MANNHEIM CURVES IN n–DIMENSIONAL EUCLIDEAN SPACE

HANDAN ÖZTEKİN AND ESRA METİN

ABSTRACT. In this study, we define the generalized Mannheim Curves in n-dimensional Euclidean Space and obtain the caracterizations of the generalized Mannheim curves.

1. INTRODUCTION

The curves are a fundamental structure of differential geometry. In differential geometry, to study the corresponding relations between the curves is very important problem. Especially, Mannheim curves are one of them. Space curves of which principal normals are the binormal of another curFve are called Mannheim curves. the notion of Mannheim curve was discovered by A. Mannheim in 1878. These curves have beeen studied by many mathematicians (see [1] and [3-11]). For instance, In [5], Liu and Wang had obtained the necessary and sufficient conditions between the curvature and the torsion for a curve to be the Mannheim partner curves. In [3], Önder, Kazaz and Uğurlu had studied some characterizations of Mannheim partner curves in Minkowski 3-space. In [4], Kızıltuğ and Yaylı had given a study on the quaternionic Mannheim curves of AW(k)-type in Euclidean space. In [6] and [9], the authors had studied the generalized Mannheim curves in Euclidean 4-space and Minkowski space-time. In [10], Önder and Kızıltuğ had studied Bertrand and Mannheim Partner D-curves on Parallel surfaces in Minkowski 3-Space.

On the other hand, the articles concerning Mannheim curves in n-dimensional space are rather few. In [11], D.W. Yoon studied non-null Mannheim curve and null Mannheim curve in an n-dimensional Lorentzian manifold. To the best of our knowledge, Mannheim curves have not been presented in n-dimensional Euclidean space. Thus, the study is proposed to serve such a need.

The main goal of this paper is to carryout some results which were given in [6] to Mannheim curves in n-dimensional Euclidean space E^n.

Communications de la Faculté des Sciences de l'Université d'Ankara. Séries A1. Mathematics and Statistics.

98

2. Preliminaries

Let E^n be an n-dimensional Euclidean space with cartesian coordinates $(x^1, x^2, \dots, x^n)$. By a parametrized curve α of class C^∞, we mean a mapping α of a certain interval I into E^n given by

$$\alpha(t) = \left[x^1(t), x^2(t), \cdots, x^n(t) \right], \forall t \in I.$$

If $\left\| \frac{d\alpha(t)}{dt} \right\| = \left\langle \frac{d\alpha(t)}{dt}, \frac{d\alpha(t)}{dt} \right\rangle^{1/2} \neq 0$ for $t \in I$, then α is called a regular curve in E^n. Here $\langle .,. \rangle$ denotes the Euclidean inner product on E^n. A regular curve α is parametrized by the arc-length parameter s. Then the tangent vector field $\frac{d\alpha}{ds}$ along α has unit length, that is, $\left\| \frac{d\alpha(s)}{ds} \right\| = 1$ for all $s \in I$.

During this paper, curves considered are regular C^∞-curves in E^n parametrized by the arc-length parameter. The Frenet equations for such a curve are given by as follow:

$$\frac{de_1(s)}{ds} = k_1(s) e_2(s)$$
$$\frac{de_2(s)}{ds} = -k_1(s) e_1(s) + k_2(s) e_3(s)$$
$$\dots$$
$$\frac{de_{n-1}(s)}{ds} = -k_{n-2}(s) e_{n-2}(s) + k_{n-1}(s) e_n(s)$$
$$\frac{de_n(s)}{ds} = -k_{n-1}(s) e_{n-1}(s)$$

for all $s \in I$. The unit vector field $e_{j+1}, j = 1, 2, ..., n-1$, along α is called the Frenet j-normal vector field along α. A straight line is called the Frenet j-normal line of α at $\alpha(s)$, if it passes throught the point $\alpha(s)$ and is collinear to the j-normal vector e_{j+1}, $j = 1, 2, ..., n-1$, of α at $\alpha(s)$, [2].

3. Mannheim Curves in E^n

Definition 3.1. *Let α be a special Frenet curve in E^n. The curve α is called a generalized Mannheim curve if there exists a distinct special Frenet curve $\widetilde{\alpha}$ in E^n such that the first normal line at each point of α is included in the space spanned by the second normal line, the third normal line ,...,n^{th} normal line of $\widetilde{\alpha}$ at corresponding point under ϕ. Here ϕ is a bijection from α to $\widetilde{\alpha}$. The curve $\widetilde{\alpha}$ is called the generalized Mannheim mate curve of α.*

Now, we give following theorems for generalized Mannheim curves in E^n:

Theorem 3.1. *Let α be a special Frenet curve in E^n. If the curve α is a generalized Mannheim curve, then the first curvature function k_1 and second curvature function k_2 of α satisfy the equality*

$$k_1(s) = \lambda \left\{ (k_1(s))^2 + (k_2(s))^2 \right\}, s \in I,$$

where λ is a positive constant number.

Proof. Let α be a generalized Mannheim curve and $\widetilde{\alpha}$ be the generalized Mannheim mate curve of α.

By the definition, a generalized Mannheim mate $\widetilde{\alpha}$ is given by

$$\widetilde{\alpha}\left(s\right) = \alpha\left(s\right) + \lambda\left(s\right) e_2\left(s\right), s \in I, \tag{3.1}$$

where α is a smooth function on I. Generally, the parameter s isn't arc-length of $\widetilde{\alpha}$. Let $\widetilde{s}$ be the arc-parameter of $\widetilde{\alpha}$ defined by

$$\widetilde{s} = \int_0^s \left\| \frac{d\widetilde{\alpha}\left(s\right)}{ds} \right\| ds.$$

We can consider a smooth function $f : I \to \widetilde{I}$ given by $f\left(s\right) = \widetilde{s}$. Then we have

$$f'\left(s\right) = \sqrt{\left[1 - \lambda\left(s\right) k_1\left(s\right)\right]^2 + \left[\lambda\prime\left(s\right)\right]^2 + \left[\lambda\left(s\right) k_2\left(s\right)\right]^2} \tag{3.2}$$

Thus , we can write the reparametrization of $\widetilde{\alpha}$ by

$$\widetilde{\alpha}\left(f\left(s\right)\right) = \alpha\left(s\right) + \lambda\left(s\right) e_2\left(s\right) \tag{3.3}$$

here ϕ is a bijection from α to $\widetilde{\alpha}$ defined by

$$\phi\left(\alpha\left(s\right)\right) = \widetilde{\alpha}\left(f\left(s\right)\right).$$

By differentiating both sides of (3.3) with respect to s, we obtain

$$f'\left(s\right) \widetilde{e_1}\left(f\left(s\right)\right) = \left(1 - \lambda\left(s\right) k_1\left(s\right)\right) e_1\left(s\right) + \lambda\prime\left(s\right) e_2\left(s\right) + \lambda\left(s\right) k_2\left(s\right) e_3\left(s\right) \tag{3.4}$$

From Definition 3.1. and since the first normal line at the each point of α is lying in the plane generated by the second normal line, the third normal line ,...,n^{th} normal line $\widetilde{\alpha}$ at the corresponding points under bijection ϕ, we have $e_2\left(s\right) = x_3\left(s\right) \widetilde{e_3}\left(f\left(s\right)\right) + x_4\left(s\right) \widetilde{e_4}\left(f\left(s\right)\right) + ... + x_n\left(s\right) \widetilde{e_n}\left(f\left(s\right)\right)$, where x_3, x_4, ..., x_n are some smooth functions on I. Thus we obtain $\lambda\prime\left(s\right) = 0$, that is, the function α is constant. Then we rewrite the equation (3.4) by

$$\widetilde{e_1}\left(f\left(s\right)\right) = \frac{\left(1 - \lambda k_1\left(s\right)\right)}{f'\left(s\right)} e_1\left(s\right) + \frac{\lambda k_2\left(s\right)}{f'\left(s\right)} e_3\left(s\right), \tag{3.5}$$

where

$$f'\left(s\right) = \sqrt{\left[1 - \lambda k_1\left(s\right)\right]^2 + \left[\lambda k_2\left(s\right)\right]^2}, s \in I.$$

By taking differentiation both sides of the equation (3.5) with respect to s, we obtain

$$f'(s)\,\tilde{k}_1(f(s))\,\tilde{e}_2(f(s)) = \left(\frac{1-\lambda k_1(s)}{f'(s)}\right)' e_1(s)$$

$$+ \left(\frac{(1-\lambda k_1(s))\,k_1(s) - \lambda(k_2(s))^2}{f'(s)}\right) e_2(s)$$

$$+ \left(\frac{\lambda k_2(s)}{f'(s)}\right)' e_3(s) + \left(\frac{\lambda k_2(s)\,k_3(s)}{f'(s)}\right) e_4(s), \quad s \in I.$$

By the fact

$$\left\langle \tilde{e}_2(f(s)), x_3(s)\,\tilde{e}_3(f(s)) + x_4(s)\,\tilde{e}_4(f(s)) + ... + x_n(s)\,\tilde{e}_n(f(s)) \right\rangle = 0, s \in I,$$

we have that coefficient of e_2 in the above equation is zero, that is,

$$(1-\lambda k_1(s))\,k_1(s) - \lambda(k_2(s))^2 = 0, s \in I.$$

Thus we obtain

$$k_1(s) = \lambda\left[(k_1(s))^2 + (k_2(s))^2\right], s \in I$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Following theorem gives a parametric representation for Mannheim curves in n-dimensional Euclidean space E^n.

Theorem 3.2. *Let α be a curve defined by*

$$\alpha(u) = \left[\lambda\int f(u)\sin(u)\,du, \lambda\int f(u)\cos(u)\,du, \lambda\int f(u)\,h_1(u)\,du,\right.$$

$$\left.\cdots, \lambda\int f(u)\,h_{n-2}(u)\,du\right],$$

where λ is a positive constant, $h_i : U \to IR, i \in \{1, 2, ..., n-2\}$ are any smooth functions, and the smooth function $f : U \to IR^+, i \neq j$ is given by

$$
f = \left(1 + \sum_{i=1}^{n-2} h_i^2\right)^{-3/2} \left[1 + \sum_{i=1}^{n-2}\left(h_i^2 + \dot{h}_i^2\right) + \sum_{\substack{i,j=1 \\ j<i}}^{n-2}\left(h_i\dot{h}_j - \dot{h}_ih_j\right)^2\right]^{-5/2}
$$

$$
\times \left\{\left[1 + \sum_{i=1}^{n-2}\left(h_i^2 + \dot{h}_i^2\right) + \sum_{\substack{i,j=1 \\ j<i}}^{n-2}\left(h_i\dot{h}_j - \dot{h}_ih_j\right)^2\right]^3 + \left(1 + \sum_{i=1}^{n-2} h_i^2\right)^3\right\}
$$

$$
\times \left\{\sum_{i=1}^{n-2}\left(h_i + \ddot{h}_i\right)^2 + \sum_{\substack{i,j=1 \\ j<i}}^{n-2}\left[\left(\left(h_i\dot{h}_j - \dot{h}_ih_j\right) - \left(\dot{h}_i\ddot{h}_j - \ddot{h}_i\dot{h}_j\right)\right)^2 + \left(h_i\ddot{h}_j - \ddot{h}_ih_j\right)^2\right]\right\}
$$

for $u \in U$. Then the curvature functions k_1 and k_2 of α satisfy

$$
k_1\left(u\right) = \lambda\left\{\left[k_1\left(u\right)\right]^2 + \left[k_2\left(u\right)\right]^2\right\}
$$

at the each point $\alpha\left(u\right)$ of α.

Proof. Let α be a curve defined by

$$
\begin{aligned}
\alpha\left(u\right) &= \left[\lambda\int f\left(u\right)\sin\left(u\right)du, \lambda\int f\left(u\right)\cos\left(u\right)du, \lambda\int f\left(u\right)h_1\left(u\right)du,\right. \\
&\quad \left.\cdots, \lambda\int f\left(u\right)h_{n-2}\left(u\right)du\right],
\end{aligned}
$$

where λ is a non-zero constant number, $h_1, h_2, ..., h_{n-2}$ are any smooth functions, f is a positive valued smooth function. Thus, we obtain

$$
\dot{\alpha}\left(u\right) = \left[\lambda f\left(u\right)\sin\left(u\right), \lambda f(u)\cos\left(u\right), \lambda f\left(u\right)h_1\left(u\right), \cdots, \lambda f\left(u\right)h_{n-2}\left(u\right)\right], u \in U
$$

where the subscript dot ($\dot{}$) denotes the differentiation with respect to u. The arclength parameter s of α is given by

$$
s = \psi\left(u\right) = \int_{u_0}^{u}\left\|\dot{\alpha}\left(u\right)\right\|du, \qquad \text{and} \qquad \left\|\dot{\alpha}\left(u\right)\right\| = \lambda f\left(u\right)\left\{1 + \sum_{i=1}^{n-2}\left(h_i\left(u\right)\right)^2\right\}^{\frac{1}{2}}.
$$

Let φ denotes the inverse function of $\psi : U \to I \subset IR$, then $u = \varphi\left(s\right)$ and

$$
\varphi'\left(s\right) = \left\|\frac{d\alpha\left(u\right)}{du}\big|_{u=\varphi(s)}\right\|, s \in I,
$$

where the prime ($'$) denotes the differentiation with respect to s.

The unit tangent vector $e_1(s)$ of the curve α at the each point $\alpha(\varphi(s))$ is given by

$$e_1(s)$$

$$= \left[\left\{ 1 + \sum_{i=1}^{n-2} (h_i(\varphi(s)))^2 \right\}^{-1/2} \sin(\varphi(s)), \left\{ 1 + \sum_{i=1}^{n-2} (h_i(\varphi(s)))^2 \right\}^{-1/2} \cos(\varphi(s)), \right.$$

$$\left. \left\{ 1 + \sum_{i=1}^{n-2} (h_i(\varphi(s)))^2 \right\}^{-1/2} h_1(\varphi(s)), \cdots, \left\{ 1 + \sum_{i=1}^{n-2} (h_i(\varphi(s)))^2 \right\}^{-1/2} h_{n-2}(\varphi(s)) \right]$$

Now, we use the following abbreviations for the sake of brevity:

$$h_1 = h_1(\varphi(s)), h_2 = h_2(\varphi(s)), ..., h_{n-2} = h_{n-2}(\varphi(s))$$

$$\dot{h}_1(\varphi(s)) = \frac{dh_1(u)}{du} |_{u=(\varphi(s))}, \dot{h}_2(\varphi(s)) = \frac{dh_2(u)}{du} |_{u=(\varphi(s))}, ..., \dot{h}_{n-2}(\varphi(s))$$

$$= \frac{dh_{n-2}(u)}{du} |_{u=(\varphi(s))},$$

$$\ddot{h}_1(\varphi(s)) = \frac{d^2 h_1(u)}{du^2} |_{u=(\varphi(s))}, \ddot{h}_2(\varphi(s)) = \frac{d^2 h_2(u)}{du^2} |_{u=(\varphi(s))}, ..., \ddot{h}_{n-2}(\varphi(s))$$

$$= \frac{d^2 h_{n-2}(u)}{du^2} |_{u=(\varphi(s))},$$

$$\varphi' = \varphi'(s) = \frac{d\varphi(s)}{du} |_s, A = 1 + \sum_{i=1}^{n-2} h_i^2, B = \sum_{i=1}^{n-2} h_i \dot{h}_i, C = \sum_{i=1}^{n-2} \dot{h}_i^2, D = \sum_{i=1}^{n-2} h_i \ddot{h}_i, E$$

$$= \sum_{i=1}^{n-2} \dot{h}_i \ddot{h}_i, F = \sum_{i=1}^{n-2} \ddot{h}_i^2$$

Then we have

$$\dot{A} = 2B, \dot{B} = C + D, \dot{C} = 2E, \acute{\varphi} = \lambda^{-1} f^{-1} A^{-1/2}$$

and

$$e_1(s) = \left[A^{-1/2} \sin(\varphi(s)), A^{-1/2} \cos(\varphi(s)), A^{-1/2} h, \cdots A^{-1/2} h_{n-2} \right]$$

By differentiation the last equation with respect to s, we find

$$k_1 = k_1(s) = \|e_1'(s)\| = \varphi' A^{-1} (A + AC - B^2)^{1/2} \qquad (3.6)$$

By the fact that $e_2 = (k_1)^{-1} e'_1$, we have

$$e_2 = \begin{bmatrix} -A^{-1/2}B(A+AC-B)^{2-1/2}\sin(\varphi(s)) + A^{1/2}(A+AC-B^2)^{-1/2}\cos(\varphi(s)) \\ -A^{-1/2}B(A+AC-B^2)^{-1/2}\cos(\varphi(s)) - A^{1/2}(A+AC-B^2)^{-1/2}\sin(\varphi(s)) \\ -A^{-1/2}B(A+AC-B^2)^{-1/2}h_1 - A^{1/2}(A+AC-B^2)^{-1/2}\dot{h}_1 \\ \vdots \\ -A^{-1/2}B(A+AC-B^2)^{-1/2}h_{n-2} - A^{1/2}(A+AC-B^2)^{-1/2}\dot{h}_{n-2} \end{bmatrix}.$$

After long process of calculation, we have

$$e'_2 + k_1 e_1 = \varphi' A^{1/2}(A+AC-B^2)^{-3/2} \begin{bmatrix} \left(\breve{P} - \breve{Q}\right)\sin(\varphi(s)) - \breve{R}\cos(\varphi(s)) \\ \left(\breve{P} - \breve{Q}\right)\cos(\varphi(s)) + \breve{R}\sin(\varphi(s)) \\ \breve{P}h_1 - \breve{R}\dot{h}_1 + \breve{Q}\ddot{h}_1 \\ \vdots \\ \breve{P}h_{n-2} - \breve{R}\dot{h}_{n-2} + \breve{Q}\ddot{h}_{n-2} \end{bmatrix},$$

(3.7)

where

$$\breve{P} = (1 + C + BE - D - CD), \ \breve{Q} = \left(A + AC - B^2\right), \ \breve{R} = (B + AE - BD)$$

Consequently, from the (3.7) and (3.8) we find

$$\begin{aligned}
\|e'_2 + k_1 e_1\|^2 &= (\varphi')^2 A \left(A + AC - B^2\right)^{-3} \{\breve{P}^2 - 2\breve{P}\breve{Q} + \breve{Q}^2 + \breve{R}^2 \\
&\quad + \breve{P}^2 \left(h_1^2 + h_2^2 + ... + h_{n-2}^2\right) \\
&\quad + \breve{R}^2 \left(\dot{h}_1^2 + \dot{h}_2^2 + ... + \dot{h}_{n-2}^2\right) + \breve{Q}^2 \left(\ddot{h}_1^2 + \ddot{h}_2^2 + ... + \ddot{h}_{n-2}^2\right) \\
&\quad - 2\breve{P}\breve{R} \left(h_1\dot{h}_1 + h_2\dot{h}_2 + ... + h_{n-2}\dot{h}_{n-2}\right) \\
&\quad - 2\breve{R}\breve{Q} \left(\dot{h}_1\ddot{h}_1 + \dot{h}_2\ddot{h}_2 + ... + \dot{h}_{n-2}\ddot{h}_{n-2}\right) \\
&\quad + 2\breve{P}\breve{Q} \left(h_1\ddot{h}_1 + h_2\ddot{h}_2 + ... + h_{n-2}\ddot{h}_{n-2}\right)\}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
(k_2)^2 &= (\varphi')^2 A \left(A + AC - B^2\right)^{-2} \\
&\quad \times \{\left(A + AC - B^2\right)(1 + F) + 2(D - 1)(1 + C + BE - D - CD) \\
&\qquad - 2E(B + AE - BD) + 1 + C - 2D - 2CD + D^2 \\
&\qquad + CD^2 + 2BE - 2BDE + AE^2\} \\
&= (\varphi')^2 A \left(A + AC - B^2\right)^{-2} \\
&\quad \times \{\left(A + AC - B^2\right)(1 + F) - 1 - C + 2D \\
&\qquad + 2CD - 2BE - AE^2 - D^2 - CD^2 + 2BDE\}.
\end{aligned}$$

Moreover, from the equation (3.6) we have

$$(k_1)^2 = \varphi' A^{-2} \left(A + AC - B^2 \right).$$

The last two equation gives us

$$(k_1)^2 + (k_2)^2 = \lambda^{-2} f^{-2} A^{-3} \left(A + AC - B^2 \right)^{-2}$$
$$\times \{ \left(A + AC - B^2 \right)^3$$
$$+ A^3 (A + AC - B^2 + AF + ACF - B^2 F - 1 - C + 2D$$
$$+ 2CD - 2BE - AE^2 - D^2 - CD^2 + 2BDE) \}$$

and

$$k_1 = \lambda^{-1} f^{-1} A^{-3/2} \left(A + AC - B^2 \right)^{1/2}.$$

Thus, by setting

$$f = \left(1 + \sum_{i=1}^{n-2} h_i^2 \right)^{-3/2}$$

$$\times \left[1 + \sum_{i=1}^{n-2} \left(h_i^2 + \dot{h}_i^2 \right) + \sum_{\substack{i,j=1 \\ j<i}}^{n-2} \left(h_i \dot{h}_j - \dot{h}_i h_j \right)^2 \right]^{-5/2}$$

$$\times \left\{ \left[1 + \sum_{i=1}^{n-2} \left(h_i^2 + \dot{h}_i^2 \right) + \sum_{\substack{i,j=1 \\ j<i}}^{n-2} \left(h_i \dot{h}_j - \dot{h}_i h_j \right)^2 \right]^3 + \left(1 + \sum_{i=1}^{n-2} h_i^2 \right)^3 \right\}$$

$$\times \left\{ \sum_{i=1}^{n-2} \left(h_i + \ddot{h}_i \right)^2 + \right.$$

$$\left. \sum_{\substack{i,j=1 \\ j<i}}^{n-2} \left[\left(\left(h_i \dot{h}_j - \dot{h}_i h_j \right) - \left(\dot{h}_i \ddot{h}_j - \ddot{h}_i \dot{h}_j \right) \right)^2 + \left(h_i \ddot{h}_j - \ddot{h}_i h_j \right)^2 \right] \right\}$$

we obtain $k_1 = \lambda \left((k_1)^2 + (k_2)^2 \right)$.   □

Theorem 3.3. *Let $\{\alpha, \widetilde{\alpha}\}$, be a generalized Mannheim mate in E^n. Let $M, \widetilde{M}$ be the curvature centers at two corresponding point of α and $\widetilde{\alpha}$, respectively. Then the ratio $\frac{\|\widetilde{\alpha}(\tilde{s})M\|}{\|\alpha(s)M\|} : \frac{\|\widetilde{\alpha}(\tilde{s})\widetilde{M}\|}{\|\alpha(s)\widetilde{M}\|}$ is not constant.*

Proof. If M is the curvature centers of α, then we can write $M = \alpha(s) + \frac{1}{k_1} \cdot e_2$ and $\quad \|\alpha(s)\,M\| = \|M - \alpha(s)\| = \frac{1}{k_1}$. Similarly, we have

$$\left\|\widetilde{\alpha}(\widetilde{s})\,\widetilde{M}\right\| = \frac{1}{\widetilde{k}_1}, \quad \left\|\alpha(s)\,\widetilde{M}\right\| = \frac{\sqrt{1 + \widetilde{k}_1^2 . \sum\limits_{i=3}^{n} \mu_i^2}}{\widetilde{k}_1}, \quad \|\widetilde{\alpha}(\widetilde{s})\,M\| = \frac{(1 + k_1)}{k_1}\sqrt{\sum\limits_{i=3}^{n} \mu_i^2}.$$

Therefore, we obtain

$$\frac{\|\widetilde{\alpha}(\widetilde{s})\,M\|}{\|\alpha(s)\,M\|} : \frac{\left\|\widetilde{\alpha}(\widetilde{s})\,\widetilde{M}\right\|}{\left\|\alpha(s)\,\widetilde{M}\right\|} = (1 + k_1)\sqrt{\sum\limits_{i=3}^{n} \mu_i^2}\sqrt{1 + \widetilde{k}_1^2 . \sum\limits_{i=3}^{n} \mu_i^2}$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

From the above Theorem 3.3, we have following Corollory:

Corollary 3.4. *The Mannheim theorem for the generalized Mannheim curves in E^n is not valid.*

Theorem 3.5. *Let α and $\widetilde{\alpha}$ be two curves parametrized by the arc-length parameter s. If $\{\alpha, \widetilde{\alpha}\}$ is a generalized Mannheim mate in E^n , then there exists following relation*

$$\sum\limits_{i=3}^{n} \xi_i \left(\mu_{i-1}\widetilde{k}_{i-1} - \mu_{i+1}\widetilde{k}_i\right) = 0,$$

where $\mu_i, \xi_i, i \in \{1, 2, ..., n\}$ are arbitrary constants and $\widetilde{k}_1, \widetilde{k}_2, ..., \widetilde{k}_{n-1}$ are curvatures of $\widetilde{\alpha}$.

Proof. Since the curve α is Mannheim curve, then we have

$$\alpha(\widetilde{s}) = \widetilde{\alpha}(\widetilde{s}) + \sum\limits_{i=3}^{n} \mu_i(\widetilde{s})\,\widetilde{e}_i(\widetilde{s}) \text{ and } e_1\frac{ds}{d\widetilde{s}} = \widetilde{e}_1 - \mu_3\widetilde{k}_2\widetilde{e}_2 + \sum\limits_{i=3}^{n} \left(\mu_{i-1}\widetilde{k}_{i-1} - \mu_{i+1}\widetilde{k}_i\right)\widetilde{e}_i.$$

$$(3.9)$$

By taking inner product with $e_2 = \sum\limits_{i=3}^{n} \xi_i\widetilde{e}_i$ second equation of the equation (3.9),

we have $, \sum\limits_{i=3}^{n} \xi_i \left(\mu_{i-1}\widetilde{k}_{i-1} - \mu_{i+1}\widetilde{k}_i\right) = 0$. Thus the proof is completed. $\qquad\square$

Theorem 3.6. *The distance between corresponding points of a generalized Mannheim curve and of its generalized Mannheim partner curve in E^n is a constant.*

Proof. The proof is trivial. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

References

[1] Blum, R., A remarkable class of Mannheim curves, Canad. Math. Bull., 9(1966), 223-228.

[2] Gluck, H. Higher curvatures of curves in Euclidean space, Amer. Math. Monthly (1966), 73, 699–704.

[3] Kahraman, T., Önder, M., Kazaz, M., Uğurlu, H.H., Some Characterizations of Mannheim Partner Curves in Minkowski 3-space, Proceedings of the Estonian Academy of Sciences, (2011), 60,4,210-220.

[4] Kızıltuğ,S.,Yaylı, Y., On the quaternionic Mannheim curves of AW(k)-type in Euclidean space, Kuwait J. Sci.(2015), 42(2), pp. 128-140.

[5] Liu, H. and Wang, F., Mannheim partner curves in 3-space, Journal of Geometry, 88(2008), 120-126.

[6] Matsuda, H. and Yorozu, S., On generalized Mannheim curves in Euclidean 4-space, Nihonkai Math.J., 20(2009), no.1, 33-56.

[7] Orbay, K. and Kasap, E., On Mannheim partner curves in E^3, Int. J. of Phys. Sci., 4(5), (2009), 261-264.

[8] Tigano O., Sulla determinazione delle curve di Mannheim, Matematiche Catania, 3(1948), 25-29.

[9] Ersoy, S., Tosun, M., Matsuda, H.,Generalized Mannheim curves in Minkowski Space-Time E_1^4, Mathematical Problems in Engineering, Volume 2011(2011), Article ID 539378,19 pages, doi./10.1155/2011/539378.

[10] Önder M. and Kızıltuğ S., Bertrand and Mannheim Partner D-curves on Parallel surfaces in Minkowski 3-Space, Int. Journal of Geometry, vol:1(2012), No.2, 34-45.

[11] Yoon, DW., Mannheim Curves in an n-dimensional Lorentz Manifold, International journal of Pure and Applied Mathematics, Vol. 96 No. 2 (2014), 165-174.

Current address: Handan ÖZTEKİN: Department of Mathematics, Firat University, 23119 Elazığ TURKEY

E-mail address: handanoztekin@gmail.com

Current address: Esra METİN: Department of Mathematics, Firat University, 23119 Elazığ TURKEY