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## ON THE PROLONGATIONS OF HOMOGENEOUS VECTOR BUNDLES

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**ABSTRACT.** In this paper, we introduce a study of prolongations of homogeneous vector bundles. We give an alternative approach for the prolongation. For a given homogeneous vector bundle  $E$ , we obtain a new homogeneous vector bundle. The homogeneous structure and its corresponding representation are derived. The prolongation of induced representation, which is an infinite dimensional linear representation, is also defined.

### 1. INTRODUCTION

In this study, we continue to work on prolongations. In our previous works [9], and [10] we have defined prolongations of finite-dimensional real representations of Lie groups and finite-dimensional real representations of Lie algebras. In [9], we obtained faithful representations on tangent bundles of Lie groups. In this work, we use the prolongations of these representations to give an alternative method to prolong a vector bundle, specially a homogeneous vector bundle, which also has group actions in its structure. We also give the definition of the prolongation of induced representations. In the literature, the well known method for prolongation is to use lifts ( for example vertical lifts or complete lifts) [15] or to use jet prolongations [14]. For example in [6], the second order tangent bundles are studied by using jets. For further information about jet manifolds, we refer to [4], [14].

Homogeneous vector bundles were studied, because of their applications to co-homologies and complex analytic Lie groups. In 1957, Raoul Bott [3] dealt with induced representations in the framework of complex analytic Lie groups. In 1964, Griffiths gave differential-geometric derivations of various properties of homogeneous complex manifolds. He gave some differential geometry applications to homogeneous vector bundles and to the study of sheaf co-homology [7]. In 1988, Purohit

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[13] showed that there is a one-to-one correspondence between homogeneous vector bundles and linear representations. Various other studies about homogeneous vector bundles can be found in the literature ([2], [8], [11]).

Moreover, in 1972 *R. W. Brockett and H. J. Sussmann* described how the tangent bundle of a homogeneous space can be viewed as a homogeneous space [5]. They associated every Lie group  $G$  with another Lie group  $G^* = Lie(G) \times G$  constructed as a semi direct product with the group operation given by

$$(a, g) \cdot (a', g') = (a + ad(g)(a'), gg') \quad (1.1)$$

where  $(a, g), (a', g) \in G^*$ . They also showed that if  $G$  acts on a manifold  $X$ , then there exists a left action of  $G^*$  on  $TX$  by

$$(a, g) \cdot v = d\sigma_g(v) + \bar{a}(g \cdot \pi(v)) \quad \text{for all } v \in TX. \quad (1.2)$$

Here  $\pi$  denotes the natural projection from  $TX$  onto  $X$  (i.e.  $\pi(v) = x$  if and only if  $v \in X$ ),  $\sigma_g : X \rightarrow X$  is the map  $x \rightarrow gx$ , and  $\bar{a}(x) = (\theta_x)_{*e}(a)$ , where  $a$  belongs to the Lie algebra  $Lie(G)$ ,  $e \in G$  represents the identity element of  $G$ , and  $\theta_x : G \rightarrow X$  smooth function that is defined as  $\theta_x(g) = gx$ . Clearly, both  $\sigma_g(v)$  and  $\bar{a}(g \cdot \pi(v))$  belong to  $T_{g \cdot \pi(v)}X$ , so the sum is well-defined.

This paper is organized as follows. In section 2 we give some basic definitions and theorems that we need for our proofs. In section 3, we give homogeneous vector bundle structure of the prolonged bundle. In section 4, we give the conclusion and future work.

## 2. PRELIMINARIES

First of all, we give the definition of a homogeneous vector bundle.

**Definition 2.1.** *Let  $G$  be a Lie group,  $F$  be a  $n$  dimensional real vector space, and  $G$  acts transitively on a manifold  $M$ . Let  $H$  be the isotropy subgroup of  $G$  at a fixed point  $p_0 \in M$  so that  $M$  becomes the coset space  $G/H$ . In addition, suppose  $G$  acts on the vector bundle  $E$  sitting over  $G/H$  so that its action on the base agrees with the usual action of  $G$  a coset. Then such a structure  $(E, \pi, M, F)$  is called a homogeneous vector bundle [13].*

**Theorem 2.2.** *Homogeneous vector bundles over  $G/H$  are in one-to-one correspondence with linear representations of  $H$  [13].*

Above mentioned representation is defined as follows:

Let  $(E, \pi, M, F)$  be a homogeneous vector bundle where  $M = G/H$ ,  $H$  be the isotropy subgroup of  $G$  at  $p_0$ , and  $F = \pi^{-1}(p_0)$ . Then, there exists a Lie group representation  $\sigma : H \rightarrow Aut(F)$  with

$$\begin{aligned} \sigma(h) : F &\rightarrow F \\ q &\rightarrow \sigma(h)(q) = hq \end{aligned} \quad (2.1)$$

where  $h \in H$ .

Conversely, if  $G$  is a Lie group with isotropy subgroup  $H$ ,  $F$  is a finite dimensional real vector space and  $\sigma : H \rightarrow \text{Aut}(F)$  is a representation, then there exists a homogeneous vector bundle  $(E, \pi, G/H, F)$  where  $E = G \times_H F$  [1, 12].

Homogeneous vector bundles also corresponds infinite dimensional real representations. This correspondence is illustrated in the following proposition.

**Proposition 2.3.** [13] *Let  $(E, \pi, M, F)$  be a homogeneous vector bundle, where  $M = G/H$ , and  $\Gamma(E)$  denotes (global) cross sections of the vector bundle  $E$ . For all  $g \in G$ ,  $\rho(g)$  can be defined by the following:*

$$\begin{aligned} \rho(g) : \Gamma(E) &\rightarrow \Gamma(E) \\ \psi &\rightarrow \rho(g)(\psi) : M \rightarrow E \\ p &\rightarrow (\rho(g)(\psi))(p) = g \cdot \psi(g^{-1}p) \end{aligned} \tag{2.2}$$

Clearly,  $\rho$  is a representation of  $G$  in  $\Gamma(E)$  which is induced by the representation  $\sigma$  that is defined in Theorem 2.2. We call the representation  $\rho$  as the induced representation of  $E$ .

**Theorem 2.4.** *If a Lie group  $G$  acts transitively and with maximal rank on a differentiable manifold  $X$ , then  $G^*$  acts transitively and with maximal rank on the tangent bundle of  $X$  [5].*

We have following remarks for above theorem:

- Remark 2.5.**
- (1) *Clearly, above result implies that the tangent bundle of a coset space  $G/H$  is again a coset space and moreover, is of the form  $G^*/K$  for some closed subgroup  $K$  of  $G^*$ .*
  - (2) *If  $H$  is a closed subgroup of  $G$ , then  $H^*$  can be identified, in an obvious way, with a closed subgroup of  $G^*$ . One verifies easily that the isotropy group of  $0_x$  corresponding to the action of  $G^*$  on  $TX$  is precisely  $H_x^*$ , where  $H_x$  is the isotropy group of  $x$  corresponding to the action of  $G$  on  $X$ . In particular, we have the diffeomorphism  $T(G/H) \simeq G^*/H^*$ .*

**Definition 2.6.** *Let  $\Phi$  be a  $n$ -dimensional Lie group representation on  $G$ . "The prolongation of the representation  $\Phi$ ", which is denoted by  $\tilde{\Phi}$  is given as in follows:*

$$\begin{aligned} \tilde{\Phi} &: TG \rightarrow GL(2n) \\ (a, v) &\rightarrow \tilde{\Phi}(a, v) = \begin{pmatrix} \Phi(a) & 0 \\ \left[ (d(\Phi)_e(v))_j^i \right] \cdot \Phi(a) & \Phi(a) \end{pmatrix} \end{aligned} \tag{2.3}$$

### 3. PROLONGATION OF A HOMOGENEOUS VECTOR BUNDLE

In this section, we'll introduce a homogeneous vector bundle with prolonged Lie group representation, which is defined in [9].

**Main Result:**

Let  $(E, \pi, G|H, F)$  be the homogeneous vector bundle with the corresponding Lie group representation  $\sigma : H \rightarrow \text{Aut}(F)$ . Using the prolongation of the representation  $\sigma$  defined in [9], we have

$$\tilde{\sigma}(dR_h(a)) = \begin{pmatrix} \sigma(h) & 0 \\ d(\sigma)_e(a).\sigma(h) & \sigma(h) \end{pmatrix} \quad (3.1)$$

where  $\tilde{\sigma} : TH \rightarrow \text{Aut}(TF)$  is the prolongation of representation  $\sigma$ . Getting composition of  $\Theta$  and  $\tilde{\sigma}$ , we define  $\sigma^* : H^* \rightarrow \text{Aut}(TF)$  as follows:

$$\sigma^*(a, h) = \begin{pmatrix} \sigma(h) & 0 \\ d(\sigma)_e(a).\sigma(h) & \sigma(h) \end{pmatrix} \quad (3.2)$$

where  $\Theta$  denotes the natural diffeomorphism  $\Theta : H^* \rightarrow TH$ . In the following configuration, we give the summary of what implications we will have next.

$$E \longleftrightarrow \sigma \longrightarrow \sigma^* \longleftarrow E^* \quad (3.3)$$

where  $E = (E, \pi, G/H, F)$ ,  $\sigma : H \rightarrow \text{Aut}(F)$ ,  $\sigma^* : H^* \rightarrow \text{Aut}(TF)$  and  $E^* = (E^*, \pi^*, G^*/H^*, TF)$ . Now we define the new induced action which will be used for defining equivalence classes.

**3.1. The Prolonged Action of  $H^*$  on  $G^* \times TF$ :** Using the above representation  $\sigma^*$ , and the natural action of  $G^*$  to its coset space  $G^*/H^*$ , we have the following induced action  $\tilde{\alpha} : (G^* \times TF) \times H^* \rightarrow G^* \times TF$  which can be obtained by the coset space definition:

$$\tilde{\alpha}(a, g, v, b, h) = ((a, g).(b, h), \sigma^*((b, h)^{-1})(v)). \quad (3.4)$$

**Proposition 3.1.** *If  $v = (\xi, u) \in TF$  and  $\sigma^*((b, h)^{-1})(v) = (\tilde{\xi}, \tilde{u})$ , then*

$$\tilde{\alpha}(a, g, v, b, h) = (a + \text{adj}(g)(b), gh, \tilde{\xi}, \tilde{u}). \quad (3.5)$$

**Proof.** Since  $(a, g).(b, h) = (a + \text{adj}(g)(b), gh)$  and  $(b, h)^{-1} = (\text{adj}(h^{-1})(-b), h^{-1})$ , we have

$$\sigma^*((b, h)^{-1})(v) = \sigma^*(\text{adj}(h^{-1})(-b), h^{-1})(\xi, u)$$

Using the definition of  $\sigma^*$  we have

$$\begin{aligned} &= \begin{pmatrix} \sigma(h^{-1}) & 0 \\ d(\sigma)_e(-\text{adj}(h^{-1})(b)) & \sigma(h^{-1}) \end{pmatrix} \cdot \begin{pmatrix} \xi \\ u \end{pmatrix} \\ &= (\sigma(h^{-1})(\xi), [(d\sigma)_e(-\text{adj}(h^{-1})(b))]_j^i \xi_i x_j + [\sigma(h^{-1})]_j^i u_i \dot{x}_j) \end{aligned} \quad (3.6)$$

$$= (\tilde{\xi}, \tilde{u}) \quad (3.7)$$

where  $v = (\xi, u) \in TF$  for all  $(a, g) \in G^*$  and  $(b, h) \in H^*$ . Therefore, using equation (3.7), we finish the proof.  $\square$

Using above action, it is possible to form an equivalence relation on  $G^* \times TF$ .

**3.2. The Prolonged Equivalence Relation:** If we use above induced action, we have the following equivalence relation on  $G^* \times TF$  by [1, 12]:

$((a, g), (\xi, u)) \simeq ((a', g'), (\xi', u'))$  if and only if there exists  $(b, h) \in H^*$  such that the following equation holds

$$((a', g'), (\xi', u')) = ((a, g), (\xi, u)) \cdot (b, h). \quad (3.8)$$

The next corollary gives the simplified form of equivalence relation that we have defined above.

**COROLLARY 3.2.** *The equivalence relation defined by the equation (3.8) can be given by the following.*

$$((a, g), (\xi, u)) \simeq ((a', g'), (\xi', u')) \Leftrightarrow \begin{cases} a' = a + \text{adj}(g)(b), \\ g' = gh, \\ \xi' = \sigma(h^{-1})(\xi), \\ u' = [(d\sigma)_e(-\text{adj}(h^{-1})(b))]_j^i \xi_i x_j \\ \quad + [\sigma(h^{-1})]_j^i u_i \dot{x}_j. \end{cases} \quad (3.9)$$

**Proof.** If  $((a, g), (\xi, u)) \simeq ((a', g'), (\xi', u'))$ , then there exist  $(b, h) \in H^*$  such that

$$\begin{aligned} ((a', g'), (\xi', u')) &= ((a, g), (\xi, u)) \cdot (b, h) \\ &= \sigma^*(a, g, (\xi, u), b, h) \end{aligned} \quad (3.10)$$

Using equation (3.6), we have  $\xi' = \sigma(h^{-1})(\xi)$  and  $u' = [(d\sigma)_e(-\text{adj}(h^{-1})(b))]_j^i \xi_i x_j + [\sigma(h^{-1})]_j^i u_i \dot{x}_j$ .

Moreover, since  $\sigma^*(a, g, (\xi, u), b, h) \in T_{(a, g) \cdot (b, h)}$ , then we have

$$\begin{aligned} (a', g') &= (a, g) \cdot (b, h) \\ &= (a + \text{adj}(g)(b), gh) \end{aligned} \quad (3.11)$$

which finishes the proof.  $\square$

**Definition 3.3.** *We denote  $E^*$  to be set of equivalence classes follows from (3.9), i.e.*

$$E^* = G^* \times_{H^*} TF$$

and we refer  $E^*$  as the prolonged vector bundle.

**Remark 3.4.** *The bundle projection of the prolonged bundle is*

$$\begin{aligned} \pi_{E^*}^* : G^* \times_{H^*} TF &\rightarrow G^*/H^* \\ ((a, g), v)H^* &\rightarrow \pi_{E^*}^*((a, g), v)H^* = (a, g)H^* \end{aligned}$$

and local trivialization of the prolonged bundle is

$$\begin{aligned} \psi^* : (\pi_{E^*}^*)^{-1}(V) &\rightarrow V \times TF \\ ((a, g), v)H^* &\rightarrow \psi^*((a, g), v)H^* = ((a, gH), \sigma^*(b, h)(v)) \end{aligned} \quad (3.12)$$

where  $V \subset T(G/H)$  is an open subset.

So far, we have given the structures of the prolonged bundle. In the following, we define a new prolongation that is obtained by the prolongation of homogeneous vector bundles.

**Definition 3.5.** *Let  $\rho : G \rightarrow \text{Aut}(\Gamma(E))$  be the induced representation of the homogeneous vector bundle  $E$ . Then  $\rho^* : TG \rightarrow \text{Aut}(\Gamma(E^*))$  is called the prolongation of the induced representation  $\rho$ , where  $E^*$  denotes the prolongation of  $E$ .*

#### 4. CONCLUSION AND FUTURE WORK

In this paper, we have introduced a study of prolongations of homogeneous vector bundles. We have used one-to-one correspondence of homogeneous vector bundles and finite dimensional Lie group representations, and we have defined prolongations of homogeneous vector bundles. We introduced the geometric structures of this new bundle, such as local trivialization of the bundle, equivalence classes and the bundle projection. We have defined the prolongations of induced representations which are infinite dimensional linear representations. In future, we plan to study on prolongations of infinite dimensional representations by using one-to-one correspondence of homogeneous vector bundles.

#### REFERENCES

- [1] Adams, J. F., *Lectures on Lie Groups*, (Benjamin Inc., New York, 1969).
- [2] Boralevi, Ada., *Sections of homogeneous vector bundles*, Journal of Algebra 323, 2301-2317, (2010).
- [3] Bott, Raoul., *Homogeneous Vector Bundles*, Annals of mathematics, Vol. 66, No. 2, (1957).
- [4] Cordero, L. A., Dodson, C.T.J., De Leon, M., *Differential Geometry of Frame Bundles*, Kluwer Academic Press, (1989).
- [5] Brockett, R.W., Sussmann, H. J., *Tangent Bundles of Homogeneous Spaces are Homogeneous Spaces*, Proceedings of the American Mathematical Society, **35**, 550-551 (1972).
- [6] Fisher, Robert and Laquer, H. Turner, *Second order tangent vectors in Riemannian geometry*. J. Korean Math. Soc.,**36**(5):959-1008, (1999).
- [7] Griffiths, Philip A., *On the Differential Geometry of Homogeneous Bundles*, (1963).
- [8] Haboush, W. J., *Homogeneous Vector Bundles and Reductive Subgroups of Reductive Algebraic Groups*, American Journal of Mathematics, Vol. 100, No. 6 , pp. 1123-1137 (1978).
- [9] Kadioglu, H., Esin, E., *On the Prolongations of Representations of Lie Groups*, Hadronic J. **33** ,183-196, (2010).
- [10] Kadioglu, H., Esin, E., Yaylı, Y. *Prolongations of Lie Algebra Representations*, Advances and Applications in Math. Sci. 10, no. 5, 533-542.(2011).
- [11] Kobayashi, S. *Homogeneous Vector Bundles and Stability*, Nagoya Math. J. Vol. 101, 37-54,(1986).

- [12] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry 1*, (Interscience Publishers, New York, 1963).
- [13] Purohit, G. N., *Vector Bundles and Induced Representations of Lie Groups*, Ganita Sandesh, **2**, 17-20(1988).
- [14] Saunders D.J., *The Geometry of Jet Bundles*, (Cambridge University Press, Cambridge- New York, 1989).
- [15] Yano, K, Ishihara, S., *Tangent and Cotangent Bundles*, (M. Dekker, New York, 1973).

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