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# $T_3$ AND $T_4$-OBJECTS IN THE TOPOLOGICAL CATEGORY OF CAUCHY SPACES

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Abstract. There are various generalization of the usual topological $T_3$ and $T_4$ axioms to topological categories defined in [2] and [7]. [7] is shown that they lead to different $T_3$ and $T_4$ concepts, in general. In this paper, an explicit characterization of each of the separation properties $T_3$ and $T_4$ is given in the topological category of Cauchy spaces. Moreover, specific relationships that arise among the various $T_i$, $i = 0, 1, 2, 3, 4$, $PreT_2$, and $T_2$ structures are examined in this category.

## 1. Introduction

In general topology and analysis, a Cauchy space is a generalization of metric spaces and uniform spaces for which the notion of Cauchy convergence still makes sense. When filters came into existence and uniform spaces were introduced, Cauchy filters appeared in topological theory as a generalization of Cauchy sequences. The theory of Cauchy spaces was initiated by H. J. Kowalsky [26]. Cauchy spaces were introduced by H. Keller [22] in 1968, as an axiomatic tool derived from the idea of a Cauchy filter in order to study completeness in topological spaces. In that paper the relation between Cauchy spaces, uniform convergence spaces, and convergence spaces was developed. In the completion theory of uniform convergence spaces and convergence vector spaces, Cauchy spaces play an essential role ([19], [25], [39]). This fact explain why most work on Cauchy spaces deals mainly with completions ([17], [18], [29]). Thus, Cauchy spaces form a useful tool for investigating completions.

In 1970, the study of regular Cauchy completions was initiated by J. Ramaley and O. Wyler [36]. Later D. C. Kent and G. D. Richardson ([23], [24]) characterized the $T_3$ Cauchy spaces which have $T_3$ completions and constructed a regular completion functor.

In 1968, Keller [22] introduced the axiomatic definition of Cauchy spaces, which is given briefly in the preliminaries section.

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Filter spaces are generalizations of Cauchy spaces. If we exclude the last of three Keller's [22] axioms for a Cauchy space, then the resulting space is what we call a filter space. In [15], it is shown that the category **FIL** of filter spaces is isomorphic to the category of filter meretopic spaces which were introduced by Katětov [21]. The category of Cauchy spaces is also known to be a bireflective, finally dense subcategory of **FIL** [35].

The notions of "closedness" and "strong closedness" in set based topological categories are introduced by Baran [2], [4] and it is shown in [9] that these notions form an appropriate closure operator in the sense of Dikranjan and Giuli [16] in some well-known topological categories. Moreover, various generalizations of each of $T_i$, $i = 0, 1, 2$ separation properties for an arbitrary topological category over **SET**, the category of sets are given and the relationship among various forms of each of these notions are investigated by Baran in [2], [7], [8], [10], [12] and [14].

Note that for a $T_1$ topological space $X$, $X$ is $T_3$ iff (a) $X/F$ is $T_2$ if it is $T_1$, where $F$ is any nonempty subset of $X$, iff (b) $X/F$ is $PreT_2$ (i.e., a topological space is called $PreT_2$ if for any two distinct points, if there is a neighborhood of one missing the other, then the two points have disjoint neighborhoods) if it is $T_1$, where $F$ be a nonempty subset of $X$, iff (c) $X/F$ is $PreT_2$ for all closed $\emptyset \neq F$ in $X$, where the equivalence of (a), (b), and (c) follow from the facts that for $T_1$ topological spaces, $T_2$ is equivalent to $PreT_2$, and $F$ is closed iff $X/F$ is $T_1$. Note also that for a topological space $X$, (d) $X$ is $T_4$ iff $X$ is $T_1$ and $X/F$ is $T_3$ if it is $T_1$, where $F$ is any nonempty subset of $X$.

In view of (c) and (d), in [2], there are four ways of generalizing each of the usual $T_3$ and $T_4$ separation axioms to arbitrary set based topological categories. Recall, also, in [2], that there are various ways of generalizing each of the usual $T_0$ and $T_2$ separation axioms to topological categories. Moreover, the relationships among various forms of $T_0$-objects and $T_2$-objects are established in [11] and [12], respectively.

The main goal of this paper is

(1) to give the characterization of each of the separation properties $T_3$ and $T_4$ in the topological category of Cauchy spaces,
(2) to examine how these generalizations are related, and
(3) to show that specific relationships that arise among the various $T_i$, $i = 0, 1, 2, 3, 4$, $PreT_2$, and $T_2$ structures are examined in the topological category of Cauchy spaces.

## 2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if $\mathcal{U}$ is concrete (i.e., faithful, amnestic and transportable), has small (i.e., sets) fibers, and for which every $\mathcal{U}$-source has an initial lift or, equivalently, for which each $\mathcal{U}$-sink has a final lift [1].

Note that a topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure [1], [10], [32], or [34].

Recall in [1] or [34], that an object $X \in \mathcal{E}$ (where $X \in \mathcal{E}$ stands for $X \in \mathrm{Ob}\,\mathcal{E}$), a topological category, is discrete iff every map $\mathcal{U}(X) \to \mathcal{U}(Y)$ lifts to a map $X \to Y$ for each object $Y \in \mathcal{E}$ and an object $X \in \mathcal{E}$ is indiscrete iff every map $\mathcal{U}(Y) \to \mathcal{U}(X)$ lifts to a map $Y \to X$ for each object $Y \in \mathcal{E}$.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. $A$ is called a subspace of $X$ if the inclusion map $i : A \to X$ is an initial lift (i.e., an embedding) and we denote it by $A \subset X$.

A filter on a set $X$ is a collection of subsets of $X$, containing $X$, which is closed under finite intersection and formation of supersets (it may contain $\emptyset$). Let $\mathbf{F}(X)$ denote the set of filters on $X$. If $\alpha,\ \beta\ \in \mathbf{F}(X)$, then $\beta \geq\ \alpha$ if and only if for each U$\in \alpha$, $\exists$V$\in \beta$ such that $V \subseteq U$, that is equivalent to $\beta \supset \alpha$. This defines a partial order relation on $\mathbf{F}(X)$. $\dot{x} = [\{x\}]$ is the filter generated by the singleton set $\{x\}$ where $[\cdot]$ means generated filter and $\alpha \cap \beta\ = [\{\ \mathrm{U} \cup \mathrm{V} \mid \mathrm{U} \in \alpha,\ \mathrm{V} \in \beta\ \}]$. If U$\cap$V$\neq \emptyset$, for all U$\in \alpha$ and V$\in\ \beta$, then $\alpha \vee \beta$ is the filter $[\{\mathrm{U}\ \cap \mathrm{V} \mid \mathrm{U} \in \alpha,\ \mathrm{V} \in \beta\ \}]$. If $\exists$U$\in \alpha$ and V$\in \beta$ such that U$\cap$V$=\emptyset$, then we say that $\alpha \vee \beta$ fails to exist.

Let $A$ be a set and $q$ be a function on $A$ that assigns to each point $x$ of $A$ a set of filters (proper or not, where a filter $\delta$ is proper iff $\delta$ does not contain the empty set, $\emptyset$, i.e., $\delta \neq [\emptyset]$) (the filters converging to $x$) is called a *convergence structure on* $A$ ($(A, q)$ a *convergence space* (in [34], it is called a convergence space)) iff it satisfies the following three conditions ([33] p. 1374 or [34] p. 142):

1. $[x] = [\{x\}] \in q(x)$ for each $x \in A$ (where $[F] = \{B \subset A : F \subset B\}$).
2. $\beta \supset \alpha \in q(x)$ implies $\beta \in q(x)$ for any filter $\beta$ on $A$.
3. $\alpha \in q(x) \Rightarrow \alpha \cap [x] \in q(x)$.

A map $f : (A, q) \to (B, s)$ between two convergence spaces is called *continuous* iff $\alpha \in q(x)$ implies $f(\alpha) \in s(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D)\ : D \in \alpha\}$). The category of convergence spaces and continuous maps is denoted by **CON** (in [34] **CONV**).

For filters $\alpha$ and $\beta$ we denote by $\alpha \cup \beta$ the smallest filter containing both $\alpha$ and $\beta$.

**Definition 2.1.** *(cf.* [22]*) Let $A$ be a set and $K \subset \mathbf{F}(A)$ be subject to the following axioms:*

*1. $[x] = [\{x\}] \in K$ for each $x \in A$ (where $[x] = \{B \subset A\ :\ x \in B\}$);*

*2. $\alpha \in K$ and $\beta \geq \alpha$ implies $\beta \in K$ (i.e., $\beta \supset \alpha \in K$ implies $\beta \in K$ for any filter $\beta$ on $A$);*

*3. if $\alpha,\ \beta \in K$ and $\alpha \vee \beta$ exists (i.e., $\alpha \cup \beta$ is proper), then $\alpha \cap \beta \in K$.*

*Then $K$ is a pre-Cauchy (Cauchy) structure if it obeys 1-2 (resp. 1-3) and the pair $(A, K)$ is called a pre-Cauchy space (Cauchy space), resp. Members of $K$ are called Cauchy filters. A map $f : (A, K) \to (B, L)$ between Cauchy spaces is* *said to be Cauchy continuous (Cauchy map) iff $\alpha \in K$ implies $f(\alpha) \in L$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) : D \in \alpha\}$). The concrete category whose objects are the pre-Cauchy (Cauchy) spaces and whose morphisms are the Cauchy continuous maps is denoted by* **PCHY** *(***CHY***), respectively.*

***2.2*** *A source $\{f_i : (A, K) \to (A_i, K_i),\ i \in I\}$ in* **CHY** *is an initial lift iff $\alpha \in K$ precisely when $f_i(\alpha) \in K_i$ for all $i \in I$* [30], [35] *or* [37].

***2.3*** *An epimorphism $f : (A, K) \to (B, L)$ in* **CHY** *(equivalently, $f$ is surjective) is a final lift iff $\alpha \in L$ implies that there exists a finite sequence $\alpha_1, ..., \alpha_n$ of Cauchy filters in $K$ such that every member of $\alpha_i$ intersects every member of $\alpha_{i+1}$ for all $i < n$ and such that $\bigcap_{i=1}^{n} f(\alpha_i) \subset \alpha$* [30], [35] *or* [37].

***2.4*** *Let $B$ be set and $p \in B$. Let $B \vee_p B$ be the wedge at $p$ (*[2] *p. 334), i.e., two disjoint copies of $B$ identified at $p$, i.e., the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in* **SET***). An epi sink $\{i_1,\ i_2 : (B, K) \to (B \vee_p B, L)\}$, where $i_1,\ i_2$ are the canonical injections, in* **CHY** *is a final lift if and only if the following statement holds. For any filter $\alpha$ on the wedge $B \vee_p B$, where either $\alpha \supset i_k(\alpha_1)$ for some $k = 1, 2$ and some $\alpha_1 \in K$, or $\alpha \in L$, we have that there exist Cauchy filters $\alpha_1, \alpha_2 \in K$ such that every member of $\alpha_1$ intersects every member of $\alpha_2$ (i.e., $\alpha_1 \cup \alpha_2$ is proper) and $\alpha \supset i_1\alpha_1 \cap i_2\alpha_2$. This is a special case of 2.3.*

***2.5*** *The discrete structure $(A, K)$ on $A$ in* **CHY** *is given by $K = \{[a] \mid a \in A\} \cup \{[\emptyset]\}$* [30] *or* [35].

***2.6*** *The indiscrete structure $(A, K)$ on $A$ in* **CHY** *is given by $K = F(A)$* [30] *or* [35].

**CHY** is a normalized topological category. The category of Cauchy spaces is Cartesian closed, and contains the category of uniform spaces as a full subcategory [35].

Let $B$ be set and $p \in B$. Let $B \vee_p B$ be the wedge at $p$. A point $x$ in $B \vee_p B$ will be denoted by $x_1(x_2)$ if $x$ is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = (p, p) = p_2$.

The principal $p-$axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed $p-$axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$.

The fold map at $p$, $\bigtriangledown_p : B \vee_p B \to B$ is given by $\bigtriangledown_p(x_i) = x$ for $i = 1, 2$ [2], [4].

Note that the maps $S_p$ and $\bigtriangledown_p$ are the unique maps arising from the above pushout diagram for which $S_p i_1 = (id, id) : B \to B^2$, $S_p i_2 = (p, id) : B \to B^2$, and $\bigtriangledown_p i_j = id, j = 1, 2$, respectively, where, $id : B \to B$ is the identity map and $p : B \to B$ is the constant map at $p$.

The infinite wedge product $\vee_p^\infty B$ is formed by taking countably many disjoint copies of $B$ and identifying them at the point $p$. Let $B^\infty = B \times B \times ...$ be the countable cartesian product of $B$. Define $A_p^\infty : \vee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, ..., p, x, p, ...)$, where $x_i$ is in the $i$-th component of the infinite wedge and

$x$ is in the $i$-th place in $(p, p, ..., p, x, p, ...)$ (infinite principal $p$-axis map), and $\bigtriangledown_p^\infty : \vee_p^\infty B \to B$ by $\bigtriangledown_p^\infty(x_i) = x$ for all $i \in I$ (infinite fold map), [2], [4].

Note, also, that the map $A_p^\infty$ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, ..., p, id, p, ...) : B \to B^\infty$, where the identity map, $id$, is in the $j$-th place [9].

**Definition 2.2.** *(cf. [2], [4]) Let $\mathcal{U} : \mathcal{E} \to$ **SET** be a topological functor, $X$ an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let $F$ be a nonempty subset of $B$. We denote by $X/F$ the final lift of the epi $\mathcal{U}-$sink $q : \mathcal{U}(X) = B \to B/F = (B \backslash F) \cup \{*\}$, where $q$ is the epi map that is the identity on $B \backslash F$ and identifying $F$ with a point $*$ [2].*

*Let $p$ be a point in $B$.*

1. *$X$ is $T_1$ at $p$ iff the initial lift of the $\mathcal{U}-$source $\{S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\bigtriangledown_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.*
2. *$p$ is closed iff the initial lift of the $\mathcal{U}-$source $\{A_p^\infty : \vee_p^\infty B \to \mathcal{U}(X^\infty) = B^\infty$ and $\nabla_p^\infty : \vee_p^\infty B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.*
3. *$F \subset X$ is closed iff $\{*\}$, the image of $F$, is closed in $X/F$ or $F = \emptyset$.*
4. *$F \subset X$ is strongly closed iff $X/F$ is $T_1$ at $\{*\}$ or $F = \emptyset$.*
5. *If $B = F = \emptyset$, then we define $F$ to be both closed and strongly closed.*

## 3. $T_2$-Objects

Recall, in [2] and [12], that there are various ways of generalizing the usual $T_2$ separation axiom to topological categories. Moreover, the relationships among various forms of $T_2$-objects are established in [12].

Let $B$ be a nonempty set, $B^2 = B \times B$ be cartesian product of $B$ with itself and $B^2 \vee_\Delta B^2$ be two distinct copies of $B^2$ identified along the diagonal. A point $(x, y)$ in $B^2 \vee_\Delta B^2$ will be denoted by $(x, y)_1$(or $(x, y)_2$) if $(x, y)$ is in the first (or second) component of $B^2 \vee_\Delta B^2$, respectively. Clearly $(x, y)_1 = (x, y)_2$ iff $x = y$ [2].

The principal axis map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map, $\nabla : B^2 \vee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$. Note that $\pi_1 S = \pi_{11} = \pi_1 A$, $\pi_2 S = \pi_{21} = \pi_2 A$, $\pi_3 A = \pi_{12}$, and $\pi_3 S = \pi_{22}$, where $\pi_k : B^3 \to B$ the $k$-th projection $k = 1,\ 2,\ 3$ and $\pi_{ij} = \pi_i + \pi_j : B^2 \vee_\Delta B^2 \to B$, for $i, j \in \{1, 2\}$ [2].

**Definition 3.1.** *(cf. [2] and [10]) Let $\mathcal{U} : \mathcal{E} \to$ **SET** be a topological functor, $X$ an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$.*

1. $X$ is $\overline{T}_0$ iff the initial lift of the $\mathcal{U}$-source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.
2. $X$ is $T_0'$ iff the initial lift of the $\mathcal{U}$-source $\{id : B^2 \vee_\Delta B^2 \to \mathcal{U}(B^2 \vee_\Delta B^2)' = B^2 \vee_\Delta B^2$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $(B^2 \vee_\Delta B^2)'$

is the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ and $\mathcal{D}(B^2)$ is the discrete structure on $B^2$. Here, $i_1$ and $i_2$ are the canonical injections.

(3) $X$ is $T_0$ iff $X$ does not contain an indiscrete subspace with (at least) two points [31] or [40].
(4) $X$ is $T_1$ iff the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete.
(5) $X$ is $Pre\overline{T}_2$ iff the initial lifts of the $\mathcal{U}$-source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ coincide.
(6) $X$ is $PreT_2'$ iff the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ coincide, where $i_1$ and $i_2$ are the canonical injections.
(7) $X$ is $\overline{T}_2$ iff $X$ is $\overline{T}_0$ and $Pre\overline{T}_2$ [2].
(8) $X$ is $T_2'$ iff $X$ is $T_0'$ and $PreT_2'$ [2].
(9) $X$ is $ST_2$ iff $\Delta$, the diagonal, is strongly closed in $X^2$ [4].
(10) $X$ is $\Delta T_2$ iff $\Delta$, the diagonal, is closed in $X^2$ [4].
(11) $X$ is $KT_2$ iff $X$ is $T_0'$ and $Pre\overline{T}_2$ [12].
(12) $X$ is $LT_2$ iff $X$ is $\overline{T}_0$ and $PreT_2'$ [12].
(13) $X$ is $MT_2$ iff $X$ is $T_0$ and $PreT_2'$ [12].
(14) $X$ is $NT_2$ iff $X$ is $T_0$ and $Pre\overline{T}_2$ [12].

**Remark 3.1.** *1. Note that for the category* **TOP** *of topological spaces,* $\overline{T}_0$, $T_0'$, $T_0$, *or* $T_1$, *or* $Pre\overline{T}_2$, $PreT_2'$, *or all of the* $T_2$*'s in Definition 3.1 reduce to the usual* $T_0$, *or* $T_1$, *or* $PreT_2$ *(where a topological space is called* $PreT_2$ *if for any two distinct points, if there is a neighborhood of one missing the other, then the two points have disjoint neighborhoods), or* $T_2$ *separation axioms, respectively* [2].

*2. For an arbitrary topological category,*

*(i) By Theorem 3.2 of* [11] *or Theorem 2.7(1) of* [12], $\overline{T}_0$ *implies* $T_0'$ *but the converse of implication is generally not true. Moreover, there are no further implications between* $\overline{T}_0$ *and* $T_0$ *(see* [11] *3.4(1) and (2)) and between* $T_0'$ *and* $T_0$ *(see* [11] *3.4(1) and (3)).*

*(ii) By Theorem 3.1(1) of* [6], *if X is* $PreT_2'$, *then X is* $Pre\overline{T}_2$. *But the converse of implication is generally not true.*

**Definition 3.2.** *A Cauchy space* $(A, K)$ *is said to be* $\mathbf{T}_2$ *if and only if* $x = y$, *whenever* $[x] \cap [y] \in K$ [38].

**Theorem 3.1.** [27] *Let* $(A, K)$ *be a Cauchy space.*

(1) $(A, K)$ *in* **CHY** *is* $\overline{T}_0$ *iff it is* $T_0$ *iff it is* $T_1$ *iff for each distinct pair x and y in A, we have* $[x] \cap [y] \notin K$.
(2) *All objects* $(A, K)$ *in* **CHY** *are* $T_0'$.
(3) *All objects* $(A, K)$ *in* **CHY** *are* $Pre\overline{T}_2$.
(4) $(A, K)$ *is* $PreT_2'$ *iff for each pair of distinct points x and y in A, we have* $[x] \cap [y] \in K$ *(equivalently, for each finite subset F of A, we have* $[F] \in K$*).*
(5) $(A, K)$ *is* $\overline{T}_2$ *iff for each distinct pair x and y in A, we have* $[x] \cap [y] \notin K$.

(6) $(A,K)$ *is* $T_2'$ *iff for each distinct points* $x$ *and* $y$ *in* $A$*, we have* $[x]\cap[y]\in K$ *(equivalently, for each finite subset* $F$ *of* $A$*, we have* $[F]\in K$*).*

**Remark 3.2.** *If a Cauchy space* $(A,K)$ *is* $\overline{T}_0$ *or* $T_0$ *(*$T_1$*) then it is* $T_0'$*. However, the converse is not true generally. For example, let* $A=\{x,y\}$ *and* $K=\{[x],\ [y],\ [\{x,\ y\}],\ [\emptyset]\}$*. Then* $(A,K)$ *is* $T_0'$ *but it is not* $\overline{T}_0$ *or* $T_0$ *(*$T_1$*)* [27]*.*

**Remark 3.3.** *If a Cauchy space* $(A,K)$ *is Pre*$T_2'$ *then it is Pre*$\overline{T}_2$*. However, the converse is not true, in general. For example, let* $A=\{x,y\}$ *and* $K=\{[x],\ [y],\ [\emptyset]\}$*. Then* $(A,K)$ *is Pre*$\overline{T}_2$ *but it is not Pre*$T_2'$ [27]*.*

**Remark 3.4.** *Let* $(A,K)$ *be in* **CHY***. By Theorem 3.1(5) and 3.6, the following are equivalent:*

*(a)* $(A,K)$ *is* $\overline{T}_2$ *and* $T_2'$*.*

*(b)* $A$ *is a point or the empty set* [27]*.*

**Corollary 3.1.** *Let* $(A,K)$ *be in* **CHY***.* $(A,K)$ *is* $ST_2$ *iff it is* $\Delta T_2$ *iff for each pair of distinct points* $x$ *and* $y$ *in* $A$ *and for any* $\alpha,\ \beta\in K$*,* $\alpha\cup\beta$ *is improper if* $\alpha\subset[x]$ *and* $\beta\subset[y]$ [27]*.*

**Remark 3.5.** *Let* $(A,K)$ *be in* **CHY***. By Remark 4.5 (2) of* [28]*,* $(A,K)$ *is* $\overline{T}_2$ *iff* $(A,K)$ *is* $ST_2$ *or* $\Delta T_2$*.*

**Remark 3.6.** *(*[3]*, p. 106) Let* $\alpha$ *and* $\beta$ *be filters on* $A$*. If* $f:A\to B$ *is a function, then* $f(\alpha\cap\beta)=f\alpha\cap f\beta$*.*

Let $(A,K)$ be in **CHY**, and $F$ be a nonempty subset of $A$. Let $q:(A,K)\to(A/F,L)$ be the quotient map that identifying $F$ to a point, $*$ [2].

**Theorem 3.2.** *If* $(A,K)$ *is* $T_2'$*, then* $(A/F,L)$ *is* $T_2'$*.*

*Proof.* Suppose $(A,K)$ is $T_2'$. Hence, for each distinct points $x$ and $y$ in $A$, we have $[x]\cap[y]\in K$ by Theorem 3.1(6). If $x$ and $y$ in $F$, then $q(x)=[*]=q(y)$ and $q([x]\cap[y])=q([x])\cap q([y])=[*]\in L$, by definition of the quotient map and Remark 3.6, where $L$ is the structure on $A/F$ induced by $q$. If $x\notin F$ and $y\notin F$, then $q(x)=[x]$, $q(y)=[y]$ and $q([x]\cap[y])=q([x])\cap q([y])=[x]\cap[y]\in L$, by definition of the quotient map and Remark 3.6. If $x\notin F$ and $y\in F$, then $q(x)=[x]$, $q(y)=[*]$ and $q([x]\cap[y])=q([x])\cap q([y])=[x]\cap[*]\in L$, by definition of the quotient map and Remark 3.6. Similarly, if $x\in F$ and $y\notin F$, then $q(x)=[*]$, $q(y)=[y]$ and $q([x]\cap[y])=q([x])\cap q([y])=[*]\cap[y]\in L$, by definition of the quotient map and Remark 3.6.

Consequently for each distinct points $a$ and $b$ in $A/F$, we have $[a]\cap[b]\in L$. Hence by Theorem 3.1(6), $(A/F,L)$ is $T_2'$. □

**Theorem 3.3.** *If* $(A,K)$ *is* $\overline{T}_2$*, then* $(A/F,L)$ *is* $\overline{T}_2$*.*

*Proof.* Suppose $(A,K)$ is $\overline{T}_2$. Let $a$ and $b$ be any distinct pair of points in $A/F$. By Theorem 3.1(5), we only need to show that $[a]\cap[b]\notin L$, where $L$ is the structure on $A/F$ induced by $q$. Suppose that $a \neq *$ and $[a], [*] \in L$ implies $\exists [a], [y] \in K$ such that $[a] \supseteq q([a])$, $[*] \supseteq q([y])$, and $x = qx = a$, $qy = *$ for any $y \in F$. If $[a] \cap [*] \in L$, then $[a] \cap [y] \in K$, by definition of the quotient map and Remark 3.6. But $[a] \cap [y] \notin K$ since $(A, K)$ is $\overline{T}_2$. Hence $[a] \cap [*] \notin L$. Similarly, if $a \neq b \neq *$ and $[a], [b] \in L$ implies $\exists [a], [b] \in K$ such that $[a] \supseteq q([a])$, $[b] \supseteq q([b])$, and $x = qx = a$, $qb = b$. If $[a] \cap [b] \in L$, then $[a] \cap [b] \in K$, by definition of the quotient map and Remark 3.6. But $[a] \cap [b] \notin K$ since $(A, K)$ is $\overline{T}_2$. Hence $[a] \cap [b] \notin L$.

Consequently for each distinct points $a$ and $b$ in $A/F$, we have $[a] \cap [b] \notin L$. Hence by Theorem 3.1(5), $(A/F, L)$ is $\overline{T}_2$. □

**Theorem 3.4.** *If $(A, K)$ is Pre$\overline{T}_2$, then $(A/F, L)$ is Pre$\overline{T}_2$.*

*Proof.* It follows from Theorem 3.1(3). □

**Theorem 3.5.** *If $(A, K)$ is Pre$T'_2$, then $(A/F, L)$ is Pre$T'_2$.*

*Proof.* It follows from Theorem 3.1(4) and by using the same argument used in the proof of Theorem 3.2. □

**Theorem 3.6.** *Let $(A, K)$ be in* **CHY**. *$\emptyset \neq F \subset A$ is closed iff for each $a \in A$ with $a \notin F$ and for all $\alpha \in K$, $\alpha \cup [F]$ is improper or $\alpha \not\subseteq [a]$* [27].

**Theorem 3.7.** *Let $(A, K)$ be in* **CHY**. *$\emptyset \neq F \subset A$ is strongly closed iff for each $a \in A$ with $a \notin F$ and for all $\alpha \in K$, $\alpha \cup [F]$ is improper or $\alpha \not\subseteq [a]$* [27].

**Lemma 3.1.** *Let $\alpha$ and $\beta$ be proper filters on $A$. Then $q\alpha \cup q\beta$ is proper iff either $\alpha \cup \beta$ is proper or $\alpha \cup [F]$ and $\beta \cup [F]$ are proper* [5].

**Theorem 3.8.** *If $(A, K)$ is $ST_2$ (or $\Delta T_2$) and $F$ is (strongly) closed, then $(A/F, L)$ is $ST_2$ (or $\Delta T_2$).*

*Proof.* Let $a$ and $b$ be any distinct pair of points in $A/F$ and $\alpha \subset [a]$, $\beta \subset [b]$ be in $L$, where $L$ is the structure on $A/F$ induced by $q$. If $\alpha \cup \beta$ is improper, then we are done by Corollary 3.1. Suppose that $\alpha \cup \beta$ is proper. $q$ is the quotient map implies $\exists \alpha_1 \in K$ and $\exists \beta_1 \in K$ such that $\alpha \supset q\alpha_1$, $\beta \supset q\beta_1$, and $qx = a$, $qy = b$. Note that $q\alpha \cup q\beta$ is proper and by Lemma 2.13 (see [5] p. 165 Lemma 2.13), either $\alpha_1 \cup \beta_1$ is proper or $\alpha_1 \cup [F]$ and $\beta_1 \cup [F]$ are proper. The first case can not hold since $x \neq y$ and $(A, K)$ is $ST_2$ (or $\Delta T_2$). Since $a \neq b$, we may assume $x \in F$. We have $\alpha_1 \in K$ and since $F$ is (strongly) closed by Theorem 3.6 (3.7), $\alpha_1 \cup [F]$ is improper. This shows that the second case also can not hold. Therefore, $\alpha \cup \beta$ must be improper and, by Definition 3.1 (9) (3.1 (10)), we have the result. □

**Theorem 3.9.** *All objects $(A, K)$ in* **CHY** *are $KT_2$.*

*Proof.* It follows from Definition 3.1, Theorem 3.1(2) and 3.3. □

**Theorem 3.10.** *$(A, K)$ in* **CHY** *is $LT_2$ iff $A$ is a point or the empty set.*

*Proof.* It follows from Definition 3.1, Theorem 3.1(1) and 3.4. □

**Theorem 3.11.** *$(A, K)$ in* **CHY** *is $MT_2$ iff $A$ is a point or the empty set.*

*Proof.* It follows from Definition 3.1, Theorem 3.1(1) and 3.4. □

**Theorem 3.12.** *$(A, K)$ in* **CHY** *is $NT_2$ iff for each distinct pair $x$ and $y$ in $A$, $[x] \cap [y] \notin K$.*

*Proof.* It follows from Definition 3.1, Theorem 3.1(1) and 3.3. □

**Remark 3.7.**
(1) *If a Cauchy space $(A, K)$ is $LT_2(MT_2)$ then it is $KT_2$. However, the converse is not true, in general. For example, let $A = \{x, y\}$ and $K = \{[x]\,,\ [y]\,,\ [\emptyset]\}$. Then $(A, K)$ is $KT_2$ but it is not $LT_2(MT_2)$.*
(2) *If a Cauchy space $(A, K)$ is $NT_2$ then it is $KT_2$. However, the converse is not true, in general. For example, let $A = \{x, y\}$ and $K = \{[x]\,,\ [y]\,,\ [\{x,\ y\}]\,,\ [\emptyset]\}$. Then $(A, K)$ is $KT_2$ but it is not $NT_2$.*
(3) *If a Cauchy space $(A, K)$ is $LT_2(MT_2)$ then it is $NT_2$. However, the converse is not true, in general. For example, let $A = \{x, y\}$ and $K = \{[x]\,,\ [y]\,,\ [\emptyset]\}$. Then $(A, K)$ is $NT_2$ but it is not $LT_2(MT_2)$.*

## 4. $T_3$-Objects

We now recall, ([2], [7] and [13]), various generalizations of the usual $T_3$ separation axiom to arbitrary set based topological categories and characterize each of them for the topological categories **CHY**.

**Definition 4.1.** *(cf. [2], [7] and [13]) Let $\mathcal{U} : \mathcal{E} \to$* **SET** *be a topological functor, $X$ an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let $F$ be a non-empty subset of $B$.*

(1) $X$ is $S\overline{T}_3$ iff $X$ is $T_1$ and $X/F$ is $Pre\overline{T}_2$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
(2) $X$ is $ST'_3$ iff $X$ is $T_1$ and $X/F$ is $PreT'_2$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
(3) $X$ is $\overline{T}_3$ iff $X$ is $T_1$ and $X/F$ is $Pre\overline{T}_2$ for all closed $F \neq \emptyset$ in $U(X)$.
(4) $X$ is $T'_3$ iff $X$ is $T_1$ and $X/F$ is $PreT'_2$ for all closed $F \neq \emptyset$ in $U(X)$.
(5) $X$ is $KT_3$ iff $X$ is $T_1$ and $X/F$ is $Pre\overline{T}_2$ if it is $T_1$, where $F \neq \emptyset$ in $U(X)$.
(6) $X$ is $LT_3$ iff $X$ is $T_1$ and $X/F$ is $PreT'_2$ if it is $T_1$, where $F \neq \emptyset$ in $U(X)$.
(7) $X$ is $ST_3$ iff $X$ is $T_1$ and $X/F$ is $ST_2$ if it is $T_1$, where $F \neq \emptyset$ in $U(X)$.
(8) $X$ is $\Delta T_3$ iff $X$ is $T_1$ and $X/F$ is $\Delta T_2$ if it is $T_1$, where $F \neq \emptyset$ in $U(X)$.

**Remark 4.1.** *1. For the category* **TOP** *of topological spaces, all of the $T_3$'s reduce to the usual $T_3$ separation axiom (cf. [2], [7] and [13]).*

*2. If $\mathcal{U} : \mathcal{E} \to \boldsymbol{B}$, where $\boldsymbol{B}$ is a topos [20], then Parts (1), (2), and (5)-(8)of Definition 4.1 still make sense since each of these notions requires only finite products and finite colimits in their definitions. Furthermore, if $\boldsymbol{B}$ has infinite products and infinite wedge products, then Definition 4.1 (4), also, makes sense.*

**Theorem 4.1.** *$(A, K)$ in* **CHY** *is $S\overline{T}_3$ iff for each distinct pair $x$ and $y$ in $A$, $[x] \cap [y] \notin K$.*

*Proof.* It follows from Definition 4.1, Theorem 3.1(1), 3.3 and 3.4. □

**Theorem 4.2.** *$(A, K)$ in* **CHY** *is $ST_3'$ iff $A$ is a point or the empty set.*

*Proof.* Suppose $(A, K)$ is $ST_3'$ and Card $A > 1$. Since $(A, K)$ is $T_1$, by Theorem 3.1(1), for each distinct pair $x$ and $y$ in $A$, we have $[x] \cap [y] \notin K$. If $\alpha$ is in $K$, $q(\alpha) \in L$, where $L$ is the structure on $A/F$ induced by $q$. Since $(A/F, L)$ is $PreT_2'$, by Theorem 3.1(4), for each pair of distinct points $a$ and $b$ in $A/F$, we have $[a] \cap [b] \in L$. If $a \neq *$ and $b \neq *$, then it is easy to see that $q(\alpha) = [a] \cap [b] \in L \Rightarrow q^{-1}(q(\alpha)) = q^{-1}([a] \cap [b]) =[a] \cap [b] \subseteq \alpha$ and consequently $\alpha = [a] \cap [b] \in K$. This contradicts the fact that $(A, K)$ is $T_1$. If $a \neq * = b$, then it follows easily that for each $y \neq *$ in $A/F$, $[\{y, *\}] \notin L$ since $F$ is closed. This contradicts the fact that $(A/F, L)$ is $PreT_2'$. Hence Card $A \leq 1$.

Conversely, $A = \{x\}$, i.e., a singleton, then clearly, by Definition 4.1, $(A, K)$ is $ST_3'$. □

**Theorem 4.3.** *$(A, K)$ in* **CHY** *is $\overline{T}_3$ iff for each distinct pair $x$ and $y$ in $A$, $[x] \cap [y] \notin K$.*

*Proof.* It follows from Definition 4.1, Theorem 3.1(1), 3.3 and 3.4. □

**Theorem 4.4.** *$(A, K)$ in* **CHY** *is $T_3'$ iff $A$ is a point or the empty set.*

*Proof.* It follows from Definition 4.1, Theorem 3.1(1) and by using the same argument used in the proof of Theorem 4.2. □

**Theorem 4.5.** *$(A, K)$ in* **CHY** *is $KT_3$ iff for each distinct pair $x$ and $y$ in $A$, $[x] \cap [y] \notin K$.*

*Proof.* It follows from Definition 4.1, Theorem 3.1(1) and 3.3. □

**Theorem 4.6.** *$(A, K)$ in* **CHY** *is $LT_3$ iff $A$ is a point or the empty set.*

*Proof.* It follows from Definition 4.1, Theorem 3.1(1) and 3.4. □

**Theorem 4.7.** *$(A, K)$ in* **CHY** *is $ST_3$ iff for each pair of distinct points $x$ and $y$ in $A$ and for any $\alpha,\ \beta \in K$, $\alpha \cup \beta$ is improper if $\alpha \subset [x]$ and $\beta \subset [y]$.*

*Proof.* It follows from Definition 4.1, Theorem 3.1(1) and Remark 4.5 (1) in [28] (i.e., $(A, K)$ is $T_1$ iff $(A, K)$ is $ST_2$ or $\Delta T_2$). □

**Theorem 4.8.** *$(A, K)$ in* **CHY** *is $\Delta T_3$ iff for each pair of distinct points $x$ and $y$ in $A$ and for any $\alpha,\ \beta \in K$, $\alpha \cup \beta$ is improper if $\alpha \subset [x]$ and $\beta \subset [y]$.*

*Proof.* It follows from Definition 4.1, Theorem 3.1(1) and Remark 4.5 (1) in [28] (i.e., $(A, K)$ is $T_1$ iff $(A, K)$ is $ST_2$ or $\Delta T_2$). □

## 5. $T_4$-Objects

We now recall various generalizations of the usual $T_4$ separation axiom to arbitrary set based topological categories that are defined in [2], [7] and [13], and characterize each of them for the topological categories **CHY**.

**Definition 5.1.** *(cf. [2], [7] and [13]) Let $\mathcal{U} : \mathcal{E} \to$ **SET** be a topological functor and $X$ an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let $F$ be a non-empty subset of $B$.*

1. $X$ is $S\overline{T}_4$ iff $X$ is $T_1$ and $X/F$ is $S\overline{T}_3$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
2. $X$ is $ST'_4$ iff $X$ is $T_1$ and $X/F$ is $ST'_3$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
3. $X$ is $\overline{T}_4$ iff $X$ is $T_1$ and $X/F$ is $\overline{T}_3$ for all closed $F \neq \emptyset$ in $U(X)$.
4. $X$ is $T'_4$ iff $X$ is $T_1$ and $X/F$ is $X/F$ is $T'_3$ for all closed $F \neq \emptyset$ in $U(X)$.

**Remark 5.1.** *1. For the category* **TOP** *of topological spaces, all of the $T_4$'s reduce to the usual $T_4$ separation axiom ([2], [7] and [13]).*

*2. If $\mathcal{U} : \mathcal{E} \to \boldsymbol{B}$, where $\boldsymbol{B}$ is a topos [20], then Definition 5.1 still makes sense since each of these notions requires only finite products and finite colimits in their definitions.*

**Theorem 5.1.** *$(A, K)$ in* **CHY** *is $S\overline{T}_4$ iff for each distinct pair $x$ and $y$ in $A$, $[x] \cap [y] \notin K$.*

*Proof.* It follows from Definition 5.1, Theorem 3.1(1) and 4.1. □

**Theorem 5.2.** *$(A, K)$ in* **CHY** *is $ST'_4$ iff $A$ is a point or the empty set.*

*Proof.* It follows from Definition 5.1, Theorem 3.1(1) and 4.2. □

**Theorem 5.3.** *$(A, K)$ in* **CHY** *is $\overline{T}_4$ iff for each distinct pair $x$ and $y$ in $A$, $[x] \cap [y] \notin K$.*

*Proof.* It follows from Definition 5.1, Theorem 3.1(1) and 4.3. □

**Theorem 5.4.** *$(A, K)$ in* **CHY** *is $T'_4$ iff $A$ is a point or the empty set.*

*Proof.* It follows from Definition 5.1, Theorem 3.1(1) and 4.4. □

**Remark 5.2.** *Let $(A, K)$ be a Cauchy space. It follows from Theorem 3.12, 4.1, 4.3, 4.5, 5.1, 5.3, Definition 3.1, 4.1 and 5.1 that $(A, K)$ is $NT2$ iff $(A, K)$ is $S\overline{T}_3$ iff $(A, K)$ is $\overline{T}_3$ iff $(A, K)$ is $KT_3$ iff $(A, K)$ is $S\overline{T}_4$ iff $(A, K)$ is $\overline{T}_4$ iff for each distinct pair $x$ and $y$ in $A$, $[x] \cap [y] \notin K$.*

**Remark 5.3.** *Let $(A, K)$ be a Cauchy space. It follows from Theorem 3.10, 3.11, 4.2, 4.4, 4.6, 5.2, 5.4, Definition 3.1, 4.1 and 5.1 that $(A, K)$ is $ST'_3$ iff $(A, K)$ is $T'_3$ iff $(A, K)$ is $LT_2$ iff $(A, K)$ is $MT_2$ iff $(A, K)$ is $LT_3$ iff $(A, K)$ is $ST'_4$ iff $(A, K)$ is $T'_4$ iff $A$ is a point or the empty set.*

We can infer the following results.

**Remark 5.4.** *Let $(A,K)$ be in $CHY$.*

*1. By Theorem 3.1(1), 4.1, 4.3, 4.5, Corollary 3.1 and Remark 5.2, $(A,K)$ is $T_1$ iff it is $T_0$ iff it is $\overline{T}_0$ iff $(A,K)$ is $S\overline{T}_3$ iff it is $\overline{T}_3$ iff it is $KT_3$ iff $(A,K)$ is $S\overline{T}_4$ iff it is $\overline{T}_4$ iff $(A,K)$ is $ST_2$ or $\Delta T_2$ iff $(A,K)$ is $ST_3$ or $\Delta T_3$ iff $(A,K)$ is $NT2$.*

*2. By Theorem 3.1(5), Remark 3.5, Theorem 4.1, 4.3, 4.5, Corollary 3.1 and Remark 5.2, $(A,K)$ is $\overline{T}_2$ iff $(A,K)$ is $S\overline{T}_3$ iff $(A,K)$ is $\overline{T}_3$ iff $(A,K)$ is $KT_3$ iff $(A,K)$ is $S\overline{T}_4$ iff $(A,K)$ is $\overline{T}_4$ iff $(A,K)$ is $ST_2$ or $\Delta T_2$ iff $(A,K)$ is $ST_3$ or $\Delta T_3$ iff $(A,K)$ is $NT2$.*

*3. By Theorem 3.1(2), 4.1, 4.3, 4.5, Corollary 3.1 and Remark 5.2, if $(A,K)$ is $S\overline{T}_3$ or $\overline{T}_3$ or $KT_3$ or $S\overline{T}_4$ or $\overline{T}_4$ or $ST_2$ or $\Delta T_2$ or $ST_3$ or $\Delta T_3$ or $NT2$, then $(A,K)$ is $T_0'$. But the converse of implication is not true, in general. For example, let $A=\{x,y\}$ and $K=\{[x], [y], [\{x,y\}], [\emptyset]\}$. Then $(A,K)$ is $T_0'$ but it is not $S\overline{T}_3$ or $\overline{T}_3$ or $KT_3$ or $S\overline{T}_4$ or $\overline{T}_4$ or $ST_2$ or $\Delta T_2$ or $ST_3$ or $\Delta T_3$ or $NT2$.*

*4. By Theorem 3.1(3), 4.1, 4.3, 4.5, Corollary 3.1 and Remark 5.2, if $(A,K)$ is $S\overline{T}_4$ or $\overline{T}_3$ or $KT_3$ or $S\overline{T}_4$ or $\overline{T}_4$ or $ST_2$ or $\Delta T_2$ or $ST_3$ or $\Delta T_3$ or $NT2$, then $(A,K)$ is $Pre\overline{T}_2$. But the converse of implication is not true, in general. For example, let $A=\{x,y\}$ and $K=\{[x], [y], [\{x,y\}], [\emptyset]\}$. Then $(A,K)$ is $Pre\overline{T}_2$ but it is not $S\overline{T}_3$ or $\overline{T}_3$ or $KT_3$ or $S\overline{T}_4$ or $\overline{T}_4$ or $ST_2$ or $\Delta T_2$ or $ST_3$ or $\Delta T_3$ or $NT2$.*

*5. By Theorem 3.1(4), 3.6, 4.1, 4.3, 4.5, Corollary 3.1 and Remark 5.2, the following are equivalent:*

*(a) $(A,K)$ is $PreT_2'$ $(T_2')$, and is $S\overline{T}_3$ or $\overline{T}_3$ or $KT_3$ or $S\overline{T}_4$ or $\overline{T}_4$ or $ST_2$ or $\Delta T_2$ or $ST_3$ or $\Delta T_3$ or $NT2$.*

*(b) $A$ is a point or the empty set.*

*6. By Definition 3.2, Theorem 4.1, 4.3, 4.5, Corollary 3.1 and Remark 5.2, $(A,K)$ is $S\overline{T}_3$ or $\overline{T}_3$ or $KT_3$ or $S\overline{T}_4$ or $\overline{T}_4$ or $ST_2$ or $\Delta T_2$ or $ST_3$ or $\Delta T_3$ or $NT2$ iff $(A,K)$ is $T_2$.*

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