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ON $CR-$SUBMANIFOLDS OF A $S-$MANIFOLD ENDOWED WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we study $CR-$submanifolds of an $S-$manifold endowed with a semi-symmetric non-metric connection. We give an example, investigating integrabilities of horizontal and vertical distributions of $CR-$sub manifolds endowed with a semi-symmetric non-metric connection. We also consider parallel horizontal distributions of $CR-$submanifolds.

1. Introduction

In 1963, Yano [23] introduced the notion of f-structure on a C^∞ m-dimensional manifold M, as a non-vanishing tensor field f of type $(1,1)$ on M which satisfies $f^3+f=0$ and has constant rank r. It is known that r is even, say $r=2n$. Moreover, TM splits into two complementary subbundles $\mathrm{Im}f$ and $\ker f$ and the restriction of f to $\mathrm{Im}f$ determines a complex structure on such subbundle. It is also known that the existence of an f-structure on M is equivalent to a reduction of the structure group to $U(n)\times O(s)$ (see [9]), where $s=m-2n$. In 1970, Goldberg and Yano [12] introduced globally frame f-manifolds (also called metric f- manifolds and f.pk-manifolds). A wide class of globally frame f-manifolds was introduced in [9] by Blair according to the following definition: a metric f-structure is said to be a K-structure if the fundamental 2-form Φ, defined usually as $\Phi(X,Y)=g(X,fY)$, for any vector fields X and Y on M, is closed and the normality condition holds, that is, $[f,f]+2\sum_{i=1}^{s} d\eta^i\otimes\xi_i=0$, where $[f,f]$ denotes the Nijenhuis torsion of f. A K-manifold is called an S-manifold if $d\eta^k=\Phi$, for all $k=1,\dots,s$. The S-manifolds have been studied by several authors (see, for instance, [2],[3],[5],[10],[11]).

On the other hand, the notion of a $CR-$submanifold of Kaehlerian manifolds was introduced by A. Bejancu in [6]. Later, the concept of $CR-$submanifolds has been developed by [4], [8], [13], [14], [16], [18], [19], [20], [22] and others.

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Let ∇ be a linear connection in an $n-$dimensional differentiable manifold M . The torsion tensor T and the curvature tensor R of ∇ are given respectively by [7]

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection. In [17], Friedmann and Schouten introduced the notion of semi-symmetric linear connections. More precisely, if ∇ is a linear connection in a differentiable manifold M, the torsion tensor T of ∇ is given by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, for any vector fields X and Y on M. The connection ∇ is said to be symmetric if the torsion tensor T vanishes, otherwise it is said to be non-symmetric. In this case, ∇ is said to be a semi-symmetric connection if its torsion tensor T is of the form $T(X,Y) = \eta(Y)X - \eta(X)Y$, for any X, Y, where η is a 1-form on M. Moreover, if g is a (pseudo)-Riemannian metric on M, ∇ is called a metric connection if $\nabla g = 0$, otherwise it is called non-metric. It is well known that the Riemannian connection is the unique metric and symmetric linear connection on a Riemannian manifold. In 1932, Hayden [15] defined a metric connection with torsion on a Riemannian manifold. In [1] Agashe and Chafle defined a semi-symmetric non-metric connection on a Riemannian manifold and studied some of its properties. Later, the concept of semi-symmetric non-metric connection has been developed by (see, for instance, [3], [21]) and others. In this paper we study $CR-$submanifolds of an $S-$manifold endowed with a semi-symmetric non-metric connection. We consider integrabilities of horizontal and vertical distributions of $CR-$submanifolds with a semi-symmetric non-metric connection. We also consider parallel horizontal distributions of $CR-$submanifolds.

The paper is organized as follows: In section 2, we give a brief introduction to $S-$manifolds. In section 3, we study $CR-$submanifolds of S-manifolds. We find necessary conditions for the induced connection on a $CR-$submanifold of an $S-$manifold with semi-symmetric non-metric connection to be also a semi-symmetric non-metric connection. In section 4, We study integrabilities of horizontal and vertical distributions of $CR-$submanifolds with a semi-symmetric non-metric connection.

2. $S-$Manifolds

A $(2n+s)-$dimensional differentiable manifold $\widetilde{M}$ is called a *metric f-manifold* if there exist an $(1,1)$ type tensor field f, s vector fields $\xi_1, \dots, \xi_s$, s 1-forms $\eta^1, \dots, \eta^s$

and a Riemannian metric g on $\widetilde{M}$ such that

$$f^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \ \eta^i(\xi_j) = \delta_{ij}, \ f\xi_i = 0, \ \eta^i \circ f = 0, \tag{2.1}$$

$$g(fX, fY) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y), \tag{2.2}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, $i, j \in \{1, \ldots, s\}$. In addition we have:

$$\eta^i(X) = g(X, \xi_i), \ g(X, fY) = -g(fX, Y). \tag{2.3}$$

Moreover, a metric f-manifold is *normal* if

$$[f, f] + 2\sum_{\alpha=1}^{s} d\eta^\alpha \otimes \xi_\alpha = 0$$

where $[f, f]$ is Nijenhuis tensor of f.

Then a 2-form F is defined by $F(X, Y) = g(X, fY)$, for any $X, Y \in \Gamma(T\widetilde{M})$, called the *fundamental 2-form.* Then $\widetilde{M}$ is said to be an S-manifold if the f structure is normal and

$$\eta^1 \wedge ... \wedge \eta^s \wedge (d\eta^\alpha)^n \neq 0, \qquad F = d\eta^\alpha$$

for any $\alpha = 1, ..., s$. In the case $s = 1$, an $S-$manifold is a Sasakian manifold.

Now, if $\widetilde{\nabla}$ denotes the Riemannian connection associated with g, then [7]

$$\left(\widetilde{\nabla}_X f\right) Y = \sum_{\alpha=1}^{s} \left\{ g\left(fX, fY\right) \xi_\alpha + \eta^\alpha\left(Y\right) f^2 X \right\}, \tag{2.4}$$

for all $X, Y \in \Gamma(T\widetilde{M})$. From (2.4), it is deduced that

$$\widetilde{\nabla}_X \xi_\alpha = -fX, \tag{2.5}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, $\alpha \in \{1, ..., s\}$.

3. $CR-$Submanifold of $S-$Manifolds

Definition 3.1. *An $(2m+s)-$dimensional Riemannian submanifold M of $S-$manifold $\widetilde{M}$ is called a $CR-$submanifold if $\xi_1, \xi_2, \ldots, \xi_s$ is tangent to M and there exists on M two differentiable distributions D and $D^\perp$ on M satisfying:*

(i) $TM = D \oplus D^\perp \oplus sp\{\xi_1, ..., \xi_s\}$;
(ii) *The distribution D is invariant under f, that is $fD_x = D_x$ for any $x \in M$;*
(iii) *The distribution $D^\perp$ is anti-invariant under f, that is, $fD_x^\perp \subseteq T_x^\perp M$ for any $x \in M$, where T_xM and $T_xM^\perp$ are the tangent space of M at x.*

We denote by $2p$ and q the real dimensions of D_x and $D_x^\perp$ respectively, for any $x \in M$. Then if $p = 0$ we have an anti-invariant submanifold tangent to $\xi_1, \xi_2, ..., \xi_s$ and if $q = 0$, we have an invariant submanifold.

Example 3.1. *In what follows,* $(R^{2n+s}, f, \eta, \xi, g)$ *will denote the manifold* R^{2n+s} *with its usual* S*-structure given by*

$$\eta^{\alpha} = \frac{1}{2}(dz_{\alpha} - \sum_{i=1}^{n} y_i dx_i), \qquad \xi_{\alpha} = 2\frac{\partial}{\partial z_{\alpha}}$$

$$f(\sum_{i=1}^{n}(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{s} Z_{\alpha}\frac{\partial}{\partial z_{\alpha}}) = \sum_{i=1}^{n}(Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{s}\sum_{i=1}^{n} Y_i y_i \frac{\partial}{\partial z_{\alpha}}$$

$$g = \sum_{\alpha=1}^{s} \eta^{\alpha} \otimes \eta^{\alpha} + \frac{1}{4}(\sum_{i=1}^{n} dx_i \otimes dx_i + dy_i \otimes dy_i),$$

$(x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_s)$ *denoting the Cartesian coordinates on* R^{2n+s}*. The consider a submanifold of* R^8 *defined by*

$$M = X(u, v, k, l, t_1, t_2) = 2(u, 0, k, v, l, 0, t_1, t_2).$$

Then local frame of TM

$$\begin{aligned} e_1 &= 2\frac{\partial}{\partial x_1}, \qquad e_2 = 2\frac{\partial}{\partial y_1}, \\ e_3 &= 2\frac{\partial}{\partial x_3}, \qquad e_4 = 2\frac{\partial}{\partial y_2}, \\ e_5 &= 2\frac{\partial}{\partial z_1} = \xi_1, \qquad e_6 = 2\frac{\partial}{\partial z_2} = \xi_2 \end{aligned}$$

and

$$e_1^* = 2\frac{\partial}{\partial x_2}, \qquad e_2^* = 2\frac{\partial}{\partial y_3}$$

from a basis of $T^{\perp}M$*. We determine* $D_1 = sp\{e_1, e_2\}$ *and* $D_2 = sp\{e_3, e_4\}$*, then* D_1*,* D_2 *are invariant and anti-invariant distribution. Thus* $TM = D_1 \oplus D_2 \oplus sp\{\xi_1, \xi_2\}$ *is a* $CR-$*submanifold of* R^8*.*

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ with respect to the induced metric g. Then Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla^*_X Y + h(X, Y) \tag{3.1}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{*\perp}_X N \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$. $\nabla^{*\perp}$ is the connection in the normal bundle, h is the second fundamental from of $\widetilde{M}$ and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by

$$g(h(X, Y), N) = g(A_N X, Y) \tag{3.3}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$.

A $CR-$submanifold is said to be $D-$totally geodesic if $h(X,Y)=0$ for any $X,Y\in\Gamma(D)$ and it is said to be $D^{\perp}-$totall geodesic if $h(Z,W)=0$ for any $Z,W\in\Gamma(D^{\perp})$.

The projection morphisms of TM to D and $D^{\perp}$ are denoted by P and Q respectively. For any $X,Y\in\Gamma(TM)$ and $N\in\Gamma(T^{\perp}M)$ we have

$$X=PX+QX+\sum_{\alpha=1}^{s}\eta^{\alpha}(X)\xi_{\alpha},\quad 1\leq\alpha\leq s \tag{3.4}$$

$$fN=BN+CN \tag{3.5}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of φN.

Now, we define a connection $\overline{\nabla}$ as

$$\overline{\nabla}_XY=\widetilde{\nabla}_XY+\sum_{\alpha=1}^{s}\eta^{\alpha}\left(Y\right)X.$$

Theorem 3.1. *Let $\widetilde{\nabla}$ be the Riemannian connection on a $S-$manifold $\widetilde{M}$. Then the linear connection which is defined as*

$$\overline{\nabla}_XY=\widetilde{\nabla}_XY+\sum_{\alpha=1}^{s}\eta^{\alpha}(Y)X,\quad\forall X,Y\in\Gamma(TM) \tag{3.6}$$

is a semi-symmetric non metric connection on $\widetilde{M}$.

Proof. Using new connection and the fact that the Riemannian connection is torsion free, the torsion tensor $\overline{T}$ of the connection $\overline{\nabla}$ is given by

$$\overline{T}(X,Y)=\sum_{\alpha=1}^{s}\{\eta^{\alpha}(Y)X-\eta^{\alpha}(X)Y\} \tag{3.7}$$

for all $X,Y\in\Gamma(TM)$. Moreover, by using (3.6) again, for all $X,Y,Z\in\Gamma(TM)$ and since $\widetilde{\nabla}$ is a metric connection, we have

$$(\overline{\nabla}_Xg)(Y,Z)=-\sum_{\alpha=1}^{s}\{g(X,Y)\eta^{\alpha}(Z)-g(X,Z)\eta^{\alpha}(Y)\}. \tag{3.8}$$

From (3.7) and (3.8) we conclude that the linear connection $\overline{\nabla}$ is a semi-symmetric non-metric connection on $\widetilde{M}$. □

Theorem 3.2. *Let M be a CR submanifold of S-manifold $\widetilde{M}$. Then*

$$(\overline{\nabla}_Xf)Y=\sum_{\alpha=1}^{s}\{g(X,Y)\xi_{\alpha}-\eta^{\alpha}(Y)(X+fX)\} \tag{3.9}$$

for all $X,Y\in\Gamma(TM)$.

Proof. From (3.6), we get

$$(\overline{\nabla}_X f)Y = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - \eta^\alpha(Y)X - \eta^\alpha(Y)fX\}$$

for all $X, Y \in \Gamma(TM)$. Therefore we obtain the result from (2.4). □

Corollary 3.1. *Let M be a CR submanifold of S-manifold $\widetilde{M}$ with semi-symmetric non-metric connection $\overline{\nabla}$. Then*

$$\overline{\nabla}_X \xi_\alpha = -fX + X \tag{3.10}$$

for all $X \in \Gamma(TM)$.

We denote by same symbol g both metrics on $\widetilde{M}$ and M. Let $\overline{\nabla}$ be the semi-symmetric non-metric connection on $\widetilde{M}$ and ∇ be the induced connection on M. Then,

$$\overline{\nabla}_X Y = \nabla_X Y + m(X,Y) \tag{3.11}$$

where m is a $\Gamma(T^\perp M)-$valued symmetric tensor field on CR- submanifold M. If ∇^* denotes the induced connection from the Riemannian connection ∇, then

$$\nabla_X Y = \nabla^*_X Y + h(X,Y), \tag{3.12}$$

where h is the second fundamental form. Using (3.1) and (3.4), we have

$$\nabla_X Y + m(X,Y) = \nabla^*_X Y + h(X,Y) + \sum_{\alpha=1}^{s}\eta^\alpha(Y)X.$$

Equating tangential and normal components from both the sides, we get

$$m(X,Y) = h(X,Y)$$

and

$$\nabla_X Y = \nabla^*_X Y + \sum_{i=1}^{s}\eta^\alpha(Y)X. \tag{3.13}$$

Thus ∇ is also a semi-symmetric non-metric connection. From (3.2) and (3.13), we have

$$\begin{aligned}\nabla_X N &= \nabla^*_X N + \sum_{\alpha=1}^{s}\eta^\alpha(N)X \\ &= -A_N X + \nabla^\perp_X N + \sum_{\alpha=1}^{s}\eta^\alpha(N)X,\end{aligned}$$

where $X \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$.

Now, Gauss and Weingarten formulas for a CR-submanifolds of a S-manifold with a semi-symmetric non-metric connection is

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3.14}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \sum_{\alpha=1}^{s} \eta^{\alpha}(N) X \tag{3.15}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, h second fundamental form of M and A_N is the Weingarten endomorphism associated with N.

Theorem 3.3. *The connection induced on $CR-$submanifold of a S-manifold with semi-symmetric non-metric connection is also a semi-symmetric non-metric connection.*

Proof. From (3.7) and (3.8), we get

$$\bar{T}(X,Y) = T(X,Y) \text{ and } (\overline{\nabla}_X g)(Y,Z) = (\nabla_X g)(Y,Z)$$

for any $X, Y \in \Gamma(TM)$, where T is the torsion tensor of ∇. □

4. Integrability and Parallel of Distributions

Lemma 4.1. *Let M be a $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then,*

$$P\nabla_X fPY - PA_{fQY}X - fP\nabla_X Y = -\sum_{\alpha=1}^{s} \eta^{\alpha}(Y)(PX + fPX), \tag{4.1}$$

$$Q\nabla_X fPY - QA_{fQY}X - th(X,Y) = -\sum_{\alpha=1}^{s} \eta^{\alpha}(Y) QX, \tag{4.2}$$

$$h(X, fPY) - fQ\nabla_X Y + \nabla^{\perp}_X fQY = nh(X,Y) - \sum_{\alpha=1}^{s} \eta^{\alpha}(Y) fQX, \tag{4.3}$$

for all $X, Y \in \Gamma(TM)$.

Proof. By direct covariant differentiation, we have

$$\overline{\nabla}_X fY = \left(\overline{\nabla}_X f\right) Y + f\left(\overline{\nabla}_X Y\right).$$

for any $X, Y \in \Gamma(TM)$. By virtue of (3.4),(3.9),(3.14) and (3.15) we get

$$\begin{aligned}
&\nabla_X fPY + h(X, fPY) + \left(-A_{fQY}X + \nabla^{\perp}_X fQY\right) \\
&\quad = \sum_{\alpha=1}^{s} \{g(X,Y)\xi_{\alpha} - \eta^{\alpha}(Y)(fX + X)\} + f\nabla_X Y + fh(X,Y).
\end{aligned}$$

Using (3.4) again, we have

$$\begin{aligned} &P\nabla_X fPY + Q\nabla_X fPY + h(X, fPY) - PA_{fQY}X - QA_{fQY}X + \nabla^{\perp}_X fQY \\ &= \sum_{\alpha=1}^{s} \{g(X,Y)P\xi_\alpha + g(X,Y)Q\xi_\alpha - \eta^\alpha(Y)PX \\ &\quad -\eta^\alpha(Y)QX - \eta^\alpha(Y)fPX - \eta^\alpha(Y)fQX\} \\ &\quad +fP\nabla_X Y + fQ\nabla_X Y + th(X,Y) + nh(X,Y). \end{aligned}$$

Equations (4.1)-(4.3) follow by comparing the horizontal, vertical and normal components. □

Lemma 4.2. *Let M be a $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then,*

$$-A_{fW}Z - fP\nabla_Z W - th(Z,W) = \sum_{\alpha=1}^{s} g(Z,W)\xi_\alpha, \tag{4.4}$$

$$\nabla^{\perp}_Z fW = fQ\nabla_Z W + nh(Z,W) \tag{4.5}$$

for any $Z, W \in \Gamma(D^{\perp})$.

Proof. From (3.9), we have

$$\left(\overline{\nabla}_Z f\right)W = \sum_{\alpha=1}^{s} \left\{g(fZ, fW)\xi_\alpha + \eta_\alpha(W)\left(f^2 Z - fZ\right)\right\}$$

for any $Z, W \in \Gamma(D^{\perp})$. Since $\eta^\alpha(W) = 0$ for $W \in \Gamma(D)$, using (2.2) we get

$$\left(\overline{\nabla}_Z f\right)W = \sum_{\alpha=1}^{s} g(fZ, fW)\xi_\alpha = \sum_{\alpha=1}^{s} g(Z,W)\xi_\alpha.$$

Therefore

$$\overline{\nabla}_Z fW - f\overline{\nabla}_Z W = \sum_{\alpha=1}^{s} g(Z,W)\xi_\alpha.$$

In above equation, using (3.14) and (3.15), we have

$$-A_{fW}Z + \nabla^{\perp}_Z fW - f\nabla_Z W - fh(Z,W) = \sum_{\alpha=1}^{s} g(Z,W)\xi_\alpha$$

$$\begin{aligned} &-A_{fW}Z + \nabla^{\perp}_Z fW - fP\nabla_Z W - fQ\nabla_Z W - th(Z,W) - nh(Z,W) \\ &= \sum_{\alpha=1}^{s} g(Z,W)\xi_\alpha. \end{aligned}$$

Now comparing tangent and normal parts in above equation, we obtain (4.4) and (4.5). □

Lemma 4.3. *Let M be a $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then,*

$$\nabla_X fY - fP\nabla_X Y = \sum_{\alpha=1}^{s} g(X,Y)\xi_\alpha + th(X,Y), \tag{4.6}$$

$$h(X,fY) = fQ\nabla_X Y + nh(X,Y) \tag{4.7}$$

for any $X,Y \in \Gamma(D)$.

Proof. From (3.9), we have

$$\left(\overline{\nabla}_X f\right) Y = \sum_{\alpha=1}^{s} \left\{ g(fX,fY)\xi_\alpha + \eta^\alpha(Y)\left(f^2X - fX\right) \right\}$$

for any $X,Y \in \Gamma(D)$. Using $\eta^\alpha(Y) = 0$ for each $Y \in \Gamma(D)$ and (2.2) we obtain

$$\begin{aligned}\left(\overline{\nabla}_X f\right) Y &= \sum_{\alpha=1}^{s} g(fX,fY)\xi_\alpha \\ &= \sum_{\alpha=1}^{s} g(X,Y)\xi_\alpha.\end{aligned}$$

Moreover, we have

$$\overline{\nabla}_X fY - f\overline{\nabla}_X Y = \sum_{\alpha=1}^{s} g(X,Y)\xi_\alpha.$$

Now using (3.14), we have

$$\nabla_X fY + h(X,fY) - f\nabla_X Y - fh(X,Y) = \sum_{\alpha=1}^{s} g(X,Y)\xi_\alpha$$

$$\begin{aligned}&\nabla_X fY + h(X,fY) - fP\nabla_X Y - fQ\nabla_X Y - th(X,Y) - nh(X,Y) \\ = &\sum_{\alpha=1}^{s} g(X,Y)\xi_\alpha.\end{aligned}$$

Now comparing tangent and normal parts, we obtain (4.6) and (4.7). □

Lemma 4.4. *Let M be a $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then,*

$$\nabla_X \xi_\alpha = -fPX + X, \quad \forall X \in \Gamma(TM) \tag{4.8}$$

$$h(X,\xi_\alpha) = -fQX, \quad \forall X \in \Gamma(TM) \tag{4.9}$$

$$A_V \xi_\alpha \in D^\perp, \qquad \forall V \in \Gamma(T^\perp M) \tag{4.10}$$

Proof. Using (3.14) in (3.10), we easily obtain

$$\overline{\nabla}_X\xi_\alpha = -fX + X \Rightarrow \nabla_X\xi_\alpha + h\left(X,\xi_\alpha\right) = -fX + X$$

which gives

$$\nabla_X\xi_\alpha + h\left(X,\xi_\alpha\right) = -fPX - fQX + X.$$

Now comparing tangent and normal parts, we get

$$\nabla_X\xi_\alpha = -fPX + X \text{ and } h\left(X,\xi_\alpha\right) = -fQX.$$

On the other hand, using (3.3) we have

$$g\left(A_V\xi_\alpha, X\right) = g\left(h\left(X,\xi_\alpha\right), V\right) = g\left(0, V\right) = 0$$

for $X \in \Gamma(D)$ and $V \in \Gamma(T^\perp M)$. Using (4.9) in the above equation, we get

$$g\left(A_V\xi_\alpha, X\right) = 0, \qquad \forall X \in \Gamma(D) \text{ which leads to } A_V\xi_\alpha \in \Gamma(D^\perp)$$

also

$$g\left(A_V\xi_\alpha, X\right) = 0, \qquad \forall X \in \Gamma(D) \Rightarrow g\left(A_V\xi_\alpha, X\right) = \eta_\alpha\left(A_V X\right) = 0$$

which gives (4.10). □

Theorem 4.1. *Let M be a $CR-$submanifold of a S-manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then the distribution D is not integrable.*

Proof. For any $X, Y \in \Gamma(D)$, we have

$$g([X,Y],\xi_i) = -g(Y,\widetilde{\nabla}_X\xi_i) + g(X,\widetilde{\nabla}_Y\xi_i).$$

Using (3.10) and (3.14), we have

$$\begin{aligned} g([X,Y],\xi_i) &= -g(Y,\overline{\nabla}_X\xi_i - X) + g(X,\overline{\nabla}_Y\xi_i - Y) \\ &= -g(Y,fX) + g(X,fY). \end{aligned}$$

Thus D is integrable if and only if $g(X,fY) = g(Y,fX)$. From (2.3), the proof is complete. □

Theorem 4.2. *Let M be a $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. The distribution $D \oplus Sp\{\xi_1,...,\xi_s\}$ is integrable if and only if*

$$h\left(X,fY\right) = h\left(Y,fX\right)$$

for any $X, Y \in \Gamma(D \oplus Sp\{\xi_1,...,\xi_s\})$.

Proof. From (4.7), we have

$$h\left(X,fPY\right) = fQ\nabla_X Y + nh\left(X,Y\right), \qquad \forall X,Y \in \Gamma(D \oplus sp\{\xi_1,...,\xi_s\}). \quad (4.11)$$

Interchanging X and Y, we have

$$h\left(Y,fPX\right) = fQ\nabla_Y X + nh\left(Y,X\right), \qquad \forall X,Y \in \Gamma(D \oplus sp\{\xi_1,...,\xi_s\}). \quad (4.12)$$

Adding (4.11) and (4.12), we obtain

$$h\left(X,fY\right) - h\left(Y,fX\right) = fQ\left[X,Y\right].$$

Then we have $[X,Y]\in\Gamma(D\oplus sp\{\xi_1,...,\xi_s\})$ if and only if $h(X,fY)=h(Y,fX)$. □

Corollary 4.1. *Let M be a $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. The distribution $D\oplus Sp\{\xi_1,...,\xi_s\}$ is integrable if and only if*

$$A_N fX=-fA_N X$$

for any $X\in\Gamma(D\oplus sp\{\xi_1,...,\xi_s\})$.

Definition 4.1. *A $CR-$submanifold is said to be mixed totally geodesic if $h(X,Z)=0$, for any $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$.*

Lemma 4.5. *Let M be a $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then M is mixed totaly geodesic if and only if one of the following satisfied;*

$$A_V X\in D \qquad \left(\forall X\in\Gamma(D),\ V\in\Gamma(T^{\perp}M)\ \right), \tag{4.13}$$

$$A_V X\in D^{\perp} \qquad \left(\forall X\in\Gamma(D^{\perp}),\ V\in\Gamma(T^{\perp}M)\ \right). \tag{4.14}$$

Proof. For $X\in\Gamma(D)$, $V\in\Gamma(T^{\perp}M)$ and $Y\in\Gamma(D^{\perp})$, consider $A_V X$, then from (3.3) we get

$$\begin{aligned} g(A_V X,Y) &= g(h(X,Y),V)\\ &= 0\Leftrightarrow A_V X\in\Gamma(D). \end{aligned}$$

Hence, we have

$$\begin{aligned} g(h(X,Y),V)=0 &\Leftrightarrow h(X,Y)=0\\ &\Leftrightarrow A_V X\in\Gamma(D) \qquad \forall X\in\Gamma(D),\ V\in\Gamma(T^{\perp}M), \end{aligned}$$

which gives (4.13). In a similar way is deduced relation (4.14). □

Definition 4.2. *The horizontal (resp.vertical) distribution on D (resp. $D^{\perp}$) is said to be parallel with respect to the connection ∇ on M if*

$$\nabla_X Y\in\Gamma(D)\ \left(resp.\,\nabla_Z W\in\Gamma(D^{\perp})\right) \text{ for any } X,Y\in\Gamma(D)\ (resp.\,Z,W\in\Gamma(D^{\perp})).$$

Theorem 4.3. *Let M be a $\xi_\alpha-$horizontal $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then, the horizontal distribution D is parallel if and only if*

$$h(X,fY)=h(Y,fX)=fh(X,Y) \tag{4.15}$$

for all $X,Y\in\Gamma(D)$.

Proof. Since every parallel is involutive then the first equality follows immediately. Now since D is parallel, we have

$$\nabla_X fY\in\Gamma(D), \qquad \forall X,Y\in\Gamma(D).$$

Then from (4.2), we have

$$th\,(X,Y) = 0 \qquad \forall X,Y \in \Gamma(D). \tag{4.16}$$

From (4.3), D is parallel if and only if

$$h\,(X,fY) = nh\,(X,Y)\,.$$

But we have

$$fh\,(X,Y) = th\,(X,Y) + nh\,(X,Y)\,,$$

and from (4.9), $fh\,(X,Y) = nh\,(X,Y)$, which completes the proof. □

Theorem 4.4. *Let M be a $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. The distribution $D^{\perp} \oplus Sp\{\xi_1,...,\xi_s\}$ is integrable if and only if*

$$A_{fX}Y - A_{fY}X = \sum_{\alpha=1}^{s}\{\eta^{\alpha}\,(X)\,Y - \eta^{\alpha}\,(Y)\,X\} \tag{4.17}$$

for all $X,Y \in \Gamma(D^{\perp} \oplus sp\{\xi_1,...,\xi_s\})$.

Proof. If $X,Y \in \Gamma(D^{\perp} \oplus sp\{\xi_1,...,\xi_s\})$, then from (4.1) and (4.2) we have

$$-PA_{fQY}X - fP\nabla_X Y = 0, \tag{4.18}$$

$$-QA_{fQY}X - th\,(X,Y) = -\sum_{\alpha=1}^{s}\eta^{\alpha}(Y)X. \tag{4.19}$$

Adding (4.18) and (4.19), we have

$$-A_{fY}X - fP\nabla_X Y - th\,(X,Y) = -\sum_{\alpha=1}^{s}\eta^{\alpha}(Y)X. \tag{4.20}$$

Now interchanging X and Y, we have

$$-A_{fX}Y - fP\nabla_Y X - th\,(X,Y) = -\sum_{\alpha=1}^{s}\eta^{\alpha}(X)Y. \tag{4.21}$$

Subtracting (4.20) and (4.21), we obtain

$$-A_{fY}X + A_{fX}Y - fP\,[X,Y] = \sum_{\alpha=1}^{s}\{-\eta^{\alpha}\,(Y)\,X + \eta^{\alpha}\,(X)\,Y\}\,.$$

Hence $P\,[X,Y] = 0$, we obtain

$$\Leftrightarrow A_{fX}Y - A_{fY}X = \sum_{\alpha=1}^{s}\{\eta^{\alpha}\,(X)\,Y - \eta^{\alpha}\,(Y)\,X\}\,.$$

Therefore $D^{\perp}$ is integrable $\Leftrightarrow$ (4.17) holds. □

Corollary 4.2. *Let M be $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non metric connection. Then, the distribution $D^{\perp}$ is integrable if and only if*

$$A_{fY}X = A_{fX}Y \tag{4.22}$$

for all $X, Y \in \Gamma(D^{\perp})$.

Proof. The proof can be obtained directly from (4.17). □

Lemma 4.6. *Let M be a $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then, the distribution $D^{\perp}$ is parallel if and only if*

$$-A_{fW}Z = \sum_{\alpha=1}^{s} g\left(Z, W\right)\xi_{\alpha} + th\left(Z, W\right) \tag{4.23}$$

for all $Z, W \in \Gamma(D^{\perp})$.

Proof. From (4.4), we have

$$-A_{fW}Z - fP\nabla_{Z}W = \sum_{\alpha=1}^{s} g\left(X, Y\right)\xi_{\alpha} + th\left(Z, W\right) \ \ \forall Z, W \in \Gamma(D^{\perp}).$$

If $D^{\perp}$ is parallel then we get

$$\nabla_{Z}W \in \Gamma(D^{\perp}) \Leftrightarrow P\nabla_{Z}W = 0,$$

which gives (4.23). □

Lemma 4.7. *Let M be a $CR-$submanifold of an $S-$manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then the distribution $D^{\perp}$ is parallel if and only if*

$$A_{fW}Z \in \Gamma(D^{\perp}) \tag{4.24}$$

for any $Z, W \in \Gamma(D^{\perp})$.

Proof. For any $Z, W \in \Gamma(D^{\perp})$, from (3.9) we have

$$\left(\overline{\nabla}_{Z}f\right)W = \sum_{\alpha=1}^{s}\left\{g\left(fZ, fW\right)\xi_{\alpha} + \eta^{\alpha}\left(W\right)\left(f^{2}Z - fZ\right)\right\}.$$

Using (3.14) and (3.15) we obtain

$$\begin{aligned}
&\overline{\nabla}_{Z}fW - f\overline{\nabla}_{Z}W \\
&= \sum_{\alpha=1}^{s}\left\{g\left(fZ, fW\right)\xi_{\alpha} + \eta^{\alpha}\left(W\right)\left(f^{2}Z - fZ\right)\right\} \\
&- A_{fW}Z + \nabla^{\perp}_{Z}fW - f\nabla_{Z}W - fh\left(Z, W\right) \\
&= \sum_{\alpha=1}^{s}\left\{g\left(fZ, fW\right)\xi_{\alpha} + \eta^{\alpha}\left(W\right)\left(f^{2}Z - fZ\right)\right\}.
\end{aligned}$$

Taking inner product with $Y \in \Gamma(D)$ in the above equation, we have

$$g\left(-A_{fW}Z,Y\right)+g\left(\nabla^{\perp}_{Z}fW,Y\right)-g\left(f\nabla_{Z}W,Y\right)-g\left(fh\left(Z,W\right),Y\right)$$
$$=\sum_{\alpha=1}^{s}\left\{g\left(fZ,fW\right)g\left(\xi_{\alpha},Y\right)+\eta^{\alpha}\left(W\right)g\left(f^{2}Z,Y\right)-\eta^{\alpha}\left(W\right)g\left(fZ,Y\right)\right\}.$$

Then we have

$$-g(A_{fW}Z,Y)=g(f\nabla_{Z}W,Y)=-g(\nabla_{Z}W,fY).$$

This imply that

$$g\left(A_{fW}Z,Y\right)=0\Leftrightarrow A_{fW}Z\in\Gamma(D^{\perp}).$$

Therefore we obtain

$$\nabla_{Z}W\in D^{\perp}\Leftrightarrow A_{fW}Z\in\Gamma(D^{\perp}),\quad \forall Z,W\in\Gamma(D^{\perp}).$$

□

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