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GENERALIZATIONS OF SOME RESULTS ON GENERALIZED POLYNOMIAL IDENTITIES

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Abstract. In this work, our aim is to generalize some of the results in [1] and [5]. Precisely, we extend the result in [1] on commuting values of the same generalized derivations to the different generalized derivations case by a short proof. Also as an application, we extend a result in [5] on images of a linear map with derivations to generalized derivations case.

1. Introduction

Throughout, rings are always associative. Let R be a ring. For $a, b \in R$, let $[a, b] = ab - ba$, the commutator of a and b. For additive subgroups A, B of R, let $[A, B]$ denote the additive subgroup of R generated by all elements $[a, b]$ for $a \in A$ and $b \in B$. An additive map $\delta\colon R \to R$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. Given $b \in R$, the map $x \mapsto bx - xb$ for $x \in R$ is called an inner derivation, denoted by $\mathrm{ad}(b)$, induced by the element b. A derivation of R is called outer if it is not inner. An additive mapping $G : R \to R$ is called a generalized derivation of R if there exists a derivation δ of R such that $G(xy) = G(x)y + x\delta(y)$ for all $x, y \in R$. Evidently, any derivation is a generalized derivation. For $a, b \in R$, it is easy to see that the mapping $ax - xb$ is a generalized derivation of R known as inner generalized derivation.

By a prime ring we mean a ring R such that for $a, b \in R$, $aRb = 0$ implies either $a = 0$ or $b = 0$. Throughout, R is always a prime ring. Let Q denote the maximal right ring of quotients of R. It is known that Q is prime and its center, denoted by C, is a field, which is called the extended centroid of R. Let $Q *_C C\{X_1, X_2, \cdots\}$ stand for the free product of the C-algebras Q and $C\{X_1, X_2, \ldots, X_n, \ldots\}$, the free C-algebra in the noncommutative indeterminates $X_1, X_2, \cdots$. An element $\phi(x_1, \ldots, x_n)$ of the free product is called a generalized polynomial identity (gpi) on R, if $\phi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$ (see [2] for more details).

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Recently, most authors study some generalized polynomial identities on a prime ring and characterize the structure of maps involving in these identities (see [1], [3], [4] and [5]). In this way, they try to find out the structure of the ring. In this work, our aim is to generalize several of the works on generalized polynomial identities.

In [1], Ali et al. study a generalized polynomial identity with a commutator of the same generalized derivation. Precisely, they characterize the structure of a nonzero generalized derivation G of R such that $\big[G(u)u, G(v)v\big] = 0$ for all $u, v \in f(R)$, the set of all evaluations in R of the multilinear polynomial over C. In section 2, we extend the result to the different generalized derivation case by a short proof, namely, we characterize the structure of two nonzero generalized derivations G and F of R such that $\big[G(u)u, F(v)v\big] = 0$ for all $u, v \in f(R)$ (see Theorem 2.1).

Motivated by the Noether-Skolem theorem, in [5] the author and T.-K. Lee characterize a linear differential map $\varphi(x) = \sum_j a_j\delta^j(x)$ for all $x \in R$ such that $\phi(R) \subseteq [R, R]$, where R is a simple ring with a nonzero derivation δ and the a_j's are finitely many elements in Q. In section 3, as an application of the result, we consider the linear map for generalized derivations, namely, we characterize a linear map $\phi(x) = \sum_j a_j G^j(x)$ for all $x \in R$ such that $\phi(R) \subseteq [R, R]$, where G is a generalized derivation of R (see Theorem 3.5).

2. A generalization of the result in [1]

Throughout this section, R is always a prime ring. Let Q be the maximal right ring of quotients of R, and C be its center. We will use the following notation for a multilinear polynomial over C:

$$f(X_1, \dots, X_n) = \sum_{\sigma \in S_n} \alpha_\sigma X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(n)}$$

for some $\alpha_\sigma \in C$, and S_n is the symmetric group of degree n. Let

$$f(R) := \{f(r_1, \dots, r_n) \mid r_1 \dots, r_n \in R\}$$

the set of all evaluations in R of the multilinear polynomial over C. We denote by s_4, the standard polynomial in four variables defined as follows:

$$s_4(X_1, X_2, X_3, X_4) = \sum_{\sigma \in S_4} (-1)^\sigma X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)} X_{\sigma(4)},$$

where $(-1)^\sigma$ is the sign of a permutation σ of the symmetric group of degree 4, S_4.

In [1], Ali et al. proved that if R is a prime ring of characteristic different 2, $f(X_1, \dots, X_n)$ is a noncentral multilinear polynomial over C and G is a nonzero generalized derivation of R such that

$$\Big[G\big(f(r_1, \dots, r_n)\big) f(r_1, \dots, r_n), G\big(f(s_1, \dots, s_n)\big) f(s_1, \dots, s_n)\Big] = 0$$

for all $r_1, \dots, r_n, s_1, \dots, s_n \in R$, then there exists $c \in Q$ such that $G(x) = cx$ for all $x \in R$, moreover either $f(X_1, \dots, X_n)^2$ is central or R satisfies s_4. As an extension of the result, in this section we precisely prove the following.

Theorem 2.1. *Let R be a prime ring of characteristic different 2 with extended centroid C and $f(X_1,\dots,X_n)$ a noncentral multilinear polynomial over C. Let G and F be nonzero generalized derivations of R such that*

$$\Big[G\big(f(r_1,\dots,r_n)\big)f(r_1,\dots,r_n),F\big(f(s_1,\dots,s_n)\big)f(s_1,\dots,s_n)\Big]=0 \qquad (1)$$

for all $r_1,\dots,r_n,s_1,\dots,s_n\in R$. Then $f(X_1,\dots,X_n)^2$ is central valued on R and moreover, one of the following statements holds:

(i) There exists $\lambda\in C$ such that $G(x)=\lambda x$ for all $x\in R$;
(ii) There exists $\mu\in C$ such that $F(x)=\mu x$ for all $x\in R$;
(iii) There exist $a,c\in Q$ such that $G(x)=ax$ and *$F(x)=cx$ for all $x\in R$ and $[a,c]=0$.*

Proof. Suppose first $G(f(r_1,\dots,r_n))f(r_1,\dots,r_n)\in C$ for all $r_1,\dots,r_n\in R$. In view of Lemma 3 in [4], $f(X_1,\dots,X_n)^2$ is central valued on R and also there exists $\lambda\in C$ such that $G(x)=\lambda x$ for all $x\in R$, as desired for (i). By the same arguments, if $F(f(s_1,\dots,s_n))f(s_1,\dots,s_n)\in C$ for all $s_1,\dots,s_n\in R$, then $f(X_1,\dots,X_n)^2$ is central valued on R and also there exists $\mu\in C$ such that $F(x)=\mu x$ for all $x\in R$, as desired for (ii). Therefore we may assume that there exist $s_1,\dots,s_n\in R$ such that $v:=F(f(s_1,\dots,s_n))f(s_1,\dots,s_n)\notin C$. Then by (1),

$$\big[G\big(f(r_1,\dots,r_n)\big)f(r_1,\dots,r_n),v\big]=0 \qquad (2)$$

for all $r_1,\dots,r_n\in R$. Let δ be an inner derivation induced by $v\in R$, i.e., $\delta(x)=[x,v]$ for all $x\in R$. So $v\notin C$ implies $\delta\neq 0$. It follows from (2) that

$$\delta\Big(G\big(f(r_1,\dots,r_n)\big)f(r_1,\dots,r_n)\Big)=0$$

for all $r_1,\dots,r_n\in R$. In view of [3], $f(X_1,\dots,X_n)^2$ is central valued on R and moreover, there exists $a\in Q$ such that $G(x)=ax$ for all $x\in R$ and $\delta(a)=0$. Then $[a,v]=0$. It means

$$\big[a,F(f(s_1,\dots,s_n))f(s_1,\dots,s_n)\big]=0 \qquad (3)$$

for all $s_1,\dots,s_n\in R$ with $F(f(s_1,\dots,s_n))f(s_1,\dots,s_n)\notin C$. However, (3) holds for all $s_1,\dots,s_n\in R$ with $F(f(s_1,\dots,s_n))f(s_1,\dots,s_n)\in C$. Thus, (3) holds for all $s_1,\dots,s_n\in R$. Then by the same arguments above for (2), we have $f(X_1,\dots,X_n)^2$ is central valued on R and moreover, there exists $c\in Q$ such that $F(x)=cx$ for all $x\in R$ and $[a,c]=0$ in view of [3]. It means there exist $a,c\in Q$ such that $G(x)=ax$ *and* $F(x)=cx$ for all $x\in R$ and $[a,c]=0$, as desired for (iii). This completes the proof. □

The main theorem of [1] is then an immediate consequence of Theorem 2.1. But the following is a sharper characterization.

Corollary 2.2. *Let R be a prime ring of characteristic different 2 with extended centroid C and $f(X_1,\ldots,X_n)$ a noncentral multilinear polynomial over C. Let G be a nonzero generalized derivation of R such that*

$$\Big[G\big(f(r_1,\ldots,r_n)\big)f(r_1,\ldots,r_n), G\big(f(s_1,\ldots,s_n)\big)f(s_1,\ldots,s_n)\Big] = 0$$

for all $r_1,\ldots,r_n,s_1,\ldots,s_n \in R$. Then $f(X_1,\ldots,X_n)^2$ is central valued on R and moreover, there exists $a \in Q$ such that $G(x) = ax$ for all $x \in R$.

3. An application of a result in [5]

Throughout this section, R is always a prime ring. Let Q be the maximal right ring of quotients of R, and C be its center. It is known that any derivation $\delta\colon R \to R$ can be uniquely extended to a derivation of Q, denoted by δ also. A derivation $\delta\colon R \to R$ is called X-inner if its extension to Q is inner; that is, $\delta = \mathrm{ad}(b)$ for some $b \in Q$. Otherwise, it is called X-outer. In [8, Theorem 4], T.-K. Lee showed that a generalized derivation G of a prime ring R is of form $G(x) = ax + \delta(x)$ for some $a \in Q$ and a derivation δ of R. Moreover $a \in Q$ and δ are uniquely determined by G. Also δ is called the associated derivation of G. A generalized derivation G is called X-inner if its associated derivation is X-inner; otherwise it is called X-outer. Following [5], let

$$Q[t;\delta] := \{a_0 + a_1t + \cdots + a_nt^n \mid a_0,\ldots,a_n \in Q, n \geq 0\},$$

be the Ore extension of Q by δ endowed with the multiplication rule: $tx = xt + \delta(x)$ for $x \in Q$. A polynomial $f(t) = a_0 + a_1t + \cdots + a_nt^n \in Q[t;\delta]$ has degree n if $a_n \neq 0$, denoted by $\deg f(t) = n$, and is called monic if $a_n = 1$.

Given $f(t) = a_0 + a_1t + \cdots + a_nt^n \in Q[t;\delta]$ and a derivation δ of R, we define $f(\delta) = (a_0)_L\,\mathrm{id}_R + (a_1)_L\delta + \cdots + (a_n)_L\delta^n$, where id_R is the identity map of R.

Definition 3.1. *A derivation $\delta\colon R \to R$ is said to be quasi-algebraic if there exist $b_1,\ldots,b_{n-1},b \in Q$ such that for all $x \in R$,*

$$\delta^n(x) + b_1\delta^{n-1}(x) + \cdots + b_{n-1}\delta(x) = bx - xb.$$

The least integer n is called the quasi-algebraic degree or the outer degree of the derivation δ and is denoted by $out-deg(\delta)$. Clearly, $out-deg(\delta) = 1$ if and only if δ is X-inner. We also let $out-deg(\delta) = \infty$ if δ is not quasi-algebraic.

Remark 3.2. Let $\delta\colon R \to R$ be a quasi-algebraic derivation. We apply Kharchenko's theorem [6, Corollaries 2 and 3] (see also [7, Theorem 2]). If $\mathrm{char}\,R = 0$, then $\delta = \mathrm{ad}(b)$ for some $b \in Q$. If $\mathrm{char}\,R = p > 0$, then $\delta, \delta^p, \delta^{p^2}, \ldots$ are linearly dependent over C modulo inner derivations of Q. Let $s \geq 0$ be the minimal integer such that

$$\delta^{p^s}, \delta^{p^{s-1}}, \cdots, \delta^p, \delta$$

are linearly dependent over C modulo inner derivations of Q. By the minimality of s, there exist $\alpha_i \in C$ and $b \in Q$ such that

$$\delta^{p^s} + \alpha_1 \delta^{p^{s-1}} + \cdots + \alpha_s \delta = ad(b).$$

By the minimality of s again, it is easy to see that $\delta(\alpha_i) = 0$ and $\delta(b) \in C$. In this case, we have $out - deg\,(\delta) = p^s$.

Definition 3.3. *Let δ be a quasi-algebraic derivation of R. We define $p(t) := t$ if $out - deg\,(\delta) = 1$ and $p(t) := t^{p^s} + \alpha_1 t^{p^{s-1}} + \cdots + \alpha_s t$ if $char\, R = p > 0$. In either case, $p(t)$ is called the associated polynomial of δ. Note that $p(\delta) = ad(b)$ for some $b \in Q$ and $p(t) \in C^{(\delta)}[t; \delta] \subseteq Q[t; \delta]$, where $C^{(\delta)} := \{\beta \in C \mid \delta(\beta) = 0\}$, the subfield of constants of δ in C. Note that $C^{(\delta)}[t; \delta] = C^{(\delta)}[t]$.*

In [5, Theorem 1.5], the author and T.-K. Lee characterize a linear differential map in the following result.

Theorem 3.4. *Let R be a simple GPI-ring with a nonzero derivation δ. Then, for a nonzero polynomial $f(t) \in Q[t; \delta]$, $f(\delta)(R) \subseteq [R, R]$ if and only if δ is quasi-algebraic and $p(t) | f(t)$, where $p(t)$ is the associated polynomial of δ.*

In this section our main is to give an application of the above result to generalization derivations.

Theorem 3.5. *Let R be a simple GPI-ring, $b_0, \ldots, b_m \in Q$ with $b_0 \neq 0$ and G a nonzero generalized derivation of R. Assume that*

$$\sum_{i=0}^{m} b_i G^i(x) \in [R, R]$$

for all $x \in R$. If G is X-outer then there exist $b, c \in Q$ and $\alpha_1, \cdots, \alpha_s \in C$ such that

$$G^{p^s} + \alpha_1 G^{p^{s-1}} + \cdots + \alpha_s G = cx - xb$$

for all $x \in R$.

Proof. In view of [8, Theorem 4], there exist $a \in Q$ and a derivation δ of R such that $G(x) = ax + \delta(x)$ for all $x \in R$. Thus by a direct computation, it is easy to see that $\sum_{i=0}^{m} b_i G^i(x) = \sum_{i=0}^{m} w_i \delta^i(x)$ for some $w_i \in Q$. Denote $f(t) = \sum_{i=0}^{m} w_i t^i(x) \in Q[t; \delta]$. So by our hypothesis, we have $f(\delta)(R) \subseteq [R, R]$. It follows from Theorem 3.4 that δ is a quasi-algebraic. Therefore by Remark (3.2), since G is X-outer we may assume that $char(R) = p$, and so there exist $\alpha_i \in C$ and $b \in Q$ such that

$$\delta^{p^s} + \alpha_1 \delta^{p^{s-1}} + \cdots + \alpha_s \delta = \mathrm{ad}(b). \tag{4}$$

On the other hand, since $char(R) = p > 0$ there exist $c_1, c_2, \cdots, c_s \in Q$ such that

$$G^p(x) = c_1x + \delta^p(x), \;\; G^{p^2}(x) = c_2x + \delta^{p^2}(x), \; \cdots, \; G^{p^s}(x) = c_sx + \delta^{p^s}(x)$$

for all $x \in R$. In view of (4), we obtain

$$c_sx+\delta^{p^s}(x)+\alpha_1(c_{s-1}x+\delta^{p^{s-1}}(x))+\cdots+\alpha_s(ax+\delta(x)) = [b,x]+\big(c_s+\alpha_1c_{s-1}+\cdots+\alpha_sa\big)x$$

for all $x \in R$. It means

$$G^{p^s(x)}+\alpha_1G^{p^{s-1}}(x)+\cdots+\alpha_sG(x) = bx-xb+\big(c_s+\alpha_1c_{s-1}+\cdots+\alpha_{s-1}c_1+\alpha_sa\big)x$$

and so

$$G^{p^s}+\alpha_1G^{p^{s-1}}+\cdots+\alpha_sG = (b+w)x-xb$$

for all $x \in R$, where $w = c_s+\alpha_1c_{s-1}+\cdots+\alpha_sa \in Q$, as desired. This completes the proof.

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