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ON FOCAL SURFACES FORMED BY TIMELIKE NORMAL RECTILINEAR CONGRUENCE

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Abstract. In this paper, we investigate the focal surfaces obtained by the timelike normal rectilinear congruence whose straight lines are normal to spacelike surface, in the Minkowski 3-Space. Then, the relation $\lambda_1 K_{\mathbf{Z}^1} + \lambda_2 K_{\mathbf{Z}^2} = 0$ between Gaussian curvatures of these focal surfaces are examined and finally the relation which concern with area preserving correspondence between the focal surfaces are given.

1. Introduction

The focal surface is the locus of points where correspond centres of curvature for all points of a surface. In general, since the centers of curvature are formed by two families of line of curvature, it consist of two sheets. it also is called *surface of centers* or *evolute*. This concept justified by the fact that a line congruence can be considered as the set of lines touching two surfaces. The focal surfaces are used as interrogation tool in order to have the information concerning quality of the surface.

Papantoniou, [2], investigated the smooth surface of $\mathbb{R}^3$ the normals of which establish a rectilinear congruence with focal surfaces. Tsagas, [6], studied the rectilinear congruences formed by the tangents to a one parametric family of curves on a surface. He gave the conditions in order that the straight lines of this rectilinear congruences establish an area preserving correspondence between its focal surfaces. The geometry of the focal surfaces was studied by [5] in the Minkowski 3-space via the bifurcation set of the family of distance squared functions on the surface. Also, Şimşek and Özdemir [7] examined the sub-parabolic lines and ridge lines in the Minkowski 3-space, which correspond to cuspidal edges and parabolic lines on the focal surfaces, respectively.

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In the present paper, we examine the focal surfaces formed by a timelike rectilinear congruence, whose straight lines are normal to the spacelike surface. Firstly, the equations of the Gauss–Codazzi and Darboux frame of the spacelike surface is given. Then, the parametric equations of the focal surface which correspond timelike rectilinear congruence whose straight lines are normal to spacelike surface are determined in the Minkowski 3-space. The equation $\lambda_1 K_{\mathbf{Z}^1} + \lambda_2 K_{\mathbf{Z}^2} = 0$ between Gaussian curvatures of these focal surfaces are studied. Besides, in case the spacelike surface is Weingarten and minimal spacelike surface, this equation is discussed. Finally, the relation preserving the area element between these focal surfaces is examined. Especially, the connection between preservation of the area element with above the equation is given.

2. The Basic Equations of The Spacelike Surfaces

The surface in the Minkowski space $\mathbb{E}_1^3$ is called *spacelike* or *timelike* if the normal vectors of the surface are timelike or spacelike, respectively.

A line congruence is two-parameter family of lines in the Minkowski space $\mathbb{E}_1^3$. *A rectilinear congruence* C which is special class of the line congruence is defined by the vector equation in $\mathbb{E}_1^3$

$$C : R(u,v) = \mathbf{r}(u,v) + \lambda \mathbf{e}(u,v), \quad \lambda \in \mathbb{R}, \tag{1}$$

where $S : \mathbf{x} = \mathbf{x}(u,v)$ is called the reference surface and $\mathbf{e} = \mathbf{e}(u,v)$ is the unit vector which determines the directions of the straight lines of C. If $\mathbf{e} = \mathbf{e}(u,v)$ is the normal vector of surface S, then C is called a *normal rectilinear congruence.* The rectilinear congruence C described by (1) is called *spacelike* (resp. *timelike*) if $\mathbf{e} = \mathbf{e}(u,v)$ is spacelike (resp. timelike), see [8].

Let $S : \mathbf{x} = \mathbf{x}(u,v)$ represents a non-spherical or non-developable regular spacelike surface in $\mathbb{E}_1^3$ and denote by g_{ij}, h_{ij} $(i,j = 1,2)$ the coefficients of the first and second fundamental forms of S. Suppose that the u-curves and v-curves of this parametrization are lines of curvature.

We consider the tangents vectors of the u-curves and v-curves as the unit vectors $\mathbf{e}_1 = \mathbf{e}_1(u,v)$, $\mathbf{e}_2 = \mathbf{e}_2(u,v)$ respectively and the unit vector $\mathbf{e}_3 = \mathbf{e}_3(u,v)$ as the normal to the surface S at the regular point (u,v). Then, we have

$$\mathbf{e}_1 = \frac{\mathbf{x}_u}{\sqrt{g_{11}}}, \quad \mathbf{e}_2 = \frac{\mathbf{x}_v}{\sqrt{g_{22}}}, \quad \mathbf{e}_3 = \mathbf{e}_2 \times \mathbf{e}_1, \tag{2}$$

where " $\times$ " is the cross product in $\mathbb{E}_1^3$ and the frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the moving frame on the spacelike surface S. This moving frame is called Darboux frame. The equations of Gauss–Codazzi of the spacelike surface are

$$K = -\frac{1}{\sqrt{g_{11}g_{22}}}\left[\left(\frac{\left(\sqrt{g_{22}}\right)_u}{\sqrt{g_{11}}}\right)_u + \left(\frac{\left(\sqrt{g_{11}}\right)_v}{\sqrt{g_{22}}}\right)_v\right] \tag{3}$$

$$k_v = \frac{(g_{11})_v}{2g_{11}}(k^* - k) \qquad (k^*)_u = \frac{(g_{22})_u}{2g_{22}}(k - k^*) \tag{4}$$

where K is the Gaussian curvature and $k=-\dfrac{h_{11}}{g_{11}}$, $k^{*}=-\dfrac{h_{22}}{g_{22}}$ are principal curvatures of the spacelike surface.

We choose arc length parameters t_1, t_2 for u-curves and v-curves, respectively. Then, we have $dt_1=\sqrt{g_{11}}du$, $dt_2=\sqrt{g_{22}}dv$. Thus, the derivative formula of the moving frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ with respect to t_1 can be stated as

$$\frac{\partial}{\partial t_1}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}=\begin{pmatrix}0 & q & k\\ -q & 0 & 0\\ k & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} \tag{5}$$

where $q=-\dfrac{(g_{11})_v}{2g_{11}\sqrt{g_{22}}}$ is geodesic curvature of the u-curves. Similarly, the derivative formula of the moving frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ with respect to t_2 is

$$\frac{\partial}{\partial t_2}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}=\begin{pmatrix}0 & q^{*} & 0\\ -q^{*} & 0 & k^{*}\\ 0 & k^{*} & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} \tag{6}$$

where $q^{*}=\dfrac{(g_{22})_u}{2g_{22}\sqrt{g_{11}}}$ is geodesic curvature of the v-curves, [9].

We denote the derivatives of any function φ with respect to t_1 and t_2 by $\dfrac{\partial\varphi}{\partial t_1}=\varphi_1$ and $\dfrac{\partial\varphi}{\partial t_2}=\varphi_2$. Since k, k^{*}, q, q^{*} are the invariant quantities of the lines of curvature on the spacelike surface, in accordance with the parameters t_1, t_2 the equations of Gauss–Codazzi take the form

$$q^2+(q^{*})^2=q_2-q_1^{*}-kk^{*} \tag{7}$$
$$q\,(k-k^{*})-k_2=0,\qquad q^{*}\,(k-k^{*})-k_1^{*}=0.$$

The unit normal vectors $\mathbf{e}_3\,(u,v)$ of the spacelike surface S, form a timelike normal rectilinear congruence such that the parametric equation of the timelike normal rectilinear congruence is

$$C:R(u,v,\lambda)=\mathbf{x}(u,v)+\lambda\mathbf{e}_3(u,v),\quad \lambda\in(-\infty,+\infty)\,. \tag{8}$$

Now, we shall find the focal surfaces formed by the timelike normal rectilinear congruence given by (8) with the following theorem.

Theorem 1. *For timelike normal rectilinear congruence C which is described by* (8)*, the parametric equations of the focal surfaces of C are*

$$\mathbf{Z}^1:\mathbf{Z}^1\,(u,v)=\mathbf{x}\,(u,v)-\frac{1}{k}\mathbf{e}_3\,(u,v) \tag{9}$$
$$\mathbf{Z}^2:\mathbf{Z}^2\,(u,v)=\mathbf{x}\,(u,v)-\frac{1}{k^{*}}\mathbf{e}_3\,(u,v)\,.$$

The focal surfaces $\mathbf{Z}^1,\mathbf{Z}^2$ of C are timelike surfaces and $\mathbf{e}_1$, $\mathbf{e}_2$ are normal to the focal surface $\mathbf{Z}^1$, $\mathbf{Z}^2$, respectively.

Proof. Denote the parametric equation of the focal surface by

$$\mathbf{Z} = \mathbf{x}(u,v) + \xi\,(u,v)\,\mathbf{e}_3(u,v). \tag{10}$$

Then $\xi\,(u,v)$ is the root of the quadratic equation

$$(\mathbf{R}_u, \mathbf{R}_v, \mathbf{R}_\xi) = 0, \tag{11}$$

where

$$\mathbf{R}_u = \mathbf{x}_u + \xi\,(\mathbf{e}_3)_u\,, \quad \mathbf{R}_v = \mathbf{x}_v + \xi\,(\mathbf{e}_3)_v\,, \quad \mathbf{R}_\xi = \mathbf{e}_3.$$

Using the equation (5) and (6), we can write

$$\mathbf{R}_u = (\sqrt{g_{11}} + \sqrt{g_{11}}\xi k)\,\mathbf{e}_1, \quad \mathbf{R}_v = (\sqrt{g_{22}} + \sqrt{g_{22}}\xi k^*)\,\mathbf{e}_2, \quad \mathbf{R}_\xi = \mathbf{e}_3. \tag{12}$$

From (12), the roots of the quadratic equation (11) are

$$\xi_1(u,v) = -\frac{1}{k} \qquad \xi_2(u,v) = -\frac{1}{k^*}.$$

Substituting in the (10), the first claim is proved. If we take derivative of the equations in the (9) with respect to t_1 and t_2, we get

$$\mathbf{Z}_1^1 = \mathbf{e}_1 + \frac{k_1}{k^2}\mathbf{e}_3 - \frac{1}{k}\,(k\mathbf{e}_1) = \frac{k_1}{k^2}\mathbf{e}_3, \tag{13}$$

$$\mathbf{Z}_2^1 = \mathbf{e}_2 + \frac{k_2}{k^2}\mathbf{e}_3 - \frac{1}{k}\,(k^*\mathbf{e}_2) = \frac{(k-k^*)}{k}\mathbf{e}_2 + \frac{k_2}{k^2}\mathbf{e}_3$$

and

$$\mathbf{Z}_1^2 = \mathbf{e}_1 + \frac{k_1^*}{(k^*)^2}\mathbf{e}_3 - \frac{1}{k^*}\,(k\mathbf{e}_1) = \frac{(k^*-k)}{k^*}\mathbf{e}_1 + \frac{k_1^*}{(k^*)^2}\mathbf{e}_3, \tag{14}$$

$$\mathbf{Z}_2^2 = \mathbf{e}_2 + \frac{k_2^*}{(k^*)^2}\mathbf{e}_3 - \frac{1}{k^*}\,(k^*\mathbf{e}_2) = \frac{k_2^*}{(k^*)^2}\mathbf{e}_3.$$

From (13) and (14) it follows that $\mathbf{e}_1$ is normal to the focal surface $\mathbf{Z}^1$ and $\mathbf{e}_2$ is normal to the focal surface $\mathbf{Z}^2$. Then, since $\mathbf{e}_1$ and $\mathbf{e}_2$ are spacelike vectors, the focal surfaces $\mathbf{Z}^1, \mathbf{Z}^2$ of C are timelike surfaces. The second claim is proved. □

Let's find the Gaussian curvatures of the timelike focal surfaces which are defined by (9). If the first and second fundamental forms of the focal surface $\mathbf{Z}^1$ are denoted by E^1, F^1 G^1 and l^1, m^1, n^1, respectively, then

$$E^1 = -g_{11}\frac{(k_1)^2}{(k)^4},\ F^1 = -\sqrt{g_{11}}\sqrt{g_{22}}\frac{k_1k_2}{(k)^4},\ G^1 = g_{22}\left(-\frac{(k_2)^2}{(k)^4} + \frac{(k-k^*)^2}{(k)^2}\right) \tag{15}$$

and

$$l^1 = g_{11}\frac{k_1}{k}, \qquad m^1 = 0, \quad n^1 = -g_{22}\frac{q^*\,(k-k^*)}{k}. \tag{16}$$

Then, substituting the values in the (15) and (16) into the formula

$$K_{\mathbf{Z}^1} = \frac{l^1n^1 - (m^1)^2}{E^1G^1 - (F^1)^2},$$

the Gaussian curvature of the timelike focal surface $\mathbf{Z}^1$ becomes

$$K_{\mathbf{Z}^1} = \frac{q^* (k)^4}{k_1 (k - k^*)}. \tag{17}$$

If we show the first and second fundamental forms of the focal surface $\mathbf{Z}^2$ as E^2, F^2, G^2 and l^2, m^2, n^2, then

$$E^2 = g_{11} \left(\frac{(k^* - k)^2}{(k^*)^2} - \frac{(k_1^*)^2}{(k^*)^4} \right), \tag{18}$$

$$F^2 = -\sqrt{g_{11}}\sqrt{g_{22}} \frac{k_1^* k_2^*}{(k^*)^4}, \quad G^2 = -g_{22} \frac{(k_2^*)^2}{(k^*)^4}$$

and

$$l^2 = g_{11} \frac{q (k^* - k)}{k^*} \tag{19}$$

$$m^2 = 0, \quad n^2 = g_{22} \frac{k_2^*}{k^*}.$$

Hence, the Gaussian curvature of $\mathbf{Z}^2$ is

$$K_{\mathbf{Z}^2} = -\frac{q (k^*)^4}{k_2^* (k^* - k)}. \tag{20}$$

Now, we investigate the spacelike surfaces the Gaussian curvatures of the timelike focal surfaces of which satisfy the equation

$$\lambda_1 K_{\mathbf{Z}^1} + \lambda_2 K_{\mathbf{Z}^2} = 0, \qquad \lambda_1, \lambda_2 \in \mathbb{R}. \tag{21}$$

Theorem 2. *Let k, k^*, q, q^* be continuous functions such that k, k^*, q, q^* and $k - k^*$ are not-zero. The solutions of the four partial differential equations* (7) *and*

$$k^3 \frac{k_1^*}{k_1} - \lambda (k^*)^3 \frac{k_2}{k_2^*} = 0 \tag{22}$$

where $\lambda \in \mathbb{R}$, determine a spacelike surface S in $\mathbb{E}_1^3$, which its parameter curves are lines of curvature and k, k^* and q, q^* are principal curvatures and geodesic curvatures of its parameter curves, respectively such that the Gaussian curvatures of the timelike focal surfaces formed by the timelike normal rectilinear congruence of S satisfy the equation (21)*. Moreover, the parametric curves on these timelike focal surfaces are conjugate.*

Proof. Let us examine the certain compatibility conditions

$$(\mathbf{e}_1)_{12} = (\mathbf{e}_1)_{21}, \qquad (\mathbf{e}_2)_{12} = (\mathbf{e}_2)_{21}. \tag{23}$$

From (23), we get

$$q_2 + kk^* = q_1^*, \quad qk^* + k_2 = 0, \quad -q^* k + k_1^* = 0. \tag{24}$$

Then, using (17), (20), and (24), the relation (21) is equivalent to the equation (22), where $\lambda = \dfrac{\lambda_2}{\lambda_1} \in \mathbb{R}$.

Since we have $m^1 = m^2 = 0$, the parametric curves on the timelike focal surfaces form a conjugate system. □

2.1. **The Relation $\lambda_1 K_{\mathbf{Z}^1} + \lambda_2 K_{\mathbf{Z}^2} = 0$ In Case of Spacelike Weingarten Surfaces.** Suppose that the surface S is a spacelike Weingarten surface, i.e., there exists a function τ on S and two functions f, g (Weingarten functions) of one variable such that

$$k = f(\tau), \qquad k^* = g(\tau). \tag{25}$$

Theorem 3. *Let S be a spacelike Weingarten surface such that its parameter curves are lines of curvature and suppose that S has the relation*

$$g(\tau) = \left(\frac{1}{\left(1/\sqrt{f(\tau)}\right) + c}\right), \quad c \in \mathbb{R}. \tag{26}$$

The Gaussian curvatures of the timelike focal surfaces obtained by the normal timelike rectilinear congruence of S satisfy the equation $K_{\mathbf{Z}^1} + K_{\mathbf{Z}^2} = 0$.

Proof. From (24) and (25) the Gaussian curvatures of the timelike focal surfaces $\mathbf{Z}^1$ and $\mathbf{Z}^2$ take the forms

$$K_{\mathbf{Z}^1} = \frac{k^3}{(k - k^*)} \frac{dk^*}{dk}, \qquad K_{\mathbf{Z}^2} = -\frac{(k^*)^3}{(k - k^*)} \frac{dk}{dk^*}. \tag{27}$$

So, the relation (21) can be written as

$$k^3 \frac{dk^*}{dk} - \lambda (k^*)^3 \frac{dk}{dk^*} = 0. \tag{28}$$

The relation (26) implies

$$\frac{1}{\sqrt{k^*}} - \frac{1}{\sqrt{k}} = c. \tag{29}$$

Hence, using (29) we can write the Gaussian curvatures of the timelike focal surfaces $\mathbf{Z}^1$ and $\mathbf{Z}^2$ as

$$K_{\mathbf{Z}^1} = \frac{\sqrt{(kk^*)^3}}{(k - k^*)}, \qquad K_{\mathbf{Z}^2} = -\frac{\sqrt{(kk^*)^3}}{(k - k^*)} \tag{30}$$

i.e., we have $\lambda = 1$ and the relation $K_{\mathbf{Z}^1} + K_{\mathbf{Z}^2} = 0$. □

In the case of spacelike Weingarten surface, the Codazzi equations (4) are reduced to

$$\frac{\partial}{\partial v}\left[\ln\left(\sqrt{g_{11}}k\right) - \int \frac{d\left(\frac{1}{k}\right)}{\frac{1}{k^*} - \frac{1}{k}}\right] = 0, \qquad \frac{\partial}{\partial u}\left[\ln\left(\sqrt{g_{22}}k^*\right) - \int \frac{d\left(\frac{1}{k^*}\right)}{\frac{1}{k} - \frac{1}{k^*}}\right] = 0. \tag{31}$$

Using equations (31) we get

$$g_{11}=\frac{\varphi^{2}\left(u\right)e^{2\int\frac{d\left(\frac{1}{k}\right)}{\frac{1}{k^{*}}-\frac{1}{k}}}}{k^{2}},\qquad g_{22}=\frac{h^{2}\left(u\right)e^{2\int\frac{d\left(\frac{1}{k^{*}}\right)}{\frac{1}{k}-\frac{1}{k^{*}}}}}{\left(k^{*}\right)^{2}}.$$

If we consider new parameters on S, u_1, v_1 are connected with u, v by the relations

$$u_{1}=\int\varphi\left(u\right)du,\qquad v_{1}=\int h\left(v\right)dv$$

and call them again u, v, then we have

$$g_{11}=\frac{e^{2\int\frac{d\left(\frac{1}{k}\right)}{\frac{1}{k^{*}}-\frac{1}{k}}}}{k^{2}},\qquad g_{22}=\frac{e^{2\int\frac{d\left(\frac{1}{k^{*}}\right)}{\frac{1}{k}-\frac{1}{k^{*}}}}}{\left(k^{*}\right)^{2}}. \tag{32}$$

2.2. **The Relation $\lambda_1 K_{\mathbf{Z}^1}+\lambda_2 K_{\mathbf{Z}^2}=0$ In Case of Spacelike Minimal Surfaces.** Suppose that the surface S is spacelike minimal surface ,i.e., the equation $k+k^*=0$ is satisfied.

Theorem 4. *The Gaussian curvatures of the timelike focal surfaces formed by the timelike normal rectilinear congruence of spacelike minimal surface, whose parameter curves are lines of curvature, satisfy the equation $K_{\mathbf{Z}^1}-K_{\mathbf{Z}^2}=0$ such that first and second fundamental forms of spacelike minimal surface are given by* (35) .

Proof. Using the formulas (27) , the Gaussian curvatures of the timelike focal surfaces become

$$K_{\mathbf{Z}^1}=-\frac{k^2}{2},\qquad K_{\mathbf{Z}^2}=-\frac{k^2}{2} \tag{33}$$

which means that $\lambda=-1$ and $K_{\mathbf{Z}^1}-K_{\mathbf{Z}^2}=0$. For the spacelike minimal surface, we get

$$g_{11}=\frac{1}{k},\qquad g_{22}=\frac{1}{k} \tag{34}$$

by means of (32). Then, we can write the first and second fundamental forms of spacelike minimal surface S as

$$\begin{aligned} g_{11}&=\frac{1}{k}, & g_{12}&=0 & g_{22}&=\frac{1}{k},\\ h_{11}&=-1, & h_{12}&=0 & h_{22}&=1. \end{aligned} \tag{35}$$

□

Theorem 5. *The timelike focal surfaces formed by the timelike normal rectilinear congruence of the spacelike minimal surface S, are singular at the points where the principal curvatures of S are constant such that theirs Gaussian curvatures satisfy the equation $K_{\mathbf{Z}^1}-K_{\mathbf{Z}^2}=0$ and the theirs first and second fundamental forms are given by* (37) *and* (38) .

Proof. When we take $k = a$, $a \in \mathbb{R}$, the equations (33) are reduced to

$$K_{\mathbf{Z}^1} = -\frac{a^2}{2}, \qquad K_{\mathbf{Z}^2} = -\frac{a^2}{2}. \tag{36}$$

Besides, from (15) and (16), the first and second fundamental forms of the timelike focal surfaces $\mathbf{Z}^1$ and $\mathbf{Z}^2$ are given by

$$E^1 = 0 \quad F^1 = 0 \quad G^1 = \frac{4}{a} \quad l^1 = 0 \quad m^1 = 0 \quad n^1 = 0 \tag{37}$$

$$E^2 = \frac{4}{a} \quad F^2 = 0 \quad G^2 = 0 \quad l^2 = 0 \quad m^2 = 0 \quad n^2 = 0. \tag{38}$$

□

Theorem 6. *Let S be a spacelike minimal surface whose the principal curvatures $k(\tau)$ and $k^*(\tau)$ are given by*

$$k(\tau) = \frac{2}{\sinh^2 \tau}$$

and

$$k^*(\tau) = -\frac{2}{\sinh^2 \tau}$$

where $\tau = t_1 + t_2$ or $\tau = t_1 - t_2$, respectively. Then, the Gaussian curvatures of these timelike focal surfaces satisfy the equations $K_{\mathbf{Z}^1} \pm K_{\mathbf{Z}^2} = 0$ and the first and second fundamental forms of the timelike focal surfaces, obtained by timelike normal rectilinear congruence of S, are given by (39) and (40) or (42) and (43), respectively.

Proof. If we take $k = f(\tau)$, $\tau = t_1 + t_2$, then $k^* = -f(\tau)$ and the equations (33) and $K_{\mathbf{Z}^1} - K_{\mathbf{Z}^2} = 0$ are valid. Furthermore, the first and second fundamental forms of the timelike focal surfaces $\mathbf{Z}^1$ and $\mathbf{Z}^2$ are reduced to

$$\begin{aligned} &E^1 = -\frac{(k_\tau)^2}{k^5} \quad F^1 = -\frac{(k_\tau)^2}{k^5} \quad G^1 = -\frac{(k_\tau)^2}{k^5} + \frac{4}{k} \\ &l^1 = \frac{k_\tau}{k^2} \quad m^1 = 0 \quad n^1 = \frac{2k_\tau}{k^2} \end{aligned} \tag{39}$$

$$\begin{aligned} &E^2 = -\frac{(k_\tau)^2}{k^5} + \frac{4}{k} \quad F^2 = -\frac{(k_\tau)^2}{k^5} \quad G^2 = -\frac{(k_\tau)^2}{k^5} \\ &l^2 = \frac{2k_\tau}{k^2} \quad m^2 = 0 \quad n^2 = \frac{k_\tau}{k^2}. \end{aligned} \tag{40}$$

Using (35), the Gauss equation (3) becomes

$$\left(\frac{\left(\sqrt{\frac{1}{k}}\right)_\tau}{\sqrt{\frac{1}{k}}} \right)_\tau = \frac{k}{2}, \qquad \tau = t_1 + t_2. \tag{41}$$

If we take $k = f(\tau)$, $\tau = t_1 - t_2$, then $k^* = -f(\tau)$ and the equations (27) are reduced to

$$K_{\mathbf{Z}^1} = -\frac{k^2}{2}, \qquad K_{\mathbf{Z}^2} = \frac{k^2}{2}$$

and so we get the equation $K_{\mathbf{Z}^1} + K_{\mathbf{Z}^2} = 0$. The first and second fundamental forms of the timelike focal surfaces $\mathbf{Z}^1$ and $\mathbf{Z}^2$ are

$$\begin{aligned} E^1 &= -\frac{(k_\tau)^2}{k^5} \quad F^1 = \frac{(k_\tau)^2}{k^5} \quad G^1 = -\frac{(k_\tau)^2}{k^5} + \frac{4}{k} \\ l^1 &= \frac{k_\tau}{k^2} \quad m^1 = 0 \quad n^1 = \frac{2k_\tau}{k^2} \end{aligned} \tag{42}$$

$$\begin{aligned} E^2 &= -\frac{(k_\tau)^2}{k^5} + \frac{4}{k} \quad F^2 = \frac{(k_\tau)^2}{k^5} \quad G^2 = -\frac{(k_\tau)^2}{k^5} \\ l^2 &= -\frac{2k_\tau}{k^2} \quad m^2 = 0 \quad n^2 = -\frac{k_\tau}{k^2}. \end{aligned} \tag{43}$$

the Gauss equation of spacelike minimal surface takes the form (41) for $\tau = t_1 - t_2$. Then, the solution of the differential equation (41) are

$$k(\tau) = \frac{2}{\sinh^2 \tau}$$

and from here

$$k^*(\tau) = -\frac{2}{\sinh^2 \tau}$$

where $\tau = t_1 + t_2$ or $\tau = t_1 - t_2$ respectively. □

3. Area Preserving Correspondence Between Timelike Focal Surfaces

In this section, we are interested in area preserving correspondence between timelike focal surfaces which correspond timelike normal rectilinear congruence of spacelike surface.

Theorem 7. *Let k, k^*, q, q^* be continuous functions such that k, k^*, q, q^* and $k-k^*$ are not-zero. The solutions of the four partial differential equations* (7) *and* (45) *determine a spacelike surface in $\mathbb{E}_1^3$, which the parameters curves of its are lines of curvature and k, k^* and q, q^* are principal curvatures and geodesic curvatures of the its parameters curves, respectively, such that the area element is preserved between the timelike focal surfaces which form the timelike normal rectilinear congruence of this spacelike surface.*

Proof. The necessary and sufficient condition to be an area preserving correspondence between timelike focal surfaces is that the following relation are satisfied:

$$E^1G^1 - \left(F^1\right)^2 = E^2G^2 - \left(F^2\right)^2. \tag{44}$$

Using (15) and (18), the relation (44) takes the form

$$\frac{k_1}{k_2^*} - \left(\frac{k}{k^*}\right)^3 = 0. \tag{45}$$

□

In case the spacelike surface is minimal, the equation (45) takes the form

$$k_1 - k_2 = 0.$$

Now, we shall find the relation between the equations (21) and (44).

Theorem 8. *Suppose that there is the equation $k_1^* = \pm k_2$. The area element is preserved between the timelike focal surfaces which correspond timelike normal rectilinear congruence of the spacelike surface if and only if the Gaussian curvatures of these timelike focal surfaces satisfy the relation $K_{\mathbf{Z}^1} \pm K_{\mathbf{Z}^2} = 0$.*

Proof. If the area element is preserved between the timelike focal surfaces, the equation (22) can be written as

$$k_1^* - \lambda k_2 = 0. \tag{46}$$

In case of $k_1^* = \pm k_2$, it can be found $\lambda = \pm 1$ and so it is satisfied $K_{\mathbf{Z}^1} \pm K_{\mathbf{Z}^2} = 0$. Conversely, if we have $\lambda = \pm 1$,i.e. $K_{\mathbf{Z}^1} \pm K_{\mathbf{Z}^2} = 0$, using the equation (22) we get the equation (45) in case of $k_1^* = \pm k_2$. □

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