xi_2, \\
\varphi_1 \circ \varphi_3 &= -\varphi_2 + \eta_3 \otimes \xi_1, \quad & \varphi_3 \circ \varphi_1 &= \varphi_2 + \eta_1 \otimes \xi_3, \\
\varphi_2 \circ \varphi_1 &= -\varphi_3 + \eta_1 \otimes \xi_2, \quad & \varphi_1 \circ \varphi_2 &= \varphi_3 + \eta_2 \otimes \xi_1,
\end{aligned}$$

then this manifold is called a 3-Sasakian manifold. The subbundle which is spanned by ξ_1, ξ_2 and ξ_3 is called vertical subbundle and orthogonal complement of the vertical subbundle is called horizontal subbundle. Both subbundles are invariant under the endomorphisms φ_1, φ_2 and φ_3. Sasakian manifolds are not necessarily Einstein. But 3-Sasakian manifolds are Einstein with Einstein constant $2(2n+1)$.

Let M be a seven dimensional 3-Sasakian manifold. Then, this manifold is spin [3]. Hence we construct a real spinor bundle on M which is an associated vector bundle

$$S := P_{Spin(7)} \times_\kappa \Delta_7,$$

where $\Delta_7 \cong \mathbb{R}^8$ and $\kappa : Spin(7) \to \mathrm{End}(\Delta_7)$ is given by restriction of the real Clifford algebra Cl_7-representation to $Spin(7) \subset Cl_7$. Then, the covariant derivative ∇^S on the spinor bundle is expressed locally as

$$\nabla^S_V \sigma = d\sigma(V) + \frac{1}{4} \sum_{i,j} g(\nabla_V e_i, e_j) \kappa(e_i e_j) \sigma$$

where σ is a local spinor section and V is a vector field. The first-order differential operator $D : \Gamma(S) \to \Gamma(S)$ is defined as

$$D\sigma := \sum_{i=1}^{7} \kappa(e_i)\nabla^S_{e_i}\sigma$$

and is called the Dirac operator of S, where $\Gamma(S)$ is the set of spinor sections [9].

3. VECTORIAL TYPE DEFORMATION OF RIEMANNIAN MANIFOLDS WITH STRUCTURE GROUP G_2

Some of the deformations of a fixed G_2 structure are conformal deformations, deformations of a G_2 structure by a vector field and infinitesimal deformations [1, 2]. Conformal deformations are studied by Fernández and Gray. How G_2 structures change after conformaly changing the metric is investigated in [1]. In [10] and [11] the relation between Dirac operators on associated spinor bundles is studied.

Vectorial type of deformations are studied by Karigiannis in [2]. He specially worked on deforming the fundamental 3-form by a vector field and obtained a new metric from the 3-form: Let (M, ω, g) be a 7-dimensional Riemannian manifold with structure group G_2. If ω is deformed by a vector field ξ the new 3-form

$$\widetilde{\omega} = \omega + \xi \lrcorner * \omega$$

is always positive-definite. Under this deformation, Karigiannis has shown that, for all vector fields X, Y the new metric is

$$\widetilde{g}(X,Y) = \frac{1}{(1+g(\xi,\xi))^{\frac{2}{3}}}\left(g(X,Y) + g(X \times \xi, Y \times \xi)\right),$$

where $\times$ is the cross product associated to the first G_2-structure ω. He has also written the new Hodge star $\widetilde{*}$ in terms of the old ω, the old $*$ and the vector field ξ corresponding to $\widetilde{\omega}$ explicitly:

$$\widetilde{*}\alpha = (1+g(\xi,\xi))^{\frac{2-k}{3}}(*\alpha + (-1)^{k-1}\xi \lrcorner (*(\xi \lrcorner \alpha)))$$

where α is a k-form [2].

In this study, we consider this type of deformations. For a fixed vector field ξ, first we observe the map

$$\begin{array}{rcl} C_\xi : \Gamma(TM) & \longrightarrow & \Gamma(TM) \\ u & \longmapsto & C_\xi(X) = (1+g(\xi,\xi))^{-1/3}(X + X \times \xi). \end{array}$$

The map C_ξ is one-to-one and C^∞-linear which has the inverse

$$C_\xi^{-1}(X) = (1+g(\xi,\xi))^{-2/3}(X - X \times \xi + g(X,\xi)\xi)$$

for a vector field X, see [12]. The new metric $\widetilde{g}$ can also be written as

$$\widetilde{g}(X,Y) = g(C_\xi(X), C_\xi(Y))$$

for any vector field X, Y. The new cross product of the new G_2-structure $\widetilde{\omega}$ is found in terms of old cross product $\times$ as

$$X \widetilde{\times} Y = k^{1/3} C_{\xi}^{-1}(X \times Y) + k^{-1/3}\left(g(X,\xi)Y - g(Y,\xi)X\right),$$

where $k = 1 + g(\xi,\xi)$ and X, Y are any vector fields. If the vector field ξ is Killing, then the new covariant derivative $\widetilde{\nabla}$ of the metric $\widetilde{g}$ determined by $\widetilde{\omega}$ is obtained as

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\left\{g(\xi, Y)\nabla_X \xi + g(\xi, X)\nabla_y \xi\right\}.$$

Note that if ξ is a parallel vector field on M, then $\widetilde{\nabla} = \nabla$ is obtained.

Deformation by a vector field of the canonical G_2-structure on seven dimensional 3-Sasakian manifolds is studied by [12]. The canonical G_2-structure is deformed by one of the charecteristic vector fields of the 3-Sasakian structure. It is shown that the new G_2-structure $\widetilde{\omega}$ is in the largest class of G_2 structures.

Similarly, deformations by a vector field of the nearly parallel G_2-structure on seven dimensional 3-Sasakian manifolds is studied by [13]. When the nearly parallel G_2-structure is deformed by a charecteristic vector field of the 3-Sasakian structure, it is shown that the new G_2-structure $\widetilde{\omega}$ is in the class $\mathcal{W}_1 \oplus \mathcal{W}_3$ or $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$.

Let (M^7, g) be a seven dimensional, compact, simply-connected 3-Sasakian manifold with 3-Sasakian structure $(\xi_i, \eta_i, \varphi_i)$ for $= 1, 2, 3$. One can get a local orthonormal frame $\{e_1, \cdots, e_7\}$ such that $e_1 = \xi_1$, $e_2 = \xi_2$, $e_3 = \xi_3$ and the endomorphisms φ_i act on $T^h := \text{span}\{e_4, e_5, e_6, e_7\}$ by matrices:

$$\varphi_1 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \varphi_2 := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \varphi_3 := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\left\{e^1, \cdots, e^7\right\}$ be the co-frame corresponding to $\{e_1, \cdots, e_7\}$. The differentials of 1-forms e^1, e^2 and e^3 is given in [8] according to this orthonormal frame.

The following 3-form on the 7-dimensional 3-Sasakian manifold

$$\omega := 3e^1 \wedge e^2 \wedge e^3 + \frac{1}{2}\left(e^1 \wedge de^1 + e^2 \wedge de^2 + e^3 \wedge de^3\right),$$

is a G_2-structure which is coclosed and this 3-form is called the canonical G_2 structure of the 7-dimensional 3-Sasakian manifold [8]. This G_2-structure can be written locally as

$$\omega = e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347} - e^{356},$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$. The vector cross products of basis elements are:

$$\begin{array}{ccc} e_1 \times e_2 = e_3, & e_1 \times e_3 = -e_2, & e_2 \times e_3 = e_1, \\ e_1 \times e_4 = -e_5, & e_1 \times e_5 = e_4, & e_4 \times e_5 = -e_1, \\ e_1 \times e_6 = -e_7, & e_1 \times e_7 = e_6, & e_6 \times e_7 = -e_1, \\ e_2 \times e_4 = -e_6, & e_2 \times e_6 = e_4, & e_4 \times e_6 = -e_2, \\ e_2 \times e_5 = e_7, & e_2 \times e_7 = -e_5, & e_5 \times e_7 = e_2, \\ e_3 \times e_4 = -e_7, & e_3 \times e_7 = e_4, & e_7 \times e_4 = e_3, \\ e_3 \times e_5 = -e_6, & e_3 \times e_6 = e_5, & e_6 \times e_5 = e_3. \end{array}$$

We consider the new G_2-structure

$$\begin{aligned} \widetilde{\omega} &= \omega + \xi_1 \lrcorner * \omega \\ &= \frac{1}{2}\left(e^1 \wedge de^1 + e^2 \wedge de^2 + e^3 \wedge de^3\right) + 4e^{123} - \frac{1}{2}e^3 \wedge de^2 + \frac{1}{2}e^2 \wedge de^3, \end{aligned}$$

in local coordinates, $\widetilde{\omega}$ can be obtained as

$$\widetilde{\omega} = e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347} - e^{356} - e^{357} + e^{346} - e^{256} - e^{247}.$$

Note that $\left\{C_{e_1}^{-1}(e_1), C_{e_1}^{-1}(e_2), \cdots, C_{e_1}^{-1}(e_7)\right\}$ is a local orthonormal frame with respect to the new metric $\tilde{g}$. From now on, the notation $\widetilde{e}_i := C_{e_1}^{-1}(e_i)$ will be used. In this frame, one can compute [8]:

$$i = 1, 2, 3; j = 4, 5, 6, 7, [e_i, e_j] = 0,$$

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = 2e_1,$$

$$[e_4, e_5] = 2e_1, \quad [e_4, e_6] = 2e_2, \quad [e_4, e_7] = 2e_3,$$

$$[e_5, e_6] = 2e_3, \quad [e_5, e_7] = -2e_2, \quad [e_6, e_7] = 2e_1.$$

From the Kozsul formula; one can obtain

$$\nabla_{e_1} e_2 = e_3,\ \nabla_{e_1} e_3 = -e_2,\ \nabla_{e_1} e_4 = -e_5,\ \nabla_{e_1} e_5 = e_4,\ \nabla_{e_1} e_6 = -e_7,$$

$$\nabla_{e_1} e_7 = e_6,\ \nabla_{e_2} e_3 = e_1,\ \nabla_{e_2} e_4 = -e_6,\ \nabla_{e_2} e_5 = e_7,\ \nabla_{e_2} e_6 = e_4,$$

$$\nabla_{e_2} e_7 = -e_5,\ \nabla_{e_3} e_4 = -e_7\ \nabla_{e_3} e_5 = -e_6, \quad \nabla_{e_3} e_6 = e_5, \quad \nabla_{e_3} e_7 = e_4.$$

Now, coclosed (canonical) G_2 structure on a 7-dimensional 3-Sasakian manifold will be deformed by one of the characteristic vector fields of the 3-Sasakian structure and then scalar curvature of the new metric is evaluated:

Theorem 3.1. *There exists a Riemannian manifold with structure group G_2 which in the widest class of G_2 structures having a nonzero Killing vector field whose scalar curvature is $-3 \cdot 2^{1/3}$.*

Proof. Let M be a seven dimensional 3-Sasakian manifold. We know that this manifold is equipped with a coclosed G_2 structure. If we deform this G_2 structure by the characteristic vector field ξ_1, one can obtain a new G_2 structure which in the widest class [12]. First we show that $\widetilde{\xi}_1$ is a Killing vector field with respect to the new metric $\widetilde{g}$. Note that, since the deformation is done by the Killing vector field ξ_1 and $\nabla_{\xi_1}\xi_1 = 0$,

$$\widetilde{\nabla}_X\widetilde{\xi}_1 = 2^{-2/3}\nabla_X\xi_1,$$

is obtained for any vector field X. By the identity $g(\nabla_X\xi_1, \xi_1) = 0$,

$$\begin{aligned} \widetilde{g}\left(\widetilde{\nabla}_X\widetilde{\xi}_1, Y\right) &= 2^{-2/3}\widetilde{g}\left(\nabla_X\xi_1, Y\right) \\ &= 2^{-2/3}g\left(C_{\xi_1}\left(\nabla_X\xi_1\right), C_{\xi_1}(Y)\right) \\ &= 2^{-1/3}g\left(\nabla_X\xi_1, Y\right). \end{aligned}$$

Since ξ_1 is a Killing vector field,

$$\widetilde{g}\left(\widetilde{\nabla}_X\widetilde{\xi}_1, Y\right) = 2^{-2/3}g\left(\nabla_X\xi_1, Y\right) = -2^{-2/3}g\left(\nabla_Y\xi_1, X\right) = -\widetilde{g}\left(\widetilde{\nabla}_Y\widetilde{\xi}_1, X\right).$$

Next to obtain the scalar curvature; we directly calculate:

$$\widetilde{R}(\widetilde{e}_1, \widetilde{e}_1) = 3 \cdot 2^{-1/3}, \quad \widetilde{R}(\widetilde{e}_2, \widetilde{e}_2) = 7 \cdot 2^{-1/3}, \quad \widetilde{R}(\widetilde{e}_3, \widetilde{e}_3) = 7 \cdot 2^{-1/3},$$

$$\widetilde{R}(\widetilde{e}_4, \widetilde{e}_4) = -5 \cdot 2^{-1/3}, \quad \widetilde{R}(\widetilde{e}_5, \widetilde{e}_5) = -5 \cdot 2^{-1/3}, \quad \widetilde{R}(\widetilde{e}_6, \widetilde{e}_6) = -5 \cdot 2^{-1/3},$$

and

$$\widetilde{R}(\widetilde{e}_7, \widetilde{e}_7) = -5 \cdot 2^{-1/3}.$$

Hence the scalar curvature is obtained as

$$\widetilde{S} = \sum_{i=1}^{7}\widetilde{R}(\widetilde{e}_i, \widetilde{e}_i) = -3 \cdot 2^{-1/3}.$$

□

On the other hand, $\widetilde{\xi}_2$ and $\widetilde{\xi}_3$ are not Killing vector fields with respect to the new metric $\widetilde{g}$.

4. DIRAC OPERATOR ON SEVEN DIMENSIONAL 3-SASAKIAN MANIFOLDS

Let (M^7, g) be a seven dimensional, compact, simply-connected 3-Sasakian manifold with 3-Sasakian structure $(\xi_i, \eta_i, \varphi_i)$ for $= 1, 2, 3$. The structure group of the frame bundle reduces to the subgroup $SU(2) \subset SO(7)$. M^7 is also a spin manifold. Then we can construct spinor bundle and write the covariant derivative and the Dirac operator of this spinor bundle. In addition, one can get a local orthonormal frame $\{e_1, \cdots, e_7\}$ as expressed before.

In local coordinates, for a vector field V, the Levi-Civita covariant derivative of the metric $\tilde{g}$ can be expressed as

$$\begin{aligned}\widetilde{\nabla}_V\widetilde{e}_i &= \nabla_V\widetilde{e}_i - \frac{1}{2}\{g(V,e_1)\nabla_{\widetilde{e}_i}e_1 + g(\widetilde{e}_i,e_1)\nabla_V e_1\}\\ &= 2^{-2/3}\left(\nabla_V e_i - \nabla_V(e_i\times e_1)\right) - 2^{-5/3}g(V,e_1)\left(\nabla_{e_i}e_1 - \nabla_{e_i\times e_1}e_1\right).\end{aligned}$$

Since

$$\begin{aligned}C_{e_1}\left(\widetilde{\nabla}_V\widetilde{e}_i\right) &= \tfrac{1}{2}\left(\nabla_V e_i + (\nabla_V e_i)\times e_1\right) - \tfrac{1}{2}\left(\nabla_V(e_i\times e_1) + (\nabla_V(e_i\times e_1))\times e_1\right)\\ &\quad -\tfrac{1}{4}g(V,e_1)\left[\nabla_{e_i}e_1 + (\nabla_{e_i}e_1)\times e_1\right]\\ &\quad +\tfrac{1}{4}g(V,e_1)\left[\nabla_{e_i\times e_1}e_1 + (\nabla_{e_i\times e_1}e_1)\times e_1\right],\end{aligned}$$

$\widetilde{w}_{ij}(V) = \widetilde{g}(\widetilde{\nabla}_V\widetilde{e}_i,\widetilde{e}_j)$ can be written as:

$$\begin{aligned}\widetilde{w}_{ij}(V) &= \tfrac{1}{2}g(\nabla_V e_i,e_j) + \tfrac{1}{2}g((\nabla_V e_i)\times e_1,e_j)\\ &\quad -\tfrac{1}{2}g(\nabla_V(e_i\times e_1),e_j) - \tfrac{1}{2}g\left((\nabla_V(e_i\times e_1))\times e_1,e_j\right)\\ &\quad -\tfrac{1}{4}g(V,e_1)g\left(\nabla_{e_i}e_1,e_j\right) - \tfrac{1}{4}g(V,e_1)g\left((\nabla_{e_i}e_1)\times e_1,e_j\right)\\ &\quad +\tfrac{1}{4}g(V,e_1)g\left(\nabla_{e_i\times e_1}e_1,e_j\right) + \tfrac{1}{4}g(V,e_1)g\left((\nabla_{e_i\times e_1}e_1)\times e_1,e_j\right)\end{aligned}$$

Hence we get the covariant derivative on the spinor bundle:

Lemma 4.1. *Let $\{\sigma_1,\cdots,\sigma_8\}$ be a local frame of spinor sections. The covariant derivative on the spinor bundle is obtained locally as*

$$\begin{aligned}\nabla_V^{\tilde{S}}\widetilde{\sigma} &= \Psi_{e_1}\left(\nabla_V^S\sigma\right) - \frac{1}{2}\sum_{k=1}^{8}x_k\Psi_{e_1}\left(\nabla_V^S\sigma_k\right)\\ &\quad +\frac{1}{4}g(V,e_1)\Psi_{e_1}\left(\kappa(e_2e_3+e_4e_5+e_6e_7)\sigma\right)\\ &\quad -\frac{1}{8}\Psi_{e_1}\left(\sum_{i,j}g\left(\nabla_V(e_i\times e_1)-(\nabla_V e_i)\times e_1,e_j\right)\kappa(e_ie_j)\sigma\right)\\ &\quad -\frac{1}{8}\Psi_{e_1}\left(\sum_{i,j}g\left((\nabla_V(e_i\times e_1))\times e_1,e_j\right)\kappa(e_ie_j)\sigma\right),\end{aligned}$$

where $\Psi_\omega(\sigma)=\widetilde{\sigma}$, $\sigma = x_1\sigma_1+\cdots+x_8\sigma_8$ is a local spinor section on the manifold (M,g) and V is a vector field.

Then one can obtain the Dirac operator as

$$\begin{aligned}
\widetilde{D}\widetilde{\sigma} &= \sum_{k=1}^{7}\widetilde{\kappa}(\widetilde{e}_k)\left(\nabla^{\widetilde{S}}_{\widetilde{e}_k}\widetilde{\sigma}\right) \\
&= \frac{1}{2}\sum_{k=1}^{7}\widetilde{\kappa}(\widetilde{e}_k)\left(\Psi_{e_1}\left(\nabla^{S}_{\tilde{e}_k}\sigma\right)\right) \\
&\quad +\frac{1}{4}\sum_{k=1}^{7}\widetilde{\kappa}(\widetilde{e}_k)\left(g(\tilde{e}_k, e_1)\Psi_{e_1}\left(\kappa(e_2e_3+e_4e_5+e_6e_7)\sigma\right)\right) \\
&\quad -\frac{1}{8}\sum_{k=1}^{7}\widetilde{\kappa}(\widetilde{e}_k)\left(\Psi_{e_1}\left\{g\left(\nabla_{\tilde{e}_k}(e_i\times e_1)-(\nabla_{\tilde{e}_k}e_i)\times e_1, e_j\right)\kappa(e_ie_j)\sigma\right\}\right) \\
&\quad -\frac{1}{8}\sum_{k=1}^{7}\widetilde{\kappa}(\widetilde{e}_k)\left(\Psi_{e_1}\left(g\left((\nabla_{\tilde{e}_k}(e_i\times e_1))\times e_1, e_j\right)\kappa(e_ie_j)\sigma\right)\right).
\end{aligned}$$

After tedious calculations, we obtain the following theorem:

Theorem 4.2. *If a spinor section σ is in the kernel of D and σ satisfies the relation*

$$\begin{aligned}
&2\left\{\kappa(e_1)d\sigma(e_1)+\kappa(e_2)d\sigma(e_3)-\kappa(e_3)d\sigma(e_2)\right\} \\
&+2\left\{-\kappa(e_4)d\sigma(e_5)+\kappa(e_5)d\sigma(e_4)-\kappa(e_6)d\sigma(e_7)+\kappa(e_7)d\sigma(e_6)\right\} \\
&=\kappa\left(e_2\wedge d\eta_2+e_3\wedge d\eta_3+\tfrac{1}{2}\left(e_3\wedge d\eta_2-e_2\wedge d\eta_3\right)+6e_{123}\right)\sigma,
\end{aligned} \tag{4.1}$$

then the spinor section $\Psi_{e_1}(\sigma)$ is in the kernel of $\widetilde{D}$.

Proof. If we use the cross product of canonical G_2 structure on 7-dimensional 3-Sasakian manifold which is presented before, the new Dirac operator can be written in terms of old Dirac operator in the following form:

$$\begin{aligned}
\widetilde{D}\widetilde{\sigma} &= 2^{-2/3}\Psi_{e_1}\{D\sigma\} \\
&\quad +2^{-2/3}\Psi_{e_1}\left(\kappa(e_1)d\sigma(e_1)+\kappa(e_2)d\sigma(e_3)-\kappa(e_3)d\sigma(e_2)\right) \\
&\quad +2^{-2/3}\Psi_{e_1}\left(-\kappa(e_4)d\sigma(e_5)+\kappa(e_5)d\sigma(e_4)-\kappa(e_6)d\sigma(e_7)+\kappa(e_7)d\sigma(e_6)\right) \\
&\quad +2^{-5/3}\Psi_{e_1}\left(\kappa\left(e_2\wedge d\eta_2+e_3\wedge d\eta_3+\frac{1}{2}\left(e_3\wedge d\eta_2-e_2\wedge d\eta_3\right)+6e_{123}\right)\sigma\right).
\end{aligned}$$

Hence we obtain the relation given in the theorem. □

The relation in the theorem is important. If we use the following real Clifford representation

$$\kappa(e_1)=I\otimes\mu_1\otimes\mu_2,\quad \kappa(e_2)=I\otimes I\otimes\mu_1,\quad \kappa(e_3)=\mu_3\otimes\mu_1\otimes\mu_3,$$
$$\kappa(e_4)=\mu_2\otimes\mu_1\otimes\mu_3,\quad \kappa(e_5)=-\mu_1\otimes I\otimes\mu_3,\quad \kappa(e_6)=\mu_1\otimes\mu_3\otimes\mu_2,$$
$$\kappa(e_7)=-\mu_1\otimes\mu_2\otimes\mu_2$$

where

$$\mu_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then one can easily check that

$$\sigma = (\cos(3x_7), \sin(3x_7), 0, 0, 0, 0, 0, 0)$$

is in kernel of Dirac operator D, locally. Unfortunately, $\Psi_{e_1}(\sigma)$ is not in the kernel of $\widetilde{D}$ (that is, $\Psi_{e_1}(\sigma)$ does not satisfy the relation (4.1)). Hence we deduce that $\Psi_{e_1}(\mathrm{Ker}D)$ does not lie in the kernel of $\widetilde{D}$.

It is known that after the deformation of a parallel G_2 structure by a parallel vector field, the kernels of Dirac operators on the new and old spinor bundles are isomorphic [14]. The change in kernels of Dirac operators on the spinor bundles of a coclosed G_2 structure after deformation by a vector field has not been investigated. This study is an example of this type of deformations.

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