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CERTAIN SUBCLASSES OF ANALYTIC AND BI-UNIVALENT FUNCTIONS INVOLVING THE q-DERIVATIVE OPERATOR

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Abstract. In this paper, we introduce and investigate two new subclasses $\mathcal{H}_{\Sigma}^{q,\alpha}$ and $\mathcal{H}_{\Sigma}^{q}(\beta)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to these classes, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z:z\in\mathbb{C}\quad\text{and}\quad|z|<1\}$. We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [3] ensures that the image of $\mathbb{U}$ under every univalent function $f\in\mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f\in\mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z))=z\qquad(z\in\mathbb{U})$$

and

$$f\left(f^{-1}(w)\right)=w\qquad\left(|w|<r_0(f);\ r_0(f)\geq\frac{1}{4}\right).$$

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In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of functions in the class Σ, see [8] (see also [2]). In fact, the aforecited work of Srivastava *et al.* [8] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [4], Xu *et al.* [9, 10] (see also the references cited in each of them).

Quantum calculus is ordinary classical calculus without the notion of limits. It defines q-calculus and h-calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. Recently, the area of q-calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [5, 6]. He was the first to develop q-integral and q-derivative in a systematic way. Later, geometrical interpretation of q-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis. A comprehensive study on applications of q-calculus in operator theory may be found in [1].

For a function $f \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the q-derivative of function f is defined by (see [5, 6])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \qquad (z \neq 0), \tag{1.2}$$

$D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (1.2), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{1.3}$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \tag{1.4}$$

As $q \to 1^-$, $[k]_q \to k$. For a function $g(z) = z^k$, we get

$$\begin{aligned} D_q\left(z^k\right) &= [k]_q z^{k-1}, \\ \lim_{q\to 1^-}\left(D_q\left(z^k\right)\right) &= kz^{k-1} = g'(z), \end{aligned}$$

where g' is the ordinary derivative.

By making use of the q-derivative of a function $f \in \mathcal{A}$, we introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ.

Firstly, in order to derive our main results, we need to following lemma.

Lemma 1. [7] *If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which*

$$\Re\left(p\left(z\right)\right) > 0, \quad p\left(z\right) = 1 + c_1 z + c_2 z^2 + \cdots$$

for $z \in \mathbb{U}$.

2. Coefficient bounds for the function class $\mathcal{H}_{\Sigma}^{q,\alpha}$

Definition 1. A function $f\left(z\right)$ given by (1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{q,\alpha}$ $(0 < q < 1,\, 0 < \alpha \leq 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(D_q f\left(z\right)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \tag{2.1}$$

and

$$\left|\arg\left(D_q g\left(w\right)\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}) \tag{2.2}$$

where the function g is given by

$$g\left(w\right) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2.3}$$

Remark 1. Note that we have the class $\lim\limits_{q \to 1^-} \mathcal{H}_{\Sigma}^{q,\alpha} = \mathcal{H}_{\Sigma}^{\alpha}$ introduced by Srivastava *et al.* [8].

Theorem 1. *Let the function $f\left(z\right)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathcal{H}_{\Sigma}^{q,\alpha}$ $(0 < q < 1,\, 0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\left[3\right]_q \alpha + \left(1-\alpha\right)\left[2\right]_q^2}} \tag{2.4}$$

and

$$|a_3| \leq \frac{4\alpha^2}{\left[2\right]_q^2} + \frac{2\alpha}{\left[3\right]_q}. \tag{2.5}$$

Proof. First of all, it follows from the conditions (2.1) and (2.2) that

$$D_q f\left(z\right) = \left[P\left(z\right)\right]^{\alpha} \quad \text{and} \quad D_q g\left(w\right) = \left[Q\left(w\right)\right]^{\alpha} \quad (z, w \in \mathbb{U}), \tag{2.6}$$

respectively, where

$$P(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad \text{and} \quad Q(w) = 1 + q_1 w + q_2 w^2 + \cdots$$

in $\mathcal{P}$. Now, upon equating the coefficients in (2.6), we get

$$[2]_q a_2 = \alpha p_1, \tag{2.7}$$

$$[3]_q a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \tag{2.8}$$

$$-[2]_q a_2 = \alpha q_1 \tag{2.9}$$

and

$$[3]_q \left(2a_2^2 - a_3\right) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2. \tag{2.10}$$

From (2.7) and (2.9), we obtain

$$p_1 = -q_1 \tag{2.11}$$

and

$$2[2]_q^2 a_2^2 = \alpha^2 \left(p_1^2 + q_1^2\right). \tag{2.12}$$

Also, from (2.8), (2.10) and (2.12), we find that

$$2[3]_q a_2^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2+q_1^2\right) = \alpha(p_2+q_2) + \frac{\alpha-1}{\alpha}[2]_q^2 a_2^2$$

Therefore, we obtain

$$a_2^2 = \frac{\alpha^2}{2[3]_q \alpha + (1-\alpha)[2]_q^2}(p_2+q_2)$$

Applying Lemma 1 for the above equality, we get the desired estimate on the coefficient $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.10) from (2.8). We thus get

$$2[3]_q a_3 - 2[3]_q a_2^2 = \alpha(p_2-q_2) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2 - q_1^2\right). \tag{2.13}$$

It follows from (2.11), (2.12) and (2.13) that

$$a_3 = \frac{\alpha^2}{2[2]_q^2}\left(p_1^2+q_1^2\right) + \frac{\alpha}{2[3]_q}(p_2-q_2).$$

Applying Lemma 1 for the above equality, we get the desired estimate on the coefficient $|a_3|$ as asserted in (2.5). □

Taking $q \to 1^-$ in Theorem 1, we obtain the following result.

Corollary 1. [8] *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1.1) *be in the class* $\mathcal{H}_{\Sigma}^{\alpha}\ (0<\alpha\leq 1)$. *Then*

$$|a_2|\leq \alpha\sqrt{\frac{2}{\alpha+2}} \quad \text{and} \quad |a_3|\leq \frac{\alpha(3\alpha+2)}{3}.$$

3. Coefficient bounds for the function class $\mathcal{H}_{\Sigma}^{q}(\beta)$

Definition 2. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{q}(\beta)$ $(0<q<1,\ 0\leq\beta<1)$ if the following conditions are satisfied:

$$f\in\Sigma \quad \text{and} \quad \Re\{D_q f(z)\}>\beta \qquad (z\in\mathbb{U}) \tag{3.1}$$

and

$$\Re\{D_q g(w)\}>\beta \qquad (w\in\mathbb{U}) \tag{3.2}$$

where the function g is defined by (2.3).

Remark 2. Note that we have the class $\lim_{q\to 1^-}\mathcal{H}_{\Sigma}^{q}(\beta)=\mathcal{H}_{\Sigma}(\beta)$ introduced by Srivastava *et al.* [8].

Theorem 2. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1.1) *be in the function class* $\mathcal{H}_{\Sigma}^{q}(\beta)$ $(0<q<1,\ 0\leq\beta<1)$. *Then*

$$|a_2|\leq \min\left\{\frac{2(1-\beta)}{[2]_q},\ \sqrt{\frac{2(1-\beta)}{[3]_q}}\right\} \tag{3.3}$$

and

$$|a_3|\leq \frac{2(1-\beta)}{[3]_q}. \tag{3.4}$$

Proof. First of all, it follows from the conditions (3.1) and (3.2) that

$$D_q f(z)=\beta+(1-\beta)P(z) \quad \text{and} \quad D_q g(w)=\beta+(1-\beta)Q(w) \quad (z,w\in\mathbb{U}), \tag{3.5}$$

respectively, where

$$P(z)=1+p_1z+p_2z^2+\cdots \quad \text{and} \quad Q(w)=1+q_1w+q_2w^2+\cdots$$

in $\mathcal{P}$. Now, upon equating the coefficients in (3.5), we get

$$[2]_q\, a_2=(1-\beta)p_1, \tag{3.6}$$

$$[3]_q\, a_3=(1-\beta)p_2, \tag{3.7}$$

$$-[2]_q\, a_2=(1-\beta)q_1 \tag{3.8}$$

and

$$[3]_q \left(2a_2^2 - a_3\right) = (1-\beta)\, q_2. \tag{3.9}$$

From (3.6) and (3.8), we obtain

$$p_1 = -q_1 \tag{3.10}$$

and

$$2\,[2]_q^2\, a_2^2 = (1-\beta)^2 \left(p_1^2 + q_1^2\right). \tag{3.11}$$

Also, from (3.7) and (3.9), we have

$$2\,[3]_q\, a_2^2 = (1-\beta)\,(p_2 + q_2). \tag{3.12}$$

Applying Lemma 1 for (3.11) and (3.12), we get the desired estimate on the coefficient $|a_2|$ as asserted in (3.3).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (3.9) from (3.7). We thus get

$$2\,[3]_q\, a_3 - 2\,[3]_q\, a_2^2 = (1-\beta)\,(p_2 - q_2), \tag{3.13}$$

which, upon substitution of the value of a_2^2 from (3.11), yields

$$a_3 = \frac{(1-\beta)^2}{2\,[2]_q^2}\left(p_1^2 + q_1^2\right) + \frac{(1-\beta)}{2\,[3]_q}\,(p_2 - q_2). \tag{3.14}$$

On the other hand, by using the equation (3.12) into (3.13), it follows that

$$a_3 = \frac{1-\beta}{2\,[3]_q}\,(p_2+q_2) + \frac{1-\beta}{2\,[3]_q}\,(p_2-q_2) = \frac{1-\beta}{[3]_q} p_2. \tag{3.15}$$

Applying Lemma 1 for (3.14) and (3.15), we get the desired estimate on the coefficient $|a_3|$ as asserted in (3.4). $\square$

Taking $q \to 1^-$ in Theorem 2, we obtain the following result.

Corollary 2. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the class $\mathcal{H}_\Sigma(\beta)$ $(0 \le \beta < 1)$. Then*

$$|a_2| \le \begin{cases} \sqrt{\frac{2(1-\beta)}{3}} & , \quad 0 \le \beta \le \frac{1}{3} \\ 1-\beta & , \quad \frac{1}{3} \le \beta < 1 \end{cases} \quad \textit{and} \quad |a_3| \le \frac{2(1-\beta)}{3}.$$

Remark 3. Corollary 2 provides an improvement of the following estimates obtained by Srivastava *et al.* [8].

Corollary 3. [8] *Let the function* $f(z)$ *given by the Taylor-Maclaurin series expansion* (1.1) *be in the class* $\mathcal{H}_{\Sigma}(\beta)$ $(0 \leq \beta < 1)$. *Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \quad \text{and} \quad |a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}.$$

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