Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.
Volume 66, Number 1, Pages 133–152(2017)
DOI: 10.1501/Commua1_0000000783
ISSN 1303–5991

ASYMPTOTIC DISTRIBUTION OF EIGENVALUES FOR FOURTH-ORDER BOUNDARY VALUE PROBLEM WITH DISCONTINUOUS COEFFICIENTS AND TRANSMISSION CONDITIONS

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Abstract. We investigate a fourth-order boundary value problem with discontinuous coefficients, functional many points and transmission conditions. In this problem, boundary conditions contain not only endpoints of the considered interval, but also a point of discontinuity, a finite number internal points and abstract linear functionals. We discuss asymptotic distribution of its eigenvalues. Finally, we obtain asymptotic formulas for the eigenvalues of the problem in sectors of the complex plane.

1. Introduction

In classical theory, boundary-value problems for ordinary differential equations are usually considered for equations with continuous coefficients and for boundary conditions which contain only end-points of the considered interval. However, this paper deals with one nonclassical boundary-value problem for ordinary differential equation with discontinuous coefficients and boundary conditions containing not only end-points of the considered interval, but also a point of discontinuity and internal points. This type problems are connected with different applied problems which include various transfer problems such as heat transfer in heterogeneous media. Naturally, transmission problems arise in various physical fields as the theory of diffraction, elasticity, heat and mass transfer [10], [16], [17], [18].

The investigation of boundary value problem for which the eigenvalue parameter appears both in the equation and boundary conditions originates from the works of G. D. Birkhoff [4], [5]. There are many papers and books that the spectral properties of such problem are investigated; see[2], [3], [6]. Some spectral properties of such problems with discontinuous coefficients and the eigenvalue parameter both in the differential equation and boundary conditions have been studied by O. Sh.

Received by the editors: March 30, 2016, Accepted: Aug. 25, 2016.
2010 *Mathematics Subject Classification.* 34A36; 34B09; 34L20.
Key words and phrases. Fourth order problem, eigenvalue parameter, asymptotic distribution, internal points, transmission conditions.

Mukhtarov, M. Kandemir and some others [7], [8], [9], [11], [12], [13]. In this study, we shall consider fourth-order differential equation

$$p(x)u^{(4)} + q(x)u = \lambda^4 u,\ x \in I, \tag{1.1}$$

with the functional-transmission boundary conditions

$$\begin{aligned} L_k(u) \quad = \quad & \sum_{s=0}^{3} \lambda^{4-s}[\alpha_{ks}u^{(s)}(-1) \\ & +\beta_{ks}u^{(s)}(-0) + \delta_{ks}u^{(s)}(+0) + \gamma_{ks}u^{(s)}(1) \\ & + \int_{-1}^{0} u^{(s)}(x)\phi_{ks}(x)dx \\ & + \int_{0}^{1} u^{(s)}(x)\phi_{ks}(x)dx \\ & + \sum_{i=1}^{2}\sum_{j=1}^{N_{ks}^i} \zeta_{ks}^{ij} u^{(s)}(a_{ksj}^i)] = 0,\ k = 1, 2, ..., 8, \end{aligned} \tag{1.2}$$

where $I = I_1 \cup I_2 = [-1, 0) \cup (0, 1]$; $p(x)$ and $q(x)$ are complex valued functions; $p(x) = p_j(x)$ and $q(x) = q_j(x)$ for $x \in I_j$, $j = 1, 2$; $\alpha_{ks},\ \beta_{ks},\ \delta_{ks}\ ,\gamma_{ks},\ \zeta_{ks}$ are complex coefficients; $a_{ksj}^i \in I_i$ internal points and $u^{(m_k)}(\mp 0)$ denotes $\underset{x\to\mp 0}{lim} u^{(m_k)}(x)$.

Denote:

$$F_{1k}u := \sum_{s=0}^{3} \lambda^{4-s} \int_{-1}^{0} u^{(s)}(x)\phi_{ks}(x)dx$$

and

$$F_{2k}u := \sum_{s=0}^{3} \lambda^{4-s} \int_{0}^{1} u^{(s)}(x)\phi_{ks}(x)dx.$$

F_{1k} and F_{2k} are abstract linear functionals. $F_{1k} + F_{2k}$ acts from $W_p^k(-1, 0) + W_p^k(0, 1)$ into complex plane $\mathbb{C}$ continuously. In virtue of the general representation of the continuous linear functionals in the $L_q(-1, 1)$ spaces and using the well-known methods of real analysis it may be shown that there exists a function $\phi_{ks}(x) \in W_p^k(-1, 0) + W_p^k(0, 1)$ such that for every $u \in W_q^k(-1, 0) + W_q^k(0, 1)$, $(\frac{1}{p} + \frac{1}{q} = 1)$.

$W_p^q(-1, 0, 1) := W_p^q(-1, 0) + W_p^q(0, 1),\ 1 < p < \infty,\ q = 0, 1, 2, ...,$ denotes the Banach spaces of complex valued functions $u = u(x)$ defined on $[-1, 0)\cup(0, 1]$, which belongs to $W_p^q(-1, 0)$ and $W_p^q(0, 1)$ on intervals $(-1, 0)$ and $(0, 1)$, respectively, with the norm

$$\|u\|_{W_p^q(-1,0,1)} = \left(\|u\|^p_{W_p^q(-1,0)} + \|u\|^p_{W_p^q(0,1)}\right)^{\frac{1}{p}}$$

where $W_p^q(-1, 0)$ and $W_p^q(0, 1)$ are the usual Sobolev space [1].

Note that, without loss of generality we consider the equation (1.1) instead of more general equation

$$p(x)u^{(4)}+p_3(x)u'''+p_2(x)u''+p_1(x)u'+p_0(x)u=\lambda^4 u,\ x\in I. \tag{1.3}$$

If $p_3\neq 0$, by using the substitution

$$u=\widetilde{u}e^{\psi(x)},$$

$$\psi(x)=\begin{cases} -\frac{1}{4p_1}\int_{-1}^{x}p_3(t)dt,\ x\in[-1,0) \\ \\ -\frac{1}{4p_2}\int_{-1}^{x}p_3(t)dt,\ x\in(0,1] \end{cases},$$

we can find that equation (1.3) takes the form

$$p(x)\widetilde{u}^{(4)}+\widetilde{p}_2(x)\widetilde{u}''+\widetilde{p}_1(x)\widetilde{u}'+\widetilde{p}_0(x)\widetilde{u}=\lambda^4\widetilde{u},$$

where $\widetilde{p}_2,\ \widetilde{p}_1,\ \widetilde{p}_0$ are continuous in I and λ is the same eigenvalue parameter. Therefore, we can write equation (1.1) instead of equation (1.3) from [14]. Also, it is easy to verify that under this substitution the form of boundary conditions (1.3) has not changed.

2. Eigenvalues of the problem

Let u_{1j} and $u_{2j},\ j=1,2,3,4$, denote some fundamental systems of solutions of the differential equation (1.1) on I_1 and I_1, respectively. By defining

$$\left\{\begin{array}{l} u_{1j}(x,\lambda)=0,\ x\in I_2 \\ u_{2j}(x,\lambda)=0,\ x\in I_1 \end{array}\right| \ j=1,2,3,4,$$

the general solution of the equation (1.1) can be written in the form

$$u(x,\lambda)=\sum_{\upsilon=1}^{2}\sum_{j=1}^{4}c_{\upsilon j}u_{\upsilon j}(x,\lambda), \tag{2.1}$$

where $c_{\upsilon j}$ are arbitrary constant numbers. Substituting (2.1) into boundary conditions (2.1) yields a system of linear homogeneous equations

$$L_k(u(x,\lambda))=\sum_{\upsilon=1}^{2}\sum_{j=1}^{4}c_{\upsilon j}L_k\left(u_{\upsilon j}\right)=0,\ k=1,2,...,8 \tag{2.2}$$

for the determination of the constants $c_{\upsilon j},\upsilon=1,2,\ j=1,2,3,4$. Consequently, the eigenvalues of the boundary value problem (1.1)-(1.2) consist of zeros of the characteristic determinant

$$\begin{aligned} \Delta(\lambda) &= \det\left(L_k\left(u_{\upsilon j}\right)\right)_{8\times 8},\ \upsilon=1,2, \\ j &= 1,2,3,4,\ k=1,2,...,8. \end{aligned} \tag{2.3}$$

First, according to considered problem, we shall divide the complex λ-plane into specific sectors, in which we shall find the asymptotic expression for solutions of the differential equation, for boundary functionals and boundary value forms with transmission conditions. Then, by substituting these obtained asymptotic expression into the equation $\Delta(\lambda)=0$ we shall find the corresponding asymptotic formulas for the eigenvalues of the problem. Note that, such formulas are not only of interest in themselves, but also they may be used for establishing the completeness and basis properties of the system of eigen-and associated functions of considered problem. In this study, we shall investigate the cases of both $\arg p_1 \neq \arg p_2$ and $\arg p_1 = \arg p_2$.

3. Asymptotic distribution of eigenvalues for the case $\arg p_1 \neq \arg p_2$

3.1. **Separation of the complex $\lambda-$plane into specific sectors.** Throughout the paper we employ the notation

$$\begin{aligned}\omega_{j1} &= (p_j)^{-\frac{1}{4}},\ \omega_{j2} = -(p_j)^{-\frac{1}{4}}\\ \omega_{j3} &= i\,(p_j)^{-\frac{1}{4}},\ \omega_{j4} = -i\,(p_j)^{-\frac{1}{4}},\ j=1,2\end{aligned}$$

where $z^{\frac{1}{4}} := |z|\, e^{\frac{i(\arg z)}{4}}$, $-\pi < \arg z < \pi$. Divide the complex $\lambda-$plane into eight sectors S_k, $k=1,2,\ldots,8$, by the rays

$$\begin{aligned}l_k \;=\; &\{\lambda \in \mathbb{C}|\ \mathrm{Re}\lambda\omega_{\upsilon j}=0,\ (-1)^k \mathrm{Im}\lambda\omega_{\upsilon j} \le 0\\ &\upsilon = 1,2,\ j=1,2,3,4\ \}.\end{aligned}$$

On all of these sectors each of the real valued functions $\mathrm{Re}\lambda\omega_{\upsilon j}$ is of a single sign, since these functions can vanish only on boundaries S_k. Let us consider one of the sectors (S_k) with fixed index k. Using the same considerations as in [14] it is easy to verify that for equation (1.1) there exists a fundamental system of particular solutions $u_{1j}(x,\lambda)$ on I_1, $j=1,2,3,4$, and $u_{2j}(x,\lambda)$ on I_2, $j=1,2,3,4$, respectively, which are analytic functions of $\lambda \in S_k$ and for sufficiently large $|\lambda|$, and which with derivatives, can be expressed in the asymptotic form

$$\begin{aligned}u_{\upsilon j}(x,\lambda) &= e^{\lambda\omega_{\upsilon j}x}(1+O(\frac{1}{\lambda}))\\ u_{\upsilon j}^{(s)}(x,\lambda) &= (\lambda\omega_{\upsilon j})^s\, e^{\lambda\omega_{\upsilon j}x}(1+O(\frac{1}{\lambda})),\\ \upsilon &= 1,2,\ j=1,2,3,4.\end{aligned} \tag{3.1}$$

Here, as usual, the expression $O(\frac{1}{\lambda})$ denotes any function of the form $\frac{f(x,\lambda)}{\lambda}$, where $|f(x,\lambda)|$ for $x \in I_j$, $j=1,2$, and sufficiently large $|\lambda|$ always remain less than a constant.

Now let l'_k, $k=1,2,\ldots,8$, be arbitrary rays, which originate from the point $\lambda=0$, distinct from the rays l and situated so as to from the sequence

$$l_1,\ l'_1,\ l_2,\ l'_2,\ l_3,\ l'_3,\ l_4,\ l'_4, ..., l_8,\ l'_8.$$

The rays l'_k divide each sector S_k into two subsectors. Therefore, we have sixteen sectors which we shall denote as Ω_i, $i = 1, 2, ..., 16$. As it seems from the construction, the sectors $\Omega = \{\Omega_1, \Omega_2, ..., \Omega_{16}\}$ can be distributed into two groups of

$$\Omega^{(i)} = \left\{\Omega_1^{(i)}, \Omega_2^{(i)}, ..., \Omega_8^{(i)}\right\}, \ i = 1, 2$$

such that, the group $\Omega^{(k)}$, $k = 1, 2$, includes those sectors Ω_i, $i = 1, 2, ..., 16$, in which

$$\mathrm{Re}\lambda\omega_{\upsilon j} \to \infty, \ \upsilon = 1, 2, \ j = 1, 2, 3, 4, \ \text{as } \lambda \to \infty.$$

3.2. **Asymptotic expressions for the characteristic determinant $\Delta(\lambda)$ in the Ω sectors.** Each of the real valued functions $\mathrm{Re}\lambda\omega_{j\upsilon}$ does not change sign also in each sector Ω_i, since each of them is a subsector of certain sector S_k.

Let $u_{\upsilon j} = u_{\upsilon j}(x, \lambda)$, $x \in I_\upsilon$, $\upsilon = 1, 2$, $j = 1, 2, ..., 8$, are functions defining as for the fundamental system in I_υ, for which satisfied asymptotic expressions (3.1). Only in one of the sectors of the groups $\Omega^{(1)}$ the conditions

$$\begin{aligned} \mathrm{Re}\lambda\omega_{11} &\to +\infty, \ \mathrm{Re}\lambda\omega_{21} \geq 0, \\ \mathrm{Re}\lambda\omega_{13} &\to +\infty, \ \mathrm{Re}\lambda\omega_{23} \geq 0 \end{aligned}$$

and only in one of the sectors of the groups $\Omega^{(2)}$ the conditions

$$\begin{aligned} \mathrm{Re}\lambda\omega_{21} &\to +\infty, \ \mathrm{Re}\lambda\omega_{11} \geq 0, \\ \mathrm{Re}\lambda\omega_{23} &\to +\infty, \ \mathrm{Re}\lambda\omega_{13} \geq 0 \end{aligned}$$

are holds for $\lambda \to \infty$. We shall denote these sectors as $\Omega_0^{(1)}$ and $\Omega_0^{(2)}$, respectively. Besides, we shall denote by $[A]$, $A \in \mathbb{C}$, any sum of the from $A + f(\lambda)$ when $f(\lambda) \to 0$ as $\lambda \to \infty$.

First, let λ vary in $\Omega_0^{(1)}$. Substituting (3.1) into (1.2), remembering that

$$\begin{aligned} \omega_{11} &= -\omega_{12}, \ \omega_{13} = -\omega_{14}, \\ \omega_{21} &= -\omega_{22}, \ \omega_{23} = -\omega_{24} \end{aligned}$$

and applying well-known Rieamann-Lebesgue Lemma [14, p. 117, Lemma 7), we have

$$\begin{aligned} & L_k(u_{11}) \\ = \ & \sum_{s=0}^{3} \lambda^{4-s} \Big((\lambda\omega_{11})^s \left(\alpha_{ks} e^{-\lambda\omega_{11}} [1] + \beta_{ks} [1]\right) \\ & + (\lambda\omega_{11})^s \int_{-1}^{0} e^{\lambda\omega_{11} x} (1 + O(\frac{1}{\lambda}))\phi_{ks}(x) dx \\ & + \sum_{j=1}^{N_{ks}^1} \zeta_{ks}^{1j} (\lambda\omega_{11})^s e^{\lambda\omega_{11} a_{ksj}^1} [1] \Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=0}^{3} \lambda^{4-s} \left(\lambda\omega_{11}\right)^{s} \Big(\alpha_{ks} e^{-\lambda\omega_{11}} [1] + \beta_{ks} [1] \\
&\quad + \int_0^1 e^{-\lambda\omega_{11} x} (1 + O(\frac{1}{\lambda})) \phi_{ks}(-x) dx + [0]\Big) \\
&= \sum_{s=0}^{3} \lambda^{4-s} \left(\lambda\omega_{11}\right)^{s} \left(\beta_{ks} [1] + [0]\right) \\
&= \lambda^{4} \left[\beta_{k0} + \omega_{11}\beta_{k1} + \omega_{11}^{2}\beta_{k2} + \omega_{11}^{3}\beta_{k3}\right],
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
& L_k(u_{12}) \\
= \; & \lambda^{4} e^{-\lambda\omega_{12}} \left[\alpha_{k0} + \omega_{12}\alpha_{k1} + \omega_{12}^{2}\alpha_{k2} + \omega_{12}^{3}\alpha_{k3}\right],
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
& L_k(u_{13}) \\
= \; & \lambda^{4} \left[\beta_{k0} + \omega_{13}\beta_{k1} + \omega_{13}^{2}\beta_{k2} + \omega_{13}^{3}\beta_{k3}\right],
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
& L_k(u_{14}) \\
= \; & \lambda^{4} e^{-\lambda\omega_{14}} \left[\alpha_{k0} + \omega_{14}\alpha_{k1} + \omega_{14}^{2}\alpha_{k2} + \omega_{14}^{3}\alpha_{k3}\right],
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
& L_k(u_{21}) \\
= \; & \sum_{s=0}^{3} \lambda^{4-s} \Big(\left(\lambda\omega_{21}\right)^{s} \left(\delta_{ks} [1] + \gamma_{ks} e^{\lambda\omega_{21}} [1]\right) \\
& + \left(\lambda\omega_{21}\right)^{s} \int_0^1 e^{\lambda\omega_{21} x} (1 + O(\frac{1}{\lambda})) \phi_{ks}(x) dx \\
& + \sum_{j=1}^{N_{ks}^{2}} \zeta_{ks}^{2j} \left(\lambda\omega_{21}\right)^{s} e^{\lambda\omega_{21} a_{ksj}^{2}} [1]\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \; & \lambda^{4} \sum_{s=0}^{3} \Big(\omega_{21}^{s} \left(\delta_{ks} [1] + \gamma_{ks} e^{\lambda\omega_{21}} [1]\right) \\
& + \omega_{21}^{s} e^{\lambda\omega_{21}} \int_0^1 e^{-\lambda\omega_{21}(1-x)} (1 + O(\frac{1}{\lambda})) \phi_{ks}(1-x) dx \\
& + \sum_{j=1}^{N_{ks}^{2}} \zeta_{ks}^{2j} \left(\omega_{21}\right)^{s} e^{\lambda\omega_{21} a_{ksj}^{2}} [1]\Big)
\end{aligned}
$$

$$\begin{aligned} &= \lambda^4 \Big(\big[\delta_{k0} + \omega_{21}\delta_{k1} + \omega_{21}^2\delta_{k2} + \omega_{21}^3\delta_{k3}\big] \\ &\quad + e^{\lambda\omega_{21}} \big[\gamma_{k0} + \omega_{21}\gamma_{k1} + \omega_{21}^2\gamma_{k2} + \omega_{21}^3\gamma_{k3}\big] \\ &\quad + \sum_{j=1}^{N_{ks}^2} e^{\lambda\omega_{21}a_{ksj}^2}[\omega_{21}^s\zeta_{ks}^{2j}]\Big), \end{aligned} \tag{3.6}$$

$$\begin{aligned} L_k(u_{22}) &= \lambda^4 \Big(\big[\delta_{k0} + \omega_{22}\delta_{k1} + \omega_{22}^2\delta_{k2} + \omega_{22}^3\delta_{k3}\big] \\ &\quad + e^{\lambda\omega_{22}} \big[\gamma_{k0} + \omega_{22}\gamma_{k1} + \omega_{22}^2\gamma_{k2} + \omega_{22}^3\gamma_{k3}\big] \\ &\quad + \sum_{j=1}^{N_{ks}^2} e^{\lambda\omega_{22}a_{ksj}^2}[\omega_{22}^s\zeta_{ks}^{2j}]\Big), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &L_k(u_{23}) \\ &= \lambda^4 \Big(\big[\delta_{k0} + \omega_{23}\delta_{k1} + \omega_{23}^2\delta_{k2} + \omega_{23}^3\delta_{k3}\big] \\ &\quad + e^{\lambda\omega_{23}} \big[\gamma_{k0} + \omega_{23}\gamma_{k1} + \omega_{23}^2\gamma_{k2} + \omega_{23}^3\gamma_{k3}\big] \\ &\quad + \sum_{j=1}^{N_{ks}^2} e^{\lambda\omega_{23}a_{ksj}^2}[\omega_{23}^s\zeta_{ks}^{2j}]\Big), \end{aligned} \tag{3.8}$$

$$\begin{aligned} &L_k(u_{24}) \\ &= \lambda^4 \Big(\big[\delta_{k0} + \omega_{24}\delta_{k1} + \omega_{24}^2\delta_{k2} + \omega_{24}^3\delta_{k3}\big] \\ &\quad + e^{\lambda\omega_{24}} \big[\gamma_{k0} + \omega_{24}\gamma_{k1} + \omega_{24}^2\gamma_{k2} + \omega_{24}^3\gamma_{k3}\big] \\ &\quad + \sum_{j=1}^{N_{ks}^2} e^{\lambda\omega_{24}a_{ksj}^2}[\omega_{24}^s\zeta_{ks}^{2j}]\Big). \end{aligned} \tag{3.9}$$

From the system that is obtained by using (3.2)-(3.9), we have the characteristic determinant in $\Omega_0^{(1)}$ as asymptotic quasi-polynomial form

$$\begin{aligned} &\Delta_1(\lambda) = \lambda^{32} e^{\lambda(\omega_{11}+\omega_{13})} \\ &\times \big([A_1]\, e^{\sigma_{11}\lambda\omega_{21}} + \cdots + [A_\rho]\, e^{\sigma_{1\rho}\lambda\omega_{21}} \\ &+ [B_1]\, e^{\sigma_{21}\lambda\omega_{23}} + \cdots + [B_\rho]\, e^{\sigma_{2\rho}\lambda\omega_{23}} \big) \end{aligned} \tag{3.10}$$

where

$$-1 = \sigma_{j1} < \sigma_{j2} < \cdots < \sigma_{j\rho} = 1,\ j = 1, 2,$$

and

$$\begin{aligned} A_1 &= A_{11} + A_{12}, ...,\ A_\rho = A_{\rho 1} + A_{\rho 2}, \\ B_1 &= B_{11} + B_{12}, ...,\ B_\rho = B_{\rho 1} + B_{\rho 2} \end{aligned}$$

some complex numbers. Let us denote

$$\Delta_{21}^{1}(\lambda) := \lambda^{32} e^{\lambda(\omega_{11}+\omega_{13})} \left([A_1] e^{\sigma_{11}\lambda\omega_{21}} \right.$$
$$\left. + [A_2] e^{\sigma_{12}\lambda\omega_{21}} + \cdots + [A_\rho] e^{\sigma_{1\rho}\lambda\omega_{21}} \right), \tag{3.11}$$

$$\Delta_{23}^{1}(\lambda) := \lambda^{32} e^{\lambda(\omega_{11}+\omega_{13})} \left([B_1] e^{\sigma_{21}\lambda\omega_{23}} \right.$$
$$\left. + [B_2] e^{\sigma_{22}\lambda\omega_{23}} + \cdots + [B_\rho] e^{\sigma_{2\rho}\lambda\omega_{23}} \right), \tag{3.12}$$

and

$$\Delta_1(\lambda) = \Delta_{21}^{1}(\lambda) + \Delta_{23}^{1}(\lambda).$$

Now, let the sector $\Omega_0^{(1)}$ divide two sectors as $\Omega_{01}^{(1)}$ and $\Omega_{02}^{(1)}$. We assume that one of the expressions $\Delta_{21}^{1}(\lambda)$ and $\Delta_{23}^{1}(\lambda)$ vanish in one of the sectors $\Omega_{01}^{(1)}$ and $\Omega_{02}^{(1)}$. Therefore, let the characteristic determinant $\Delta_1(\lambda)$ has the asymptotic representation in the form (3.11) in $\Omega_{01}^{(1)}$ and in the form (3.12) in $\Omega_{01}^{(1)}$. Here, all determinants are different from each other. Also, it is easy to see that A_{11} and A_{12} determinants for first coefficient of (3.11)

$$A_{11} = \begin{vmatrix} [\beta_{10} + \omega_{11}\beta_{11} + \omega_{11}^2\beta_{12} + \omega_{11}^3\beta_{13}] & \cdots & [\delta_{10} + \omega_{24}\delta_{11} + \omega_{24}^2\delta_{12} + \omega_{24}^3\delta_{13}] \\ [\beta_{20} + \omega_{11}\beta_{21} + \omega_{11}^2\beta_{22} + \omega_{11}^3\beta_{23}] & \cdots & [\delta_{20} + \omega_{24}\delta_{21} + \omega_{24}^2\delta_{22} + \omega_{24}^3\delta_{23}] \\ \vdots & \vdots & \vdots \\ [\beta_{80} + \omega_{11}\beta_{81} + \omega_{11}^2\beta_{82} + \omega_{11}^3\beta_{83}] & \cdots & [\delta_{80} + \omega_{24}\delta_{81} + \omega_{24}^2\delta_{82} + \omega_{24}^3\delta_{83}] \end{vmatrix},$$

$$A_{12} = \begin{vmatrix} [\beta_{10} + \omega_{11}\beta_{11} + \omega_{11}^2\beta_{12} + \omega_{11}^3\beta_{13}] & \cdots & [\gamma_{10} + \omega_{24}\gamma_{11} + \omega_{24}^2\gamma_{12} + \omega_{24}^3\gamma_{13}] \\ [\beta_{20} + \omega_{11}\beta_{21} + \omega_{11}^2\beta_{22} + \omega_{11}^3\beta_{23}] & \cdots & [\gamma_{20} + \omega_{24}\gamma_{21} + \omega_{24}^2\gamma_{22} + \omega_{24}^3\gamma_{23}] \\ \vdots & \vdots & \vdots \\ [\beta_{80} + \omega_{11}\beta_{81} + \omega_{11}^2\beta_{82} + \omega_{11}^3\beta_{83}] & \cdots & [\gamma_{80} + \omega_{24}\gamma_{81} + \omega_{24}^2\gamma_{82} + \omega_{24}^3\gamma_{83}] \end{vmatrix}.$$

We can obtain that the other determinants of (3.11) in the same way. B_{11} and B_{12} determinants for first coefficient of (3.12)

$$B_{11} = \begin{vmatrix} [\beta_{10} + \omega_{11}\beta_{11} + \omega_{11}^2\beta_{12} + \omega_{11}^3\beta_{13}] \\ [\beta_{20} + \omega_{11}\beta_{21} + \omega_{11}^2\beta_{22} + \omega_{11}^3\beta_{23}] \\ \vdots \\ [\beta_{80} + \omega_{11}\beta_{81} + \omega_{11}^2\beta_{82} + \omega_{11}^3\beta_{83}] \end{vmatrix}$$

$$\begin{array}{cc} \cdots & \left[\gamma_{10}+\omega_{24}\gamma_{11}+\omega_{24}^2\gamma_{12}+\omega_{24}^3\gamma_{13}\right] \\ \cdots & \left[\gamma_{20}+\omega_{24}\gamma_{21}+\omega_{24}^2\gamma_{22}+\omega_{24}^3\gamma_{23}\right] \\ \vdots & \vdots \\ \cdots & \left[\gamma_{80}+\omega_{24}\gamma_{81}+\omega_{24}^2\gamma_{82}+\omega_{24}^3\gamma_{83}\right] \end{array}\right|,$$

$$B_{12}=\left|\begin{array}{c} \left[\beta_{10}+\omega_{11}\beta_{11}+\omega_{11}^2\beta_{12}+\omega_{11}^3\beta_{13}\right] \\ \left[\beta_{20}+\omega_{11}\beta_{21}+\omega_{11}^2\beta_{22}+\omega_{11}^3\beta_{23}\right] \\ \vdots \\ \left[\beta_{80}+\omega_{11}\beta_{81}+\omega_{11}^2\beta_{82}+\omega_{11}^3\beta_{83}\right] \end{array}\right.$$

$$\begin{array}{cc} \cdots & \left[\gamma_{10}+\omega_{24}\gamma_{11}+\omega_{24}^2\gamma_{12}+\omega_{24}^3\gamma_{13}\right] \\ \cdots & \left[\gamma_{20}+\omega_{24}\gamma_{21}+\omega_{24}^2\gamma_{22}+\omega_{24}^3\gamma_{23}\right] \\ \vdots & \vdots \\ \cdots & \left[\gamma_{80}+\omega_{24}\gamma_{81}+\omega_{24}^2\gamma_{82}+\omega_{24}^3\gamma_{83}\right] \end{array}\right|.$$

The other determinants of (3.12) can be obtained in the same way. It can be shown analogically that, the characteristic determinant $\Delta_2(\lambda)$ in the sector $\Omega_0^{(2)}$ has the next asymptotic quasi-polynomial representation

$$\begin{aligned} &\Delta_2(\lambda)=\lambda^{32}e^{\lambda(\omega_{21}+\omega_{23})} \\ &\times\left(\left[M_1\right]e^{\mu_{11}\lambda\omega_{11}}+\cdots+\left[M_\varphi\right]e^{\mu_{1\varphi}\lambda\omega_{11}}\right. \\ &\left.+\left[N_1\right]e^{\mu_{21}\lambda\omega_{13}}+\cdots+\left[N_\varphi\right]e^{\mu_{2\varphi}\lambda\omega_{13}}\right) \end{aligned} \tag{3.13}$$

where

$$-1=\mu_{j1}<\mu_{j2}<\cdots<\mu_{j\varphi}=1,\ j=1,2,$$

$$\begin{aligned} M_1 &= M_{11}+M_{12},...,\ M_\varphi=M_{\varphi1}+M_{\varphi2}, \\ N_1 &= N_{11}+N_{12},...,\ N_\varphi=N_{\varphi1}+N_{\varphi2}. \end{aligned}$$

Now, let us denote

$$\begin{aligned} &\Delta_{11}^2(\lambda):=\lambda^{32}e^{\lambda(\omega_{21}+\omega_{23})}\left(\left[M_1\right]e^{\mu_{11}\lambda\omega_{11}}\right. \\ &\left.+\left[M_2\right]e^{\mu_{12}\lambda\omega_{11}}+\cdots+\left[M_\varphi\right]e^{\mu_{1\varphi}\lambda\omega_{11}}\right), \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\Delta_{13}^2(\lambda):=\lambda^{32}e^{\lambda(\omega_{21}+\omega_{23})}\left(\left[N_1\right]e^{\mu_{21}\lambda\omega_{13}}\right. \\ &\left.+\left[N_2\right]e^{\mu_{22}\lambda\omega_{13}}+\cdots+\left[N_\varphi\right]e^{\mu_{2\varphi}\lambda\omega_{13}}\right), \end{aligned} \tag{3.15}$$

and

$$\Delta_2(\lambda)=\Delta_{11}^2(\lambda)+\Delta_{13}^2(\lambda).$$

Let the sector $\Omega_0^{(2)}$ divide two sectors as $\Omega_{01}^{(2)}$ and $\Omega_{02}^{(2)}$. We assume that one of the expressions $\Delta_{11}^2(\lambda)$ and $\Delta_{13}^2(\lambda)$ vanish in one of the sectors $\Omega_{01}^{(2)}$ and $\Omega_{02}^{(2)}$. Therefore, let the characteristic determinant $\Delta_2(\lambda)$ has the asymptotic representation in the form (3.14) in $\Omega_{01}^{(2)}$ and in the form (3.15) in $\Omega_{02}^{(2)}$. Here, all determinants are different from each other and some of them in the form. M_{11} and M_{12} determinants for first coefficient of (3.14)

$$M_{11}=\left|\begin{matrix}\left[\alpha_{10}+\omega_{11}\alpha_{11}+\omega_{11}^2\alpha_{12}+\omega_{11}^3\alpha_{13}\right]\\ \left[\alpha_{20}+\omega_{11}\alpha_{21}+\omega_{11}^2\alpha_{22}+\omega_{11}^3\alpha_{23}\right]\\ \vdots\\ \left[\alpha_{80}+\omega_{11}\alpha_{81}+\omega_{11}^2\alpha_{82}+\omega_{11}^3\alpha_{83}\right]\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots & \left[\delta_{10}+\omega_{24}\delta_{11}+\omega_{24}^2\delta_{12}+\omega_{24}^3\delta_{13}\right]\\ \cdots & \left[\delta_{20}+\omega_{24}\delta_{21}+\omega_{24}^2\delta_{22}+\omega_{24}^3\delta_{23}\right]\\ \vdots & \vdots\\ \cdots & \left[\delta_{80}+\omega_{24}\delta_{81}+\omega_{24}^2\delta_{82}+\omega_{24}^3\delta_{83}\right]\end{matrix}\right|,$$

$$M_{12}=\left|\begin{matrix}\left[\alpha_{10}+\omega_{11}\alpha_{11}+\omega_{11}^2\alpha_{12}+\omega_{11}^3\alpha_{13}\right]\\ \left[\alpha_{20}+\omega_{11}\alpha_{21}+\omega_{11}^2\alpha_{22}+\omega_{11}^3\alpha_{23}\right]\\ \vdots\\ \left[\alpha_{80}+\omega_{11}\alpha_{81}+\omega_{11}^2\alpha_{82}+\omega_{11}^3\alpha_{83}\right]\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots & \left[\delta_{10}+\omega_{24}\delta_{11}+\omega_{24}^2\delta_{12}+\omega_{24}^3\delta_{13}\right]\\ \cdots & \left[\delta_{20}+\omega_{24}\delta_{21}+\omega_{24}^2\delta_{22}+\omega_{24}^3\delta_{23}\right]\\ \vdots & \vdots\\ \cdots & \left[\delta_{80}+\omega_{24}\delta_{81}+\omega_{24}^2\delta_{82}+\omega_{24}^3\delta_{83}\right]\end{matrix}\right|.$$

We can obtain that the other determinants of (3.14) in the same way. N_{11} and N_{12} determinants for first coefficient of ((3.15)

$$N_{11}=\left|\begin{matrix}\left[\beta_{10}+\omega_{11}\beta_{11}+\omega_{11}^2\beta_{12}+\omega_{11}^3\beta_{13}\right]\\ \left[\beta_{20}+\omega_{11}\beta_{21}+\omega_{11}^2\beta_{22}+\omega_{11}^3\beta_{23}\right]\\ \vdots\\ \left[\beta_{80}+\omega_{11}\beta_{81}+\omega_{11}^2\beta_{82}+\omega_{11}^3\beta_{83}\right]\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots & \left[\delta_{10}+\omega_{24}\delta_{11}+\omega_{24}^2\delta_{12}+\omega_{24}^3\delta_{13}\right]\\ \cdots & \left[\delta_{20}+\omega_{24}\delta_{21}+\omega_{24}^2\delta_{22}+\omega_{24}^3\delta_{23}\right]\\ \vdots & \vdots\\ \cdots & \left[\delta_{80}+\omega_{24}\delta_{81}+\omega_{24}^2\delta_{82}+\omega_{24}^3\delta_{83}\right]\end{matrix}\right|,$$

$$N_{12}=\left|\begin{matrix}\left[\alpha_{10}+\omega_{11}\alpha_{11}+\omega_{11}^2\alpha_{12}+\omega_{11}^3\alpha_{13}\right]\\ \left[\alpha_{20}+\omega_{11}\alpha_{21}+\omega_{11}^2\alpha_{22}+\omega_{11}^3\alpha_{23}\right]\\ \vdots\\ \left[\alpha_{80}+\omega_{11}\alpha_{81}+\omega_{11}^2\alpha_{82}+\omega_{11}^3\alpha_{83}\right]\end{matrix}\right.$$

$$\begin{array}{cc} \cdots & [\delta_{10}+\omega_{24}\delta_{11}+\omega_{24}^2\delta_{12}+\omega_{24}^3\delta_{13}] \\ \cdots & [\delta_{20}+\omega_{24}\delta_{21}+\omega_{24}^2\delta_{22}+\omega_{24}^3\delta_{23}] \\ \vdots & \vdots \\ \cdots & [\delta_{80}+\omega_{24}\delta_{81}+\omega_{24}^2\delta_{82}+\omega_{24}^3\delta_{83}] \end{array}\Bigg| .$$

The other determinants of (3.15) can be obtained in the same way.

3.3. **Asymptotic distribution of eigenvalues for** $\arg p_1 \neq \arg p_2$**.** Now we can obtain the asymptotic formulas for the eigenvalues of the boundary value problem for $\arg p_1 \neq \arg p_2$.

Theorem 1. *We assume that the following conditions be satisfied*

1) $\arg p_1 \neq \arg p_2$.
2) $q(x) \in L_p(-1,1),\ p > 1$.
3) $A_i, B_i \neq 0,\ i = 1$ and $i = \rho$; $M_i, N_i, \neq 0,\ i = 1$ and $i = \varphi$.
4) The linear functionals $F_{1k} + F_{2k}$ in the spaces $W_p^k(-1,0) + W_p^k(0,1)$ are continuous.

Then, the boundary value problem (1.1)-(1.2) has in each sector S_k an precisely numerable number eigenvalues, whose asymptotic distribution may be expressed by the following formulas.

$$\lambda_n^j = p_j^{\frac{1}{4}}\pi ni(1+O(\frac{1}{n})),\ j = 1,2, \tag{3.16}$$

$$\lambda_n^{j+2} = -p_j^{\frac{1}{4}}\pi ni(1+O(\frac{1}{n})),\ j = 1,2, \tag{3.17}$$

$$\lambda_n^{j+4} = p_j^{\frac{1}{4}}\pi n(1+O(\frac{1}{n})),\ j = 1,2, \tag{3.18}$$

$$\lambda_n^{j+6} = -p_j^{\frac{1}{4}}\pi n(1+O(\frac{1}{n})),\ j = 1,2. \tag{3.19}$$

Proof. By the rays l'_j, the complex λ-plane is divided into eight sectors $D_j,\ j = 1,2,...,8$. Let D_j be that sector which contains the rays l_j. We shall distribute these sectors into two groups

$$D^{(i)} = \left\{D_1^{(i)}, D_2^{(i)}, ..., D_8^{(i)}\right\}, i = 1,2.$$

Obviously that sector of the group $D^{(k)}$ contains two sectors of the group $\Omega^{(k)}$ by $D_0^{(k)}$ denote that sectors of the group $D^{(k)}$ which contain $\Omega_0^{(k)}$, $k = 1,2$. As seems from the consideration in subsection 3.1 and 3.2 the asymptotic expressing (3.10) and (3.13) hold also in the sectors $D_0^{(1)}$ and $D_0^{(2)}$, respectively. Let $D_1^{(1)}$ and $D_1^{(2)}$ are the other sectors of the groups $D^{(1)}$ and $D^{(2)}$ respectively. Only in one of the sectors of the groups $D^{(1)}$ the conditions

$$\begin{aligned} \mathrm{Re}\lambda\omega_{12} &\to +\infty,\ \mathrm{Re}\lambda\omega_{22} \geq 0, \\ \mathrm{Re}\lambda\omega_{14} &\to +\infty,\ \mathrm{Re}\lambda\omega_{24} \geq 0 \end{aligned}$$

and only in one of the sectors of the groups $D^{(2)}$ the conditions

$$\begin{aligned} \mathrm{Re}\lambda\omega_{22} &\to +\infty,\ \mathrm{Re}\lambda\omega_{12} \geq 0, \\ \mathrm{Re}\lambda\omega_{24} &\to +\infty,\ \mathrm{Re}\lambda\omega_{14} \geq 0\ . \end{aligned}$$

hold for $\lambda \to \infty$. By the similar way as in subsection 3.1 and 3.2, one can prove that the characteristic determinants have the asymptotic quasi-polynomial representation given by

$$\begin{aligned} &\Delta_3(\lambda) = \lambda^{32} e^{-\lambda(\omega_{11}+\omega_{13})} \big([K_1]\, e^{\eta_{11}\lambda\omega_{21}} \\ &+ \cdots + [K_r]\, e^{\eta_{1r}\lambda\omega_{21}} \\ &+ [T_1]\, e^{\eta_{21}\lambda\omega_{23}} + \cdots + [T_r]\, e^{\eta_{2r}\lambda\omega_{23}}\big) \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\Delta_4(\lambda) = \lambda^{32} e^{-\lambda(\omega_{21}+\omega_{23})} \big([U_1]\, e^{\xi_{11}\lambda\omega_{11}} \\ &+ \cdots + [U_\varrho]\, e^{\xi_{1\varrho}\lambda\omega_{11}} \\ &+ [V_1]\, e^{\xi_{21}\lambda\omega_{13}} + \cdots + [V_\varrho]\, e^{\xi_{2\varrho}\lambda\omega_{13}}\big) \end{aligned} \tag{3.21}$$

in the sectors $D_0^{(1)}$ and $D_0^{(2)}$, respectively, where

$$-1 = \eta_{j1} < \eta_{j2} < \cdots < \eta_{jr} = 1,\ j = 1, 2,$$

$$\begin{aligned} K_1 &= K_{11} + K_{12}, ...,\ K_r = K_{r1} + K_{r2}, \\ T_1 &= T_{11} + T_{12}, ...,\ T_r = T_{r1} + T_{r2} \end{aligned}$$

and

$$-1 = \xi_{j1} < \xi_{j2} < \cdots < \xi_{j\varrho} = 1,\ j = 1, 2,$$

$$\begin{aligned} U_1 &= U_{11} + U_{12}, ...,\ U_\varrho = U_{\varrho 1} + U_{\varrho 2}, \\ V_1 &= V_{11} + V_{12}, ...,\ V_\varrho = V_{\varrho 1} + V_{\varrho 2}. \end{aligned}$$

Let us denote

$$\begin{aligned} &\Delta_{21}^3(\lambda) := \lambda^{32} e^{-\lambda(\omega_{11}+\omega_{13})} \big([K_1]\, e^{\eta_{11}\lambda\omega_{21}} \\ &+ [K_2]\, e^{\eta_{12}\lambda\omega_{21}} + \cdots + [K_r]\, e^{\eta_{1r}\lambda\omega_{21}}\big), \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\Delta_{23}^3(\lambda) := \lambda^{32} e^{-\lambda(\omega_{11}+\omega_{13})} \big([T_1]\, e^{\eta_{21}\lambda\omega_{23}} \\ &+ [T_2]\, e^{\eta_{22}\lambda\omega_{23}} + \cdots + [T_r]\, e^{\eta_{2r}\lambda\omega_{23}}\big), \end{aligned} \tag{3.23}$$

and

$$\Delta_3(\lambda) = \Delta_{21}^3(\lambda) + \Delta_{23}^3(\lambda)\,.$$

Let the sector $D_0^{(1)}$ is divided into two sectors as $D_{01}^{(1)}$ and $D_{02}^{(1)}$. We assume that one of the expressions $\Delta_{21}^3(\lambda)$ and $\Delta_{23}^3(\lambda)$ vanish in one of the sectors $D_{01}^{(1)}$ and $D_{02}^{(1)}$. Therefore, let the characteristic determinant $\Delta_3(\lambda)$ has the asymptotic representation in the form (3.22) in $D_{01}^{(1)}$ and in the form (3.23) in $D_{02}^{(1)}$. By the similar way for the sector $D_0^{(2)}$ the characteristic determinant $\Delta_4(\lambda)$ has the asymptotic quasi-polynomial representation in the form in $D_{01}^{(2)}$

$$\Delta_{11}^4(\lambda) := \lambda^{32} e^{-\lambda(\omega_{21}+\omega_{23})} \left([U_1]\, e^{\xi_{11}\lambda\omega_{11}} + [U_2]\, e^{\xi_{12}\lambda\omega_{11}} + \cdots + [U_\varrho]\, e^{\xi_{1\varrho}\lambda\omega_{11}}\right), \tag{3.24}$$

and in $D_{02}^{(2)}$

$$\Delta_{13}^4(\lambda) := \lambda^{32} e^{-\lambda(\omega_{21}+\omega_{23})} \left([V_1]\, e^{\xi_{21}\lambda\omega_{13}} + [V_2]\, e^{\xi_{22}\lambda\omega_{13}} + \cdots + [V_\varrho]\, e^{\xi_{2\varrho}\lambda\omega_{13}}\right) \tag{3.25}$$

and

$$\Delta_4(\lambda) = \Delta_{11}^4(\lambda) + \Delta_{13}^4(\lambda).$$

Hence, let the characteristic determinant $\Delta_4(\lambda)$ has the asymptotic representation in the form (3.24) in $D_{01}^{(2)}$ and in the form (3.25) in $D_{02}^{(2)}$. Here, all determinants are different from each other and some determinants are in the following form

$$K_{11} = \begin{vmatrix} [\alpha_{10}+\omega_{11}\alpha_{11}+\omega_{11}^2\alpha_{12}+\omega_{11}^3\alpha_{13}] & \cdots & [\delta_{10}+\omega_{24}\delta_{11}+\omega_{24}^2\delta_{12}+\omega_{24}^3\delta_{13}] \\ [\alpha_{20}+\omega_{11}\alpha_{21}+\omega_{11}^2\alpha_{22}+\omega_{11}^3\alpha_{23}] & \cdots & [\delta_{20}+\omega_{24}\delta_{21}+\omega_{24}^2\delta_{22}+\omega_{24}^3\delta_{23}] \\ \vdots & \vdots & \vdots \\ [\alpha_{80}+\omega_{11}\alpha_{81}+\omega_{11}^2\alpha_{82}+\omega_{11}^3\alpha_{83}] & \cdots & [\delta_{80}+\omega_{24}\delta_{81}+\omega_{24}^2\delta_{82}+\omega_{24}^3\delta_{83}] \end{vmatrix},$$

$$K_{12} = \begin{vmatrix} [\alpha_{10}+\omega_{11}\alpha_{11}+\omega_{11}^2\alpha_{12}+\omega_{11}^3\alpha_{13}] & \cdots & [\gamma_{10}+\omega_{24}\gamma_{11}+\omega_{24}^2\gamma_{12}+\omega_{24}^3\gamma_{13}] \\ [\alpha_{20}+\omega_{11}\alpha_{21}+\omega_{11}^2\alpha_{22}+\omega_{11}^3\alpha_{23}] & \cdots & [\gamma_{20}+\omega_{24}\gamma_{21}+\omega_{24}^2\gamma_{22}+\omega_{24}^3\gamma_{23}] \\ \vdots & \vdots & \vdots \\ [\alpha_{80}+\omega_{11}\alpha_{81}+\omega_{11}^2\alpha_{82}+\omega_{11}^3\alpha_{83}] & \cdots & [\gamma_{80}+\omega_{24}\gamma_{81}+\omega_{24}^2\gamma_{82}+\omega_{24}^3\gamma_{83}] \end{vmatrix}.$$

The other determinants can be obtained in the same way. According to the condition 3 of the theorem, principal term of first and last coefficients of the asymptotic quasipolynomials (3.10), (3.13), (3.20) and (3.23) are different from zero, that is $A_i, B_i \neq 0,\ i = 1$ and $i = \rho$; $M_i, N_i \neq 0,\ i = 1$ and $i = \varphi$; $K_i, T_i \neq 0,\ i = 1$ and $i = r$; $U_i, V_i \neq 0,\ i = 1$ and $i = \varrho$.

Since $\Delta(\lambda) = \Delta_j(\lambda)$ when λ vary in sector $D_j^{(i)}$ and all quasi-polynomials $\Delta_j(\lambda)$ have the same form. Therefore, it is enough to investigate only one of them. Hence, we shall investigate the equation $\Delta(\lambda) = 0$ only in the sector $\Omega_0^{(1)}$. We know that $\Omega_0^{(1)}$ consists of the sectors $\Omega_{01}^{(1)}$ and $\Omega_{02}^{(1)}$. Therefore, from (3.11), we can write the equation

$$[A_1]\, e^{\sigma_{11}\lambda\omega_{21}} + [A_2]\, e^{\sigma_{12}\lambda\omega_{21}} + \cdots + [A_\rho]\, e^{\sigma_{1\rho}\lambda\omega_{21}} = 0 \tag{3.26}$$

in $\Omega_{01}^{(1)}$ and from (3.12), the equation

$$[B_1]\, e^{\sigma_{21}\lambda\omega_{23}} + [B_2]\, e^{\sigma_{22}\lambda\omega_{23}} + \cdots + [B_\rho]\, e^{\sigma_{2\rho}\lambda\omega_{23}} = 0 \tag{3.27}$$

in $\Omega_{02}^{(1)}$. By virtue of the [15, p. 100, Lemma 1] the equations (3.26) and (3.27) have an infinite number of roots λ_n which contain in strips

$$E_{01} = \left\{ \lambda \in \mathbb{C} |\, |\mathrm{Re}\lambda\omega_{21}| < \frac{h_1}{2} \right\}$$

and

$$E_{02} = \left\{ \lambda \in \mathbb{C} |\, |\mathrm{Re}\lambda\omega_{23}| < \frac{h_2}{2} \right\}$$

in the sectors $\Omega_{01}^{(1)}$ and $\Omega_{02}^{(1)}$, respectively, of finite width $h_1, h_2 > 0$ and have the asymptotic expressions

$$\begin{aligned} \left|\lambda_n^2\omega_{21}\right| &= \left|\frac{2\pi n}{\sigma_{1\rho} - \sigma_{11}}(1 + O(\frac{1}{n}))\right| \\ &= \left|\pi n(1 + O(\frac{1}{n}))\right| \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} \left|\lambda_n^6\omega_{23}\right| &= \left|\frac{2\pi n}{\sigma_{2\rho} - \sigma_{21}}(1 + O(\frac{1}{n}))\right| \\ &= \left|\pi n(1 + O(\frac{1}{n}))\right|. \end{aligned} \tag{3.29}$$

Taking into account $\lambda_n^2 \in E_{01}$, $\lambda_n^6 \in E_{02}$ and $\lambda_n^2 \in \Omega_{01}^{(1)}$, $\lambda_n^6 \in \Omega_{02}^{(1)}$ from (3.26) and (3.27)

$$\begin{aligned} \lambda_n^2 &= (\omega_{21})^{-1}\, \pi n i(1 + O(\frac{1}{n})) \\ &= p_2^{\frac{1}{4}}\pi n i(1 + O(\frac{1}{n})),\ n = \mp 1, \mp 2, ... \end{aligned}$$

and

$$\begin{aligned} \lambda_n^6 &= (\omega_{23})^{-1}\, \pi n i(1 + O(\frac{1}{n})) \\ &= p_2^{\frac{1}{4}}\pi n(1 + O(\frac{1}{n})),\ n = \mp 1, \mp 2, ..., \end{aligned}$$

where there is only one possible choice for the sign of the integer n. Similarly, from (3.14) and (3.15), we can write the following asymptotic expression in $\Omega_{01}^{(2)}$ and $\Omega_{02}^{(2)}$, respectively,

$$\lambda_n^1 = p_1^{\frac{1}{4}}\pi n i(1+O(\frac{1}{n})),\ n=\mp 1,\mp 2,...,$$

and

$$\lambda_n^5 = p_1^{\frac{1}{4}}\pi n(1+O(\frac{1}{n})),\ n=\mp 1,\mp 2,....$$

The other formulas in (3.16)-(3.19) can be obtained by the same procedure, which we used in proving above asymptotic formulas. □

4. Asymptotic distribution of eigenvalues for the case $\arg p_1 = \arg p_2$

4.1. **Separation of the complex $\lambda-$plane into specific sectors.** In the case $\arg p_1 = \arg p_2$, the lines

$$\begin{aligned} l_1 &= \{\lambda \in \mathbb{C} |\, \mathrm{Re}\lambda\omega_{11} = 0\}, \\ l_3 &= \{\lambda \in \mathbb{C} |\, \mathrm{Re}\lambda\omega_{21} = 0\} \end{aligned}$$

and the lines

$$\begin{aligned} l_2 &= \{\lambda \in \mathbb{C} |\, \mathrm{Re}\lambda\omega_{13} = 0\}, \\ l_4 &= \{\lambda \in \mathbb{C} |\, \mathrm{Re}\lambda\omega_{23} = 0\} \end{aligned}$$

coincide, then the lines $d_1 = l_1 = l_3$ and $d_2 = l_2 = l_4$ divide the complex $\lambda-$plane into four sectors $S_j',\ j=1,2,3,4$. On all of these sectors each of the real valued functions $\mathrm{Re}\lambda\omega_{vj}$ is a single sign, since these functions can vanish only on boundaries S_j'. Now let $d_k',\ k=1,2,3,4$, be arbitrary rays, which originate from the point $\lambda = 0$, distinct from the rays d and situated so as to from the sequence

$$d_1,\ d_1',\ d_2,\ d_2',\ d_3,\ d_3',\ d_4,\ d_4'.$$

The rays d_k' divide each sector S_j' into two subsectors. Therefore, we have eight sectors which we shall denote as G_i, $i=1,2,...,8$. As it seems from the construction, the sectors $G = \{G_1, G_2, ..., G_8\}$ can be distributed into two groups of

$$G^{(i)} = \left\{G_1^{(i)}, G_2^{(i)}, G_3^{(i)}, G_4^{(i)}\right\},\ i=1,2,$$

such that the group $G_4^{(k)},\ k=1,2,$ includes those sectors $G^{(i)}$, $i=1,2,...,8$, in which

$$\mathrm{Re}\lambda\omega_{vj} \to \infty,\ v=1,2,\ j=1,2,3,4,\ \text{as}\ \lambda\to\infty.$$

Only in one of the sectors of the groups $G^{(1)}$ the conditions

$$\mathrm{Re}\lambda\omega_{11}\,(\mathrm{Re}\lambda\omega_{21}) \to +\infty,\ \mathrm{Re}\lambda\omega_{13}\,(\mathrm{Re}\lambda\omega_{23}) \geq 0,$$

and only in one of the sectors of the groups $G^{(2)}$ the conditions

$$\mathrm{Re}\lambda\omega_{13}\,(\mathrm{Re}\lambda\omega_{23}) \to +\infty,\ \mathrm{Re}\lambda\omega_{11}\,(\mathrm{Re}\lambda\omega_{21}) \geq 0,$$

hold for $\lambda \to \infty$. These sectors denote as $G_0^{(1)}$ and $G_0^{(2)}$ accordingly.

4.2. **Asymptotic expressions for the characteristic determinant $\Delta(\lambda)$ in the G sectors.** First, we shall consider λ vary in $G_0^{(1)}$. Let us substitute (3.1) into (1.2). Therefore, we have the characteristic determinant as asymptotic quasi-polynomial form

$$\begin{aligned}\Delta_5(\lambda) &:= \lambda^{32} e^{\lambda(\omega_{11}+\omega_{21})} \\ &\times \left([Q_{11}]\, e^{\tau_{11}\lambda\omega_{14}} + \cdots + [Q_{1l}]\, e^{\tau_{1l}\lambda\omega_{14}} \right. \\ &\left. + [Q_{21}]\, e^{\tau_{21}\lambda\omega_{24}} + \cdots + [Q_{2l}]\, e^{\tau_{2l}\lambda\omega_{24}} \right)\end{aligned}$$

where

$$-1 = \tau_{j1} < \tau_{j2} < \cdots < \tau_{jl} = 1, \ j = 1, 2.$$

Let us denote

$$\begin{aligned}\Delta_{51}(\lambda) &:= \lambda^{32} e^{\lambda(\omega_{11}+\omega_{21})} \\ &\times \left([Q_{11}]\, e^{\tau_{11}\lambda\omega_{14}} + \cdots + [Q_{1l}]\, e^{\tau_{1l}\lambda\omega_{14}} \right),\end{aligned} \tag{4.1}$$

$$\begin{aligned}\Delta_{52}(\lambda) &:= \lambda^{32} e^{\lambda(\omega_{11}+\omega_{21})} \\ &\times \left([Q_{21}]\, e^{\tau_{21}\lambda\omega_{24}} + \cdots + [Q_{2l}]\, e^{\tau_{2l}\lambda\omega_{24}} \right)\end{aligned} \tag{4.2}$$

and

$$\Delta_5(\lambda) = \Delta_{51}(\lambda) + \Delta_{51}(\lambda).$$

Let divide the sector $G_0^{(1)}$ into two sectors as $G_{01}^{(1)}$ and $G_{02}^{(1)}$. We assume that one of the expressions $\Delta_{51}(\lambda)$ and $\Delta_{52}(\lambda)$ vanish in one of the sectors $G_{01}^{(1)}$ and $G_{02}^{(1)}$. Hence, let the characteristic determinant $\Delta_5(\lambda)$ has the asymptotic representation in the form (4.1) in $G_{01}^{(1)}$ and in the form ((4.2) in $G_{02}^{(1)}$ where

$$Q_{11} = \left| \begin{matrix} [\beta_{10} + \omega_{11}\beta_{11} + \omega_{11}^2\beta_{12} + \omega_{11}^3\beta_{13}] \\ [\beta_{20} + \omega_{11}\beta_{21} + \omega_{11}^2\beta_{22} + \omega_{11}^3\beta_{23}] \\ \vdots \\ [\beta_{80} + \omega_{11}\beta_{81} + \omega_{11}^2\beta_{82} + \omega_{11}^3\beta_{83}] \end{matrix} \right.$$

$$\left. \begin{matrix} \cdots & [\delta_{10} + \omega_{24}\delta_{11} + \omega_{24}^2\delta_{12} + \omega_{24}^3\delta_{13}] \\ \cdots & [\delta_{20} + \omega_{24}\delta_{21} + \omega_{24}^2\delta_{22} + \omega_{24}^3\delta_{23}] \\ \vdots & \vdots \\ \cdots & [\delta_{80} + \omega_{24}\delta_{81} + \omega_{24}^2\delta_{82} + \omega_{24}^3\delta_{83}] \end{matrix} \right|,$$

$$Q_{1l} = \left| \begin{matrix} [\beta_{10} + \omega_{11}\beta_{11} + \omega_{11}^2\beta_{12} + \omega_{11}^3\beta_{13}] \\ [\beta_{20} + \omega_{11}\beta_{21} + \omega_{11}^2\beta_{22} + \omega_{11}^3\beta_{23}] \\ \vdots \\ [\beta_{80} + \omega_{11}\beta_{81} + \omega_{11}^2\beta_{82} + \omega_{11}^3\beta_{83}] \end{matrix} \right.$$

$$\begin{array}{cc} \cdots & [\delta_{10}+\omega_{24}\delta_{11}+\omega_{24}^2\delta_{12}+\omega_{24}^3\delta_{13}] \\ \cdots & [\delta_{20}+\omega_{24}\delta_{21}+\omega_{24}^2\delta_{22}+\omega_{24}^3\delta_{23}] \\ \vdots & \vdots \\ \cdots & [\delta_{80}+\omega_{24}\delta_{81}+\omega_{24}^2\delta_{82}+\omega_{24}^3\delta_{83}] \end{array}\right|,$$

$$Q_{21}=\left|\begin{array}{c} [\beta_{10}+\omega_{11}\beta_{11}+\omega_{11}^2\beta_{12}+\omega_{11}^3\beta_{13}] \\ [\beta_{20}+\omega_{11}\beta_{21}+\omega_{11}^2\beta_{22}+\omega_{11}^3\beta_{23}] \\ \vdots \\ [\beta_{80}+\omega_{11}\beta_{81}+\omega_{11}^2\beta_{82}+\omega_{11}^3\beta_{83}] \end{array}\right.$$

$$\begin{array}{cc} \cdots & [\delta_{10}+\omega_{24}\delta_{11}+\omega_{24}^2\delta_{12}+\omega_{24}^3\delta_{13}] \\ \cdots & [\delta_{20}+\omega_{24}\delta_{21}+\omega_{24}^2\delta_{22}+\omega_{24}^3\delta_{23}] \\ \vdots & \vdots \\ \cdots & [\delta_{80}+\omega_{24}\delta_{81}+\omega_{24}^2\delta_{82}+\omega_{24}^3\delta_{83}] \end{array}\right|,$$

$$Q_{2l}=\left|\begin{array}{c} [\beta_{10}+\omega_{11}\beta_{11}+\omega_{11}^2\beta_{12}+\omega_{11}^3\beta_{13}] \\ [\beta_{20}+\omega_{11}\beta_{21}+\omega_{11}^2\beta_{22}+\omega_{11}^3\beta_{23}] \\ \vdots \\ [\beta_{80}+\omega_{11}\beta_{81}+\omega_{11}^2\beta_{82}+\omega_{11}^3\beta_{83}] \end{array}\right.$$

$$\begin{array}{cc} \cdots & [\gamma_{10}+\omega_{24}\gamma_{11}+\omega_{24}^2\gamma_{12}+\omega_{24}^3\gamma_{13}] \\ \cdots & [\gamma_{20}+\omega_{24}\gamma_{21}+\omega_{24}^2\gamma_{22}+\omega_{24}^3\gamma_{23}] \\ \vdots & \vdots \\ \cdots & [\gamma_{80}+\omega_{24}\gamma_{81}+\omega_{24}^2\gamma_{82}+\omega_{24}^3\gamma_{83}] \end{array}\right|.$$

By the same procedure in the sector $G_0^{(2)}$, we have the characteristic determinant as asymptotic representation

$$\begin{aligned} &\Delta_6(\lambda)=\lambda^{32}e^{\lambda(\omega_{13}+\omega_{23})} \\ &\times\left([R_{11}]\,e^{t_{11}\lambda\omega_{12}}+\cdots+[R_{1m}]\,e^{t_{1m}\lambda\omega_{12}}\right. \\ &\left.+[R_{21}]\,e^{t_{21}\lambda\omega_{22}}+\cdots+[R_{2m}]\,e^{t_{2m}\lambda\omega_{22}}\right) \end{aligned}$$

where

$$-1=t_{j1}<t_{j2}<\cdots<t_{jm}=1,\ j=1,2.$$

Considering the above idea, we can write the following equalities in sectors $G_{01}^{(2)}$ and $G_{02}^{(2)}$

$$\begin{aligned} &\Delta_{61}(\lambda):=\lambda^{32}e^{\lambda(\omega_{13}+\omega_{23})} \\ &\times\left([R_{11}]\,e^{t_{11}\lambda\omega_{12}}+\cdots+[R_{1m}]\,e^{t_{1m}\lambda\omega_{12}}\right), \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\Delta_{62}(\lambda):=\lambda^{32}e^{\lambda(\omega_{13}+\omega_{23})} \\ &\times\left([R_{21}]\,e^{t_{21}\lambda\omega_{22}}+\cdots+[R_{2m}]\,e^{t_{2m}\lambda\omega_{22}}\right), \end{aligned} \tag{4.4}$$

respectively, and

$$\Delta_6(\lambda) = \Delta_{61}(\lambda) + \Delta_6(\lambda).$$

The numbers R_{vj} can be seen by the same procedure in sectors $G_{01}^{(2)}$ and $G_{02}^{(2)}$.

4.3. **Asymptotic distribution of eigenvalues for** $\arg p_1 = \arg p_2$**.** Now we can prove the next theorem for the problem (1.1)-(1.2).

Theorem 2. *We assume that the following conditions be satisfied*

1) $\arg p_1 = \arg p_2$.
2) $q(x) \in L_p(-1,1),\ p > 1$.
3) $Q_{j1}, Q_{jl}, R_{j1}, R_{jm}, \neq 0,\ j = 1,2$.
4) The linear functionals $F_{1k} + F_{2k}$ in the spaces $W_p^k(-1,0) + W_p^k(0,1)$ are continuous.

Then, the boundary value problem (1.1)-(1.2) has an precisely number of eigenvalues whose asymptotic distribution may be expressed by the following formulas

$$\lambda_n^1 = -p_1^{\frac{1}{4}}\pi n(1 + O(\frac{1}{n})),$$

$$\lambda_n^2 = -p_2^{\frac{1}{4}}\pi n(1 + O(\frac{1}{n})),$$

$$\lambda_n^3 = -p_1^{\frac{1}{4}}\pi ni(1 + O(\frac{1}{n})),$$

$$\lambda_n^4 = -p_2^{\frac{1}{4}}\pi ni(1 + O(\frac{1}{n})).$$

in each sector S_j'.

Proof. According to condition (3) of the Theorem, the principal terms of the first and last coefficients of the asymptotic quasi-polynomials (4.1), (4.2), (4.3) and (4.2) are different from zero. These quasi-polynomials in sectors $G_{01}^{(1)}$, $G_{02}^{(1)}$, $G_{01}^{(2)}$ and $G_{02}^{(2)}$ have an infinite number of roots $\{\lambda_n^1\}$, $\{\lambda_n^2\}$, $\{\lambda_n^3\}$ and $\{\lambda_n^4\}$, respectively, and they are contained in strips

$$E_{1j} = \left\{ \lambda \in \mathbb{C} \middle| \left|\mathrm{Re}\lambda\omega_{j4}\right| < \frac{h_{1j}}{2} \right\},\ j = 1,2,$$

$$E_{2j} = \left\{ \lambda \in \mathbb{C} \middle| \left|\mathrm{Re}\lambda\omega_{j2}\right| < \frac{h_{2j}}{2} \right\},\ j = 1,2,$$

respectively, where $h_{ij} > 0$. Again, in view of the [15, p. 100, Lemma 1] eigenvalues of the problem have the asymptotic representation

$$\begin{aligned} \left|\lambda_n^j\omega_{j4}\right| &= \left|\frac{2\pi n}{\tau_{jl} - \tau_{j1}}(1 + O(\frac{1}{n}))\right| \\ &= \left|\pi n(1 + O(\frac{1}{n}))\right|,\ j = 1,2, \end{aligned}$$

$$\begin{aligned} \left|\lambda_n^{j+2}\omega_{j2}\right| &= \left|\frac{2\pi n}{t_{jm}-t_{j1}}(1+O(\frac{1}{n}))\right| \\ &= \left|\pi n(1+O(\frac{1}{n}))\right|,\ j=1,2. \end{aligned}$$

Therefore, we have the sought asymptotic formulas

$$\begin{aligned} \lambda_n^{j} &= (\omega_{j4})^{-1}\pi n i(1+O(\frac{1}{n})) \\ &= -p_j^{\frac{1}{4}}\pi n(1+O(\frac{1}{n})), \\ j &= 1,2,\ n=\mp 1,\mp 2,..., \end{aligned}$$

$$\begin{aligned} \lambda_n^{j+2} &= (\omega_{j2})^{-1}\pi n i(1+O(\frac{1}{n})) \\ &= -p_j^{\frac{1}{4}}\pi n i(1+O(\frac{1}{n})), \\ j &= 1,2,\ n=\mp 1,\mp 2,.... \end{aligned}$$

for eigenvalues of the problem (1.1)-(1.2). □

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