Available online: July 30, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 2, Pages 2272–2282 (2019)
DOI: 10.31801/cfsuasmas.529703
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

ON THE LIMIT OF DISCRETE q-HERMITE I POLYNOMIALS

SAKINA ALWHISHI, REZAN SEVİNİK ADIGÜZEL, AND MEHMET TURAN

Abstract. The main purpose of this paper is to introduce the limit relations between the discrete q-Hermite I and Hermite polynomials such that the orthogonality property and the three-terms recurrence relations remain valid. The discrete q-Hermite I polynomials are the q-analogues of the Hermite polynomials which form an important class of the classical orthogonal polynomials. The q-difference equation of hypergeometric type, Rodrigues formula and generating function are also considered in the limiting case.

1. Introduction

Hermite polynomials are one of the important orthogonal family of the classical orthogonal polynomials which have several applications in various science, such as in mathematical physics, in particular in quantum mechanics [15], mathematics [12], statistics [32]. These polynomials were studied by Charles Hermite in 1864 and named after him. They satisfy the following differential equation of hypergeometric type [1, 19, 20, 25, 26, 27]

$$y''(z) - 2zy'(z) + 2ny(z) = 0, \tag{1}$$

where $n \in \mathbb{N}_0$.

Discrete version of the Hermite polynomials is also important and has enormous applications in several problems on theoretical and mathematical physics, e.g., in the continued fractions, Eulerian series [16], algebras and quantum groups [21, 22, 31], discrete mathematics, algebraic combinatorics (coding theory, design theory, various theories of group representation) [8], q-Schrödinger equation and q-harmonic oscillators [3, 4, 5, 6, 7, 9, 11, 23].

In fact, discrete q-Hermite I polynomials are one of the significant polynomial family among the q-classical polynomials in the Hahn sense. The Hermite polynomials and their q-analogues can be obtained in a suitable limit case from the other

Received by the editors: February 21, 2019; Accepted: July 03, 2019.

2010 *Mathematics Subject Classification.* 33C45, 33D45.

Key words and phrases. Discrete q-Hermite I polynomials, Hermite polynomials, Rodrigues formula, q-difference equation of hypergeometric type.

orthogonal polynomials [2, 19, 20]. Moreover, Hermite polynomials are obtained from the discrete q-Hermite I polynomials in the limiting case as $q \to 1$.

The main aim of this study is to consider the limit relation between the discrete q-Hermite I polynomials and the classical Hermite polynomials such that the orthogonality property and the three-terms recurrence relations (TTRR) remain valid. In fact, some kind of limit relation is given in [19, 20]. The importance of this study is to deal with the limit relation which preserves the orthogonality and TTRR. Moreover, the q-difference equation of hypergeometric type, Rodrigues formula and generating function are also considered in detailed.

In the next chapter, some important characteristic properties, such as polynomial solutions of the q-difference equation [1, 10, 18, 24, 28], Rodrigues formula [14, 30], TTRR [13, 29], generating function [19, 20] and orthogonality relation [1, 19, 20, 25, 26, 27] of the discrete q-Hermite I polynomials are introduced. Some necessary basic definitions related with q-calculus are also established in the preliminaries part [1, 17, 19, 20, 25, 26, 27].

Chapter 3 includes the main results of this study where limit relation between the discrete q-Hermite I polynomials and the classical Hermite polynomials [19, 20] are introduced in detailed.

2. Preliminaries

In this part, some preliminaries for the discrete q-Hermite I polynomials are presented. See for example [1, 19, 20, 25, 26, 27]. Although, most of the properties given in this part are known, for the sake of completeness they are listed here.

Let $0 < q < 1$. The equation

$$-D_q D_{q^{-1}} y(x) + \frac{x}{1-q} D_{q^{-1}} y(x) + \lambda y(x) = 0 \tag{2}$$

is called the q-Hermite difference equation. Here, $D_q f(x)$ is the q-Jackson derivative [19, 20, 26] of f defined by

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x}, & x \neq 0, \\ f'(0), & x = 0. \end{cases}$$

Observe that if f is differentiable, then

$$\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx}.$$

For $\lambda = \lambda_n := -\dfrac{q^{1-n}[n]_q}{1-q}$, where $[n]_q = 1 + q + \cdots + q^{n-1}$ denotes the q-integer with $n \in \mathbb{N}_0$, one solution of (2) is a polynomial of degree n. The monic polynomial solution of (2) is called the discrete q-Hermite I polynomial and it is denoted by $h_n(x;q)$. It is known that all q-derivatives of discrete q-Hermite I polynomials are also solutions of an equation of the same kind. More precisely, $v_{kn} := D_q^k h_n(x;q)$ is a polynomial solution of the equation

$$-D_q D_{q^{-1}} v_{kn} + \frac{x}{1-q} D_{q^{-1}} v_{kn} + \mu_{kn} v_{kn} = 0$$

where $\mu_{kn} = \lambda_{n-k}$ for all $n \in \mathbb{N}_0$ and $k = 0, 1, \ldots, n$.

The discrete q-Hermite I polynomials may also be obtained from the Rodrigues' formula:

$$h_n(x;q) = (1-q^{-1})^n q^{\binom{n}{2}} \frac{D_{q^{-1}}^n[\rho_q(x)]}{\rho_q(x)} \tag{3}$$

where $\binom{n}{2} = \frac{n(n-1)}{2}$ is the usual binomial and $\rho_q(x)$ is the q-weight function given by

$$\rho_q(x) = (qx, -qx; q)_\infty, \tag{4}$$

in which $(a;q)_\infty = \prod_{s=0}^{\infty}(1-aq^s)$, $a \in \mathbb{C}$, is the infinite q-product [19, 20, 26] and $(a_1, \ldots, a_r; q)_\infty = (a_1;q)_\infty \ldots (a_r;q)_\infty$.

The q-Hermite I polynomials $h_n(x;q)$ satisfy the three-term recurrence relation

$$xh_n(x;q) = h_{n+1}(x;q) + q^{n-1}(1-q^n)h_{n-1}(x;q), \quad n = 0, 1, \ldots \tag{5}$$

where $h_{-1}(x;q) := 0$ and $h_0(x;q) = 1$.

A generating function of the discrete q-Hermite I polynomials is

$$\frac{(t^2;q^2)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{h_n(x;q)}{(q;q)_n} t^n \tag{6}$$

where $(a;q)_k = \prod_{s=0}^{k-1}(1-aq^s), a \in \mathbb{C}$, is the q-shifted factorial [19, 20, 26]. One can derive, from (6), that

$$h_{2n+1}(0;q) = 0, \quad h_{2n}(0;q) = (-1)^n q^{n(n-1)}(q;q^2)_n \quad \text{and} \quad h_n(-x;q) = (-1)^n h_n(x;q)$$

for all $n = 0, 1, \ldots$

The discrete q-Hermite I polynomials have the following hypergeometric representation:

$$h_n(x;q) = x^n \, {}_2\phi_0 \left(\left. \begin{matrix} q^{-n}, q^{-n+1} \\ - \end{matrix} \right| q^2; \frac{q^{2n-1}}{x^2} \right) \tag{7}$$

where ${}_r\phi_s$ is the q-hypergeometric function [17, 19, 20, 26] given by

$${}_r\phi_s \left(\left. \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right| q; z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} \frac{z^k}{(q;q)_k}.$$

The set $\{h_n(x;q)\}_{n=0}^{\infty}$ of q-Hermite I polynomials is orthogonal on the interval $(-1, 1)$ with respect to the q-weight function $\rho_q(x)$. More precisely, the q-Hermite

I polynomials $h_n(x;q)$ satisfy

$$\int_{-1}^{1} \rho_q(x)\, h_n(x;q)\, h_m(x;q)\, d_q x = \mathcal{M}_n^2 \delta_{nm}, \tag{8}$$

where $\mathcal{M}_n^2 = (1-q)(q,-1,-q;q)_\infty (q;q)_n q^{\binom{n}{2}}$ is the square of norm of $h_n(x;q)$.

This result can be generalized for the kth order q-derivative of $h_n(x;q)$. That is, the set $\{v_{kn}\}_{n=0}^{\infty}$ is also orthogonal on the same interval with respect to the same q-weight function. The orthogonality relation is

$$\int_{-1}^{1} \rho_q(x) v_{kn}(x) v_{km}(x)\, d_q x = \mathcal{M}_{kn}^2 \delta_{mn}$$

where

$$\mathcal{M}_{kn}^2 = (1-q)^{n-k+1} q^{\binom{n-k}{2}} (q,-1,-q;q)_\infty \frac{([n]_q!)^2}{[n-k]_q!}\, \delta_{kp}$$

and $p = \min\{k,n\}$. Note that $\mathcal{M}_{0n} = \mathcal{M}_n$.

3. Limit Relations

In this part, it will be shown that, in the limit case as $q \to 1$, the discrete q-Hermite I polynomials tend to the Hermite polynomials with some suitable transformation of the independent variable, which preserves the orthogonality and three-terms recurrence relation. Under this transformation, the limiting cases of all the properties, given in the previous section, for the discrete q-Hermite I polynomials are studied. We start by the following key lemma which is given in [20] without proof.

Lemma 1. *Let $x = z\sqrt{1-q^2}$. Then,*

$$\lim_{q\to 1} \frac{h_n(x;q)}{(1-q^2)^{n/2}} = \frac{H_n(z)}{2^n},$$

where $H_n(z)$ is the classical Hermite polynomials of degree n.

Proof. Let $x = z\sqrt{1-q^2}$ and set $u(z) = y(z\sqrt{1-q^2})$ in (2) with $\lambda = -q^{1-n}[n]_q/(1-q)$. Using

$$D_{q^{-1}} y(x)\big|_{x=z\sqrt{1-q^2}} = \frac{1}{\sqrt{1-q^2}} D_{q^{-1}} u(z),$$

and

$$D_q D_{q^{-1}} y(x)\big|_{x=z\sqrt{1-q^2}} = \frac{1}{1-q^2} D_q D_{q^{-1}} u(z),$$

the equation becomes

$$-\frac{1}{1-q^2} D_q D_{q^{-1}} u(z) + \frac{z}{1-q} D_{q^{-1}} u(z) - \frac{q^{1-n}}{1-q} [n]_q u(z) = 0. \tag{9}$$

Since $[n]_q \to n$ and $D_q f \to f'$ as $q \to 1$, multiplying the last equation by $-(1-q^2)$ and taking the limit as $q \to 1$, one derives

$$u''(z) - 2zu'(z) + 2nu(z) = 0, \tag{10}$$

which is the Hermite differential equation given in (1). As the monic polynomial solutions of (9) and (10) are

$$\frac{h_n\left(z\sqrt{1-q^2};q\right)}{(1-q^2)^{n/2}} \qquad \text{and} \qquad \frac{H_n(z)}{2^n},$$

respectively, one obtains

$$\lim_{q\to1} \frac{h_n\left(z\sqrt{1-q^2};q\right)}{(1-q^2)^{n/2}} = \frac{H_n(z)}{2^n}. \tag{11}$$

□

Theorem 2. *Let $x = z\sqrt{1-q^2}$. Then,*

$$\lim_{q\to1} \rho_q(x) = \rho(z),$$

where $\rho_q(x)$ is the the q-weight function defined by (4) of the discrete q-Hermite I polynomials and $\rho(z)$ is the weight function of the classical Hermite polynomials.

Proof. Under the given transformation, the q-weight function defined in (4), $\rho_q(x) = (qx, -qx; q)_\infty = (q^2x^2; q^2)_\infty$ becomes

$$\rho_q\left(z\sqrt{1-q^2}\right) = \left(q^2(1-q^2)z^2; q^2\right)_\infty = \frac{\left((1-q^2)z^2; q^2\right)_\infty}{1-(1-q^2)z^2}. \tag{12}$$

Using Euler's identity [2],

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}, \quad |q| < 1, \tag{13}$$

one derives

$$\lim_{q\to1} \frac{1}{((1-q^2)z; q^2)_\infty} = e^z. \tag{14}$$

Therefore, (12) leads to

$$\lim_{q\to1} \rho_q\left(z\sqrt{1-q^2}\right) = \lim_{q\to1} \frac{\left((1-q^2)z^2; q^2\right)_\infty}{1-(1-q^2)z^2} = e^{-z^2} = \rho(z)$$

which is the weight function for the Hermite polynomials. □

Theorem 3. *Let $x = z\sqrt{1-q^2}$. Then, Rodrigues formula given in (3) for the discrete q-Hermite I polynomials tends to Rodrigues formula for the classical Hermite polynomials in the limit case as $q \to 1$.*

Proof. Applying the transformation to the Rodrigues formula (3) for the discrete q-Hermite I polynomials leads to

$$h_n\left(z\sqrt{1-q^2};q\right) = (1-q^{-1})^n q^{\binom{n}{2}} \frac{D^n_{q^{-1}}[\rho_q(x)]\Big|_{x=z\sqrt{1-q^2}}}{\rho_q\left(z\sqrt{1-q^2}\right)}.$$

Using the fact that

$$D^n_{q^{-1}} f(x)\Big|_{x=\alpha z} = \alpha^{-n} D_{q^{-1}} f(\alpha z),$$

and dividing both sides of the last expression by $(1-q^2)^{n/2}$, one obtains

$$\frac{h_n\left(z\sqrt{1-q^2};q\right)}{(1-q^2)^{n/2}} = \frac{(-1)^n q^{\binom{n}{2}-n}}{(1+q)^n} \frac{D^n_{q^{-1}}\rho_q\left(z\sqrt{1-q^2}\right)}{\rho_q\left(z\sqrt{1-q^2}\right)}.$$

Taking the limit of both sides as $q \to 1$ and using (11), we see that

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n}\left(e^{-z^2}\right)$$

which is the Rodrigues formula for Hermite polynomials. □

Theorem 4. *Let $x = z\sqrt{1-q^2}$. Then, the three-term recurrence relation given in* (5) *for the discrete q-Hermite I polynomials tends to the three-term recurrence relation for the classical Hermite polynomials as $q \to 1$.*

Proof. For the three-term recurrence relation OF the discrete q-Hermite I polynomials given by (5), using the transformation and dividing the resulting equation by $(1-q^2)^{\frac{n+1}{2}}$, one derives

$$\frac{h_{n+1}\left(z\sqrt{1-q^2};q\right)}{(1-q^2)^{\frac{n+1}{2}}} - z\frac{h_n\left(z\sqrt{1-q^2};q\right)}{(1-q^2)^{\frac{n}{2}}} + \frac{q^{n-1}[n]_q}{1+q}\frac{h_{n-1}\left(z\sqrt{1-q^2};q\right)}{(1-q^2)^{\frac{n-1}{2}}} = 0.$$

Taking the limit as $q \to 1$, and multiplying both sides by 2^{n+1}, we obtain

$$H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0,$$

which is the three-term recurrence relation for Hermite polynomials. □

Theorem 5. *Let $x = z\sqrt{1-q^2}$. Then, the generating function* (6) *for the discrete q-Hermite I polynomials satisfies the following limit relation*

$$\lim_{q\to 1} \frac{(t^2;q^2)_\infty}{xt;q)_\infty} = e^{2zt-t^2}$$

where e^{2zt-t^2} is the generating function for the classical Hermite polynomials.

Proof. In (13), replacing x by $(1-q^2)zt$ and taking the limit as $q \to 1$, result in

$$\lim_{q\to 1} \frac{1}{((1-q^2)zt;q)_\infty} = e^{2zt}.$$

Also, from (14), we have

$$\lim_{q\to 1} \left((1-q^2)t^2;q^2\right)_\infty = e^{-t^2}.$$

Now, using the transformation $x = z\sqrt{1-q^2}$ and replacing t by $\sqrt{1-q^2}t$ in the generating function relation (6) for the discrete q-Hermite I polynomials, we get

$$\begin{aligned}\frac{\left((1-q^2)t^2;q^2\right)_\infty}{((1-q^2)zt;q)_\infty} &= \sum_{n=0}^{\infty} \frac{h_n\left(z\sqrt{1-q^2};q\right)}{(q;q)_n} \left(t\sqrt{1-q^2}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{h_n\left(z\sqrt{1-q^2};q\right)}{(1-q^2)^{n/2}} \frac{(1-q)^n}{(q;q)_n} [(1+q)t]^n.\end{aligned}$$

Taking the limit of both sides as $q \to 1$, we obtain

$$e^{2zt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{2^n} \frac{(2t)^n}{n!} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n$$

which is the generating function relation for Hermite polynomials. □

Theorem 6. *Let $x = z\sqrt{1-q^2}$. Then, the hypergeometric representation* (7) *of the discrete q-Hermite I polynomials tends to the hypergeometric representation of the classical Hermite polynomials as $q \to 1$.*

Proof. Note that the hypergeometric representation (7) of the discrete q-Hermite I polynomials is

$$h_n(x;q) = x^n \sum_{k=0}^{\infty} \frac{\left(q^{-n};q^2\right)_k \left(q^{-n+1};q^2\right)_k}{\left(q^2;q^2\right)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{-1} \left(\frac{q^{2n-1}}{x^2}\right)^k.$$

Using the transformation of the independent variable and dividing both sides by $(1-q^2)^{n/2}$, one obtains

$$\begin{aligned}\frac{h_n\left(z\sqrt{1-q^2};q\right)}{(1-q^2)^{n/2}} &= z^n \sum_{k=0}^{\infty} \frac{\left(q^{-n};q^2\right)_k \left(q^{-n+1};q^2\right)_k}{\left(q^2;q^2\right)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{-1} \left(\frac{q^{2n-1}}{(1-q^2)z^2}\right)^k \\ &= z^n \sum_{k=0}^{\infty} \frac{(q^{-n};q^2)_k}{(1-q^2)^k} \frac{(q^{-n+1};q^2)_k}{(1-q^2)^k} \frac{(1-q^2)^k}{(q^2;q^2)_k} q^{(2n-1)k-\binom{k}{2}} \left(\frac{-1}{z^2}\right)^k.\end{aligned}$$

Taking the limit as $q \to 1$, and noting that

$$\lim_{q\to 1} \frac{(q^\alpha;q)_k}{(1-q)^k} = (\alpha)_k, \qquad \alpha \in \mathbb{C},$$

where $(\alpha)_k$ denotes the Pochammer's symbol, we get

$$\frac{H_n(z)}{2^n} = z^n \sum_{k=0}^{\infty} \frac{\left(\frac{-n}{2}\right)_k \left(\frac{-n+1}{2}\right)_k}{(1)_k} \left(\frac{-1}{z^2}\right)^k$$

or

$$H_n(z) = (2z)^n \sum_{k=0}^{\infty} \left(\frac{-n}{2}\right)_k \left(\frac{-n+1}{2}\right)_k \frac{\left(-z^{-2}\right)^k}{k!} = (2z)^n\, {}_2F_0\left(\left.\begin{matrix}\frac{-n}{2}, \frac{1-n}{2}\\ -\end{matrix}\right| -\frac{1}{z^2}\right)$$

which is the hypergeometric representation of the Hermite polynomials. □

Finally, the limit relation of the orthogonality relation (8) for the discrete q-Hermite I polynomials is given in the next theorem.

Theorem 7. *Let $x = z\sqrt{1-q^2}$. Then, the orthogonality relation* (8) *of the discrete q-Hermite I polynomials tends to the orthogonality relation of the classical Hermite polynomials as $q \to 1$.*

Proof. For the orthogonality relation, we first note that the substitution $x = z\sqrt{1-q^2}$ in (8) leads to

$$\int_{-\frac{1}{\sqrt{1-q^2}}}^{\frac{1}{\sqrt{1-q^2}}} \rho_q\left(z\sqrt{1-q^2}\right) h_n\left(z\sqrt{1-q^2};q\right) h_m\left(z\sqrt{1-q^2};q\right) \sqrt{1-q^2}\,d_q x = \mathcal{M}_n^2\,\delta_{nm}.$$

Divide both sides by $(1-q^2)^{\frac{n+m+1}{2}}$

$$\int_{-\frac{1}{\sqrt{1-q^2}}}^{\frac{1}{\sqrt{1-q^2}}} \rho_q\left(z\sqrt{1-q^2}\right) \frac{h_n\left(z\sqrt{1-q^2};q\right)}{(1-q^2)^{\frac{n}{2}}} \frac{h_m\left(z\sqrt{1-q^2};q\right)}{(1-q^2)^{\frac{m}{2}}}\,d_q z = \frac{\mathcal{M}_n^2}{(1-q^2)^{\frac{n+m+1}{2}}}\,\delta_{nm}.$$

Take the limit as $q \to 1$ to obtain

$$\int_{-\infty}^{\infty} e^{-z^2} H_n(z) H_m(z)\,dz = \lim_{q\to 1} \frac{2^{2n}\mathcal{M}_n^2}{(1-q^2)^{\frac{2n+1}{2}}}\,\delta_{nm}. \tag{15}$$

To complete the proof, we shall evaluate the limit on the right side. Now,

$$\begin{aligned}
\lim_{q\to 1} \frac{\mathcal{M}_n^2}{(1-q^2)^{\frac{2n+1}{2}}} &= \lim_{q\to 1} \frac{q^{\binom{n}{2}}(1-q)(q;q)_n(q,-1,-q;q)_\infty}{(1-q^2)^{\frac{2n+1}{2}}} \\
&= \lim_{q\to 1}(1-q)\frac{(q;q)_n}{(1-q)^n}\frac{(q^2;q^2)_\infty(-1;q)_\infty}{(1+q)^n\sqrt{1-q^2}} \\
&= \lim_{q\to 1}(1-q)[n]_q!\,\frac{(q^2;q^2)_\infty\,(-1;q)_\infty}{(1+q)^n\sqrt{1-q^2}}.
\end{aligned}$$

Recall the q-gamma function defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}\,(1-q)^{1-x}$$

which is the q-analogue of the gamma function and satisfies $\lim_{q\to 1}\Gamma_q(x) = \Gamma(x)$. (See [2].) Since

$$\Gamma_{q^2}\left(\frac{1}{2}\right) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sqrt{1-q^2},$$

one has

$$\lim_{q\to 1}\frac{\mathcal{M}_n^2}{(1-q^2)^{\frac{2n+1}{2}}} = \lim_{q\to 1}\,[n]_q!\,\frac{\Gamma_{q^2}\left(\frac{1}{2}\right)(q;q^2)_\infty(-1;q)_\infty}{(1+q)^{n+1}} = \frac{n!}{2^n}\lim_{q\to 1}\Gamma_{q^2}\left(\frac{1}{2}\right)(q;q^2)_\infty(-q;q)_\infty$$

where we have used the fact that $(-1;q)_\infty = 2(-q;q)_\infty$. Clearly,

$$(q;q^2)_\infty(-q;q)_\infty = \frac{(q;q^2)_\infty(q^2;q^2)_\infty}{(q;q)_\infty} = 1.$$

Also,

$$\lim_{q\to 1}\Gamma_{q^2}\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Hence,

$$\lim_{q\to 1}\frac{\mathcal{M}_n^2}{(1-q^2)^{\frac{2n+1}{2}}} = \frac{n!\sqrt{\pi}}{2^n}.$$

As a result, (15) gives us

$$\int_{-\infty}^{\infty} e^{-z^2}H_n(z)H_m(z)\,dz = 2^n\,n!\sqrt{\pi}\,\delta_{nm}$$

which is the orthogonality relation for the Hermite polynomials. □

Acknowledgment. The authors express their sincere gratitude to the anonymous referees for their valuable comments and suggestions which improved the paper.

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Current address: Sakina Alwhishi: Al Maghrib University, Department of Mathematics, Libya
E-mail address: sakinaalwhishi@gmail.com
ORCID Address: http://orcid.org/0000-0001-8424-7024
Current address: Rezan Sevinik Adıgüzel: Atilim University, Department of Mathematics, Incek 06836, Ankara, Turkey, mehmet.turan@atilim.edu.t
E-mail address: rezan.adiguzel@atilim.edu.tr
ORCID Address: http://orcid.org/0000-0002-9181-8566
Current address: Mehmet Turan: Atilim University, Department of Mathematics, Incek 06836, Ankara, Turkey
E-mail address: mehmet.turan@atilim.edu.tr
ORCID Address: https://orcid.org/0000-0002-1718-3902