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ON MULTIPLICATIVE (GENERALIZED)-DERIVATIONS IN SEMIPRIME RINGS

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ABSTRACT. In this paper, we study commutativity of a prime or semiprime ring using a map $F : R \longrightarrow R$, multiplicative (generalized)-derivation and a map $H : R \longrightarrow R$, multiplicative left centralizer, under the following conditions: For all $x, y \in R$, *i)* $F(xy) \pm H(xy) = 0$, *ii)* $F(xy) \pm H(yx) = 0$, *iii)* $F(x)F(y) \pm H(xy) = 0$, *iv)* $F(xy) \pm H(xy) \in Z$, *v)* $F(xy) \pm H(yx) \in Z$, *vi)* $F(x)F(y) \pm H(xy) \in Z$.

1. Introduction

Let R be a ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ (resp. $x \circ y$) means that $xy - yx$ (resp. $xy + yx$). We use many times the commutator identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$, for all $x, y, z \in R$. Recall that R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$ and R is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. Therefore, it is known that if R is semiprime, then $aRb = (0)$ yields $ab = 0$ and $ba = 0$. In [3], Bresar was introduced the generalized derivation as the following: Let $F : R \longrightarrow R$ be a additive map and $g : R \longrightarrow R$ be a derivation. If $F(xy) = F(x)y + xg(y)$ holds for all $x, y \in R$, then F is called a generalized derivation associated with g. It is symbolized by (F, g). Hence the concept of generalized derivation involves the concept of derivation. In [4] Daif defined multiplicative derivation as the following. Let $D : R \longrightarrow R$ be a map. If $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$, then D is said to be multiplicative derivation. Thus the concept of multiplicative derivation involves the concept of derivation. Next, in [5], Daif and El-Sayiad gave multiplicative generalized derivation as the following. Let $F : R \longrightarrow R$ be a map and $d : R \longrightarrow R$ be a derivation. If $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, then F is called a multiplicative generalized derivation associated with d. Hence the concept of multiplicative generalized derivation involves the concept of generalized derivation. Let $H : R \longrightarrow R$ be a map. If $H(xy) = H(x)y$

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holds for all $x, y \in R$, then H is called a multiplicative left centralizer ([6]). In [11], Dhara and Ali gave definition of multiplicative (generalized)-derivation as the following. Let $F, f : R \longrightarrow R$ be two maps. If for all $x, y \in R$, $F(xy) = F(x)y + xf(y)$, then F is called a multiplicative (generalized)-derivation associated with f. Hence the concept of multiplicative (generalized)-derivation involves the concept of multiplicative generalized derivation.

With the generalization of derivation, it is given following conditions of commutativity of prime or semiprime ring. As a first time, in Ashraf and Rehman's paper [7], if $d(xy) \pm xy \in Z(R)$ holds for all $x, y \in I$, then R is commutative where R is a prime ring, I is nonzero two sided ideal of R and $d : R \longrightarrow R$ is a derivation. In papers ([8], [12], [9], [11], [1], [10], [14]), studied following conditions. *i)* $F(xy) \pm xy \in Z(R)$, $F(xy) \pm yx \in Z(R)$, $F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$, where R is a prime ring, I is a nonzero two sided ideal of R, $d : R \longrightarrow R$ is a derivation and $F : R \longrightarrow R$ is a generalized derivation ([8]). *ii)* $d([x, y]) = \pm[x, y]$ for all $x, y \in I$, where R is a semiprime ring, I is a nonzero ideal of R, $d : R \longrightarrow R$ is a derivation. ([9]). *iii)* $F([x, y]) = \pm[x, y]$ for all $x, y \in I$, where R is a prime ring, I is a nonzero two sided ideal of R, $d : R \longrightarrow R$ is a derivation and $F : R \longrightarrow R$ is a generalized derivation ([10]). *iv)* $F([x, y]) \pm [x, y] \in Z(R)$ for all $x, y \in I$, where R is a prime ring, I is a nonzero two sided ideal of R, (F, d) is a generalized derivation and $d(Z(R))$ is nonzero ([11]). *v)* $F(xy) \in Z(R)$, $F([x, y]) = 0$, $F(xy) \pm yx \in Z(R)$, $F(xy) \pm [x, y] \in Z(R)$ for all $x, y \in I$, where R is a semiprime ring, I is a nonzero left ideal of R and (F, d) is a generalized derivation ([12]). *vi)* $F(xy) \pm xy = 0$, $F(xy) \pm yx = 0$, $F(x)F(y) \pm xy = 0$, $F(x)F(y) \pm yx = 0$, $F(xy) \pm xy \in Z(R)$, $F(xy) \pm yx \in Z(R)$, $F(x)F(y) \pm xy \in Z(R)$, $F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in I$, where R is a semiprime ring, I is a nonzero left ideal of R and F is a multiplicative (generalized)-derivation ([1]). *vii)* $F(x)F(y) \pm [x, y] \in Z(R)$, $F(x)F(y) \pm x \circ y \in Z(R)$, $F[x, y] \pm [x, y] = 0$, $F[x, y] \pm [x, y] \in Z(R)$, $F(x \circ y) \pm (x \circ y) = 0$, $F(x \circ y) \pm (x \circ y) \in Z(R)$, $F[x, y] \pm [F(x), y] \in Z(R)$, $F(x \circ y) \pm (F(x) \circ y) \in Z(R)$ for all $x, y \in I$, where R is a semiprime ring, I is a nonzero left ideal of R and F is a multiplicative (generalized)-derivation ([14]).

Let R be a semiprime ring, $F : R \longrightarrow R$ be a multiplicative (generalized)-derivation associated with the map f and the map $H : R \longrightarrow R$ be a multiplicative left centralizer. In this paper, we study following conditions. *i)* $F(xy) \pm H(xy) = 0$, for all $x, y \in R$. *ii)* $F(xy) \pm H(yx) = 0$, for all $x, y \in R$. *iii)* $F(x)F(y) \pm H(xy) = 0$, for all $x, y \in R$. *iv)* $F(xy) \pm H(xy) \in Z(R)$, for all $x, y \in R$. *v)* $F(xy) \pm H(yx) \in Z(R)$, for all $x, y \in R$. *vi)* $F(x)F(y) \pm H(xy) \in Z(R)$, for all $x, y \in R$. Moreover, given some corollaries for prime rings.

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2. Results

Lemma 1. [13, Lemma 3] *Let R be a prime ring and d be a derivation of R such that $[d(a), a] = 0$, for all $a \in R$. Then R is commutative or d is zero.*

Lemma 2. *Let R be a semiprime ring. If F is a multiplicative (generalized)-derivation associated with the map f, then f is a multiplicative derivation, that is, $f(xy) = f(x)y + xf(y)$ for all $x, y \in R$.*

Proof. Since F is a multiplicative (generalized)-derivation we have

$$F(x(yz)) = F(x)yz + xf(yz), \ \forall x, y, z \in R$$

and

$$F((xy)z) = F(x)yz + xf(y)z + xyf(z), \ \forall x, y, z \in R.$$

Hence we get,

$$xf(yz) = xf(y)z + xyf(z), \ \forall x, y, z \in R.$$

From the last equation, we find that $R(f(yz) - f(y)z - yf(z)) = (0)$, for all $y, z \in R$. Since the semiprimeness of R, we have, $f(yz) = f(y)z + yf(z)$, for all $y, z \in R$. □

Lemma 3. *Let R be a semiprime ring and F be a multiplicative (generalized)-derivation associated with f. If $F(xy) = 0$ holds for all $x, y \in R$, then $F = 0$.*

Proof. By the assumption, we have

$$F(xy) = 0, \ \forall x, y \in R.$$

If we replace x by xz with $z \in R$, we get

$$F(xzy) = 0, \ \forall x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we get

$$F(xz)y + xzf(y) = 0, \ \forall x, y, z \in R.$$

Using the hypothesis we find that

$$xzf(y) = 0, \ \forall x, y, z \in R.$$

Since R is a semiprime ring, we obtain $xf(z) = 0$, for all $x, z \in R$. This means $f = 0$. From the definition of F, we get $F(xy) = F(x)y$, for all $x, y \in R$. By the hypothesis we see that

$$F(x)y = 0, \ \forall x, y \in R.$$

From the semiprimeness of R, we find that $F = 0$. □

Lemma 4. *Let R be a semiprime ring and F be a multiplicative (generalized)-derivation associated with f. If $F(xy) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$.*

Proof. By the hypothesis, we have

$$F(xy) \in Z(R),\ \forall x, y \in R.$$

Taking yz instead of y with $z \in R$, we get

$$F(xyz) \in Z(R),\ \forall x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$F(xy)z + xyf(z) \in Z(R),\ \forall x, y, z \in R.$$

From the hypothesis, we get

$$[xyf(z), z] = 0,\ \forall x, y, z \in R.$$

Replacing x by rx with $r \in R$, so we have

$$[r, z]xyf(z) = 0,\ \forall x, y, z, r \in R.$$

In this equation replacing x by $f(z)x$, we find that

$$[r, z]f(z)xyf(z) = 0,\ \forall x, y, z, r \in R.$$

This implies that, for all $x, y, s \in R$,

$$[x, y]f(y)s[x, y]f(y) = [x, y]f(y)sxyf(y) - [x, y]f(y)syxf(y) = 0.$$

Since R is a semiprime ring, we find that

$$[x, y]f(y) = 0,\ \forall x, y \in R.$$

Replacing x by xy with $y \in R$, we have

$$[x, y]yf(y) = 0,\ \forall x, y \in R.$$

Hence, we see that

$$[x, y][f(y), y] = 0,\ \forall x, y \in R.$$

If we replace x by $f(y)x$ and using the semiprimeness of R, we get $[f(y), y] = 0$ for all $y \in R$. □

Lemma 5. *Let R be a ring, F be a multiplicative (generalized)-derivation associated with f and H be a multiplicative left centralizer. If the map $G : R \longrightarrow R$ is defined as $G(x) = F(x) \mp H(x)$ for all $x \in R$, then G is a multiplicative (generalized)-derivation associated with f.*

Proof. We suppose that, for all $x \in R$

$$G(x) = F(x) \mp H(x).$$

So we have, for all $x, y \in R$

$$\begin{aligned} G(xy) &= F(xy) \mp H(xy) = F(x)y + xf(y) \mp H(x)y \\ &= (F(x) \mp H(x))y + xf(y) \\ &= G(x)y + xf(y). \end{aligned}$$

Then G is a multiplicative (generalized)-derivation associated with f. □

Theorem 1. *Let R be a semiprime ring, $F : R \longrightarrow R$ be a multiplicative (generalized)-derivation associated with f and $H : R \longrightarrow R$ be a multiplicative left centralizer. If $F(xy) \mp H(xy) = 0$ holds for all $x, y \in R$, then $f = 0$. Moreover, $F(xy) = F(x)y$ holds for all $x, y \in R$ and $F = \pm H$.*

Proof. By the hypothesis, we have

$$F(xy) - H(xy) = 0, \ \forall x, y \in R.$$

So we have

$$G(xy) = 0, \ \forall x, y \in R$$

where $G(x) = F(x) - H(x)$. Using Lemma 3 and Lemma 5, we get

$$G = 0.$$

So we have

$$F = H. \tag{2.1}$$

Using the definition of F and (2.1) in the hypothesis, we get

$$0 = F(xy) - H(xy) = F(x)y + xf(y) - H(x)y = xf(y), \ \forall x, y \in R.$$

Since R is a semiprime ring, we obtain $f = 0$. Thus, we get $F(xy) = F(x)y$ for all $x, y \in R$. Similar proof shows that the same conclusion holds as $F(xy) + H(xy) = 0$, for all $x, y \in R$. In this case, we obtain $F = -H$. Therefore the proof is completed. □

Theorem 2. *Let R be a semiprime ring, $F : R \longrightarrow R$ be a multiplicative (generalized)-derivation associated with f and $H : R \longrightarrow R$ be a multiplicative left centralizer. If $F(xy) \mp H(yx) = 0$ holds for all $x, y \in R$, then $f = 0$. Moreover, $F(xy) = F(x)y$, for all $x, y \in R$ and $[F(x), x] = 0$, for all $x \in R$.*

Proof. By the hypothesis, we have

$$F(xy) - H(yx) = 0, \ \forall x, y \in R. \tag{2.2}$$

Replacing y by yz with $z \in R$ in (2.2), we obtain

$$F(xyz) - H(yzx) = 0, \ \forall x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$(F(xy) - H(yx))z + xyf(z) + H(y)[x, z] = 0, \ \forall x, y, z \in R.$$

Using (2.2) in the last equation, we get

$$xyf(z) + H(y)[x, z] = 0, \ \forall x, y, z \in R. \tag{2.3}$$

If we replace z by x in (2.3), we get

$$xyf(x) = 0, \ \forall x, y \in R.$$

Since R is a semiprime ring, we obtain $xf(x) = f(x)x = 0$, for all $x \in R$. Hence we get,

$$[f(x), x] = 0, \ \forall x \in R. \tag{2.4}$$

If we replace x by xr with $r \in R$ in (2.3), we get the following equation. For all $x, y, z, r \in R$,

$$\begin{aligned} 0 &= xryf(z) + H(y)[xr, z] \\ &= xryf(z) + H(y)x[r, z] + H(y)[x, z]r + xyf(z)r - xyf(z)r \\ &= xryf(z) + H(y)x[r, z] - xyf(z)r + (xyf(z) + H(y)[x, z])r. \end{aligned}$$

So, using (2.3) in this equation, we find that

$$x[r, yf(z)] + H(y)x[r, z] = 0, \ \forall x, y, z, r \in R.$$

In this equation replacing r by $f(z)$ and using (2.4), we get

$$x[f(z), y]f(z) = 0, \ \forall x, y, z \in R.$$

Since R is a semiprime ring, we have

$$[f(z), y]f(z) = 0, \ \forall y, z \in R. \tag{2.5}$$

Replacing y by yt with $t \in R$ in (2.5) and using (2.5), we find that

$$[f(z), y]tf(z) = 0, \ \forall y, z, t \in R.$$

This yields following equation.

$$[f(z), y]t[f(z), y] = 0, \ \forall y, z, t \in R.$$

From the semiprimeness of R, we find that

$$[f(z), y] = 0, \ \forall y, z \in R. \tag{2.6}$$

Replacing x by $f(x)$ in (2.3) and using (2.6), we get, for all $x, y, z \in R$, $f(x)yf(z) = 0$. From the semiprimeness of R, this means

$$f = 0. \tag{2.7}$$

Hence, from the definition of F, we get

$$F(xy) = F(x)y, \ \forall x, y \in R. \tag{2.8}$$

Applying (2.7) to (2.3), we have

$$H(y)[x, z] = 0, \ \forall x, y, z \in R.$$

Replacing y by yz in the last equation and using respectively (2.2) and (2.8), we get

$$F(z)y[x, z] = 0, \ \forall x, y, z \in R. \tag{2.9}$$

If we replace x by $F(z)$ in (2.9), we obtain

$$F(z)y[F(z), z] = 0, \ \forall y, z \in R.$$

Hence for $y, z \in R$, we get

$$[F(z), z]y[F(z), z] = (F(z)z - zF(z))y[F(z), z] = 0.$$

Consequently, since R is a semiprime ring, we find that $[F(z), z] = 0$, for all $z \in R$. Similar proof shows that the same conclusion holds as $F(xy) + H(yx) = 0$, for all $x, y \in R$. Therefore the proof is completed. □

Theorem 3. *Let R be a semiprime ring, $F : R \longrightarrow R$ be a multiplicative (generalized)-derivation associated with f and $H : R \longrightarrow R$ be a multiplicative left centralizer. If $F(x)F(y) \mp H(xy) = 0$ holds for all $x, y \in R$, then $f = 0$. Moreover, $F(xy) = F(x)y$ for all $x, y \in R$ and $[F(x), x] = 0$, for all $x \in R$.*

Proof. By the hypothesis we have

$$F(x)F(y) - H(xy) = 0,\ \forall x, y \in R. \tag{2.10}$$

Replacing y by yz with $z \in R$ in (2.10), we get

$$F(x)F(yz) - H(xyz) = 0,\ \forall x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$(F(x)F(y) - H(xy))z + F(x)yf(z) = 0,\ \forall x, y, z \in R.$$

Using (2.10) in the last equation, we get

$$F(x)yf(z) = 0,\ \forall x, y, z \in R. \tag{2.11}$$

Replacing x by ux with $u \in R$ in (2.11) and using (2.11), from the definition of F, we obtain

$$uf(x)yf(z) = 0,\ \forall x, y, z, u \in R.$$

In the last equation replacing y by $yr, r \in R$ and using that R is a semiprime ring, so we have $f = 0$. Thus, we get $F(xy) = F(x)y$ for all $x, y \in R$. In (2.10) replacing x by xy, we have

$$F(x)yF(y) - H(xy)y = 0,\ \forall x, y \in R. \tag{2.12}$$

Multiplying (2.10) by y on the right, we have

$$F(x)F(y)y - H(xy)y = 0,\ \forall x, y \in R. \tag{2.13}$$

Subtracting (2.12) from (2.13), we get

$$F(x)[F(y), y] = 0,\ \forall x, y \in R.$$

Replacing x by xr with $r \in R$ in the last equation, we have

$$F(x)r[F(y), y] = 0,\ \forall x, y, r \in R.$$

In this case, for $x, r \in R$, we find that

$$[F(x), x]r[F(x), x] = (F(x)x - xF(x))r[F(x), x] = 0.$$

Thus, since R is a semiprime ring, we obtain $[F(x), x] = 0$, for all $x \in R$. Similar proof shows that the same conclusion holds as $F(x)F(y) + H(xy) = 0$, for all $x, y \in R$. □

Theorem 4. *Let R be a semiprime ring, $F : R \longrightarrow R$ be a multiplicative (generalized)-derivation associated with f and $H : R \longrightarrow R$ be a multiplicative left centralizer. If $F(xy) \mp H(xy) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$.*

Proof. By the supposition, we have

$$F(xy) \mp H(xy) \in Z(R),\ \forall x, y \in R.$$

So we have

$$G(xy) \in Z(R),\ \forall x, y \in R.$$

Using Lemma 4 and Lemma 5, we get

$$[f(x), x] = 0,\ \forall x \in R.$$

□

Theorem 5. *Let R be a semiprime ring, $F : R \longrightarrow R$ be a multiplicative (generalized)-derivation associated with f and $H : R \longrightarrow R$ be a multiplicative left centralizer. If $F(xy) \mp H(yx) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$.*

Proof. By the hypothesis, we have

$$F(xy) - H(yx) \in Z(R),\ \forall x, y \in R. \tag{2.14}$$

If we replace y by yz with $z \in R$ in (2.14), we get

$$F(xyz) - H(yzx) \in Z(R),\ \forall x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we find that

$$(F(xy) - H(yx))z + xyf(z) + H(y)[x, z] \in Z(R),\ \forall x, y, z \in R.$$

From the (2.14), we have

$$[xyf(z), z] + [H(y)[x, z], z] = 0,\ \forall x, y, z \in R. \tag{2.15}$$

Replacing x by xz in (2.15), we find that

$$[xzyf(z), z] + [H(y)[x, z], z]z = 0,\ \forall x, y, z \in R. \tag{2.16}$$

Multiplying (2.15) by z on the right, we find that

$$[xyf(z), z]z + [H(y)[x, z], z]z = 0,\ \forall x, y, z \in R. \tag{2.17}$$

Subtracting (2.16) and (2.17) side by side, so we get

$$[x[yf(z), z], z] = 0,\ \forall x, y, z \in R.$$

In the last equation, we replace x by rx with $r \in R$. Hence we get

$$[r, z]x[yf(z), z] = 0,\ \forall x, y, z, r \in R.$$

In this equation, replacing r by $yf(z)$ and using that semiprimeness of R, we obtain $[yf(z), z] = 0$, for all $y, z \in R$. If we take $f(z)y$ instead of y and using last equation, we have $[f(z), z]yf(z) = 0$, for all $y, z \in R$. From the last equation we have, $[f(z), z]y[f(z), z] = 0$, for all $y, z \in R$. Since R is a semiprime ring, we find that $[f(z), z] = 0$, for all $z \in R$.

Similar proof shows that if $F(xy) + H(yx) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$. □

Theorem 6. *Let R be a semiprime ring, $F : R \longrightarrow R$ be a multiplicative (generalized)-derivation associated with f and $H : R \longrightarrow R$ be a multiplicative left centralizer. If $F(x)F(y) \mp H(xy) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$.*

Proof. By the supposition, we have

$$F(x)F(y) - H(xy) \in Z(R),\ \forall x, y \in R. \tag{2.18}$$

Replacing y by yz with $z \in R$ in (2.18), we get

$$F(x)F(yz) - H(xyz) \in Z(R),\ \forall x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$(F(x)F(y) - H(xy))z + F(x)yf(z) \in Z(R),\ \forall x, y, z \in R.$$

Using (2.18), we get

$$[F(x)yf(z), z] = 0,\ \forall x, y, z \in R. \tag{2.19}$$

Replacing x by xz in (2.19) and using (2.19), hence we have

$$[xf(z)yf(z), z] = 0,\ \forall x, y, z \in R.$$

In the last equation, replacing x by $f(z)x$ and using this equation, we find that

$$[f(z), z]xf(z)yf(z) = 0,\ \forall x, y, z \in R.$$

This implies that

$$[f(z), z]x[f(z), z]y[f(z), z] = 0,\ \forall x, y, z \in R.$$

It gives that, $(R[f(z), z])^3 = 0$ for all $z \in R$. Since there is no nilpotent left ideal in semiprime rings ([2]), it gives that, $R[f(z), z] = 0$ for all $z \in R$. Hence using semiprimeness of R, we conclude that $[f(z), z] = 0$, for all $z \in R$. Similar proof shows that if $F(x)F(y) + H(xy) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$. □

By Lemma 2, every multiplicative (generalized)-derivation $F : R \longrightarrow R$ associated with an additive map f is always a multiplicative generalized derivation in semiprime ring. Thus our next corollary is about multiplicative generalized derivation.

Corollary 1. *Let R be a prime ring and $F : R \longrightarrow R$ be a multiplicative generalized derivation associated with a nonzero derivation d and $H : R \longrightarrow R$ be a multiplicative left centralizer. If one of the following conditions holds, for all $x, y \in R$, then R is commutative.*

i) $F(xy) \mp H(xy) \in Z(R)$,
ii) $F(xy) \mp H(yx) \in Z(R)$,
iii) $F(x)F(y) \mp H(xy) \in Z(R)$.

Proof. By Theorem 4, Theorem 5 and Theorem 6, we have $[d(x), x] = 0$ for all $x \in R$. Then by Lemma 1, R must be commutative. □

Using the examples of similar in [1], the following examples show that the importance hypothesis of semiprimeness.

Example 1. *Let* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$, *where* $\mathbb{Z}$ *is the set of all integers and the maps* $F, f, H : R \longrightarrow R$ *defined by*

$$F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \lambda b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ f\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda a^2 & \lambda b^2 \\ 0 & 0 & \lambda c \\ 0 & 0 & 0 \end{pmatrix}$$

$$H\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda a & \lambda b \\ 0 & 0 & \lambda c \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } \lambda \in \mathbb{Z}.$$

Since $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$, *$R$ is not semiprime. Moreover, it is easy to show that, F is a multiplicative (generalized)-derivation associated with f and $H(xy) = H(x)y$, $F(xy) - H(xy) = 0$ holds for all $x, y \in R$. But, we observe that $f(R) \neq 0$ and $F(xy) \neq F(x)y$ for $x, y \in R$. Hence the semiprimeness hypothesis in the Theorem 1 is crucial.*

Example 2. *Let* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$, *where* $\mathbb{Z}$ *is the set of all integers and the maps* $F, f, H : R \longrightarrow R$ *defined by*

$$F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda a & 0 \\ 0 & 0 & \lambda c \\ 0 & 0 & 0 \end{pmatrix}, \ f\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda ab & \lambda b^2 \\ 0 & 0 & -\lambda c \\ 0 & 0 & 0 \end{pmatrix}$$

$$H\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda^2 a & \lambda^2 b \\ 0 & 0 & \lambda^2 c \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } \lambda \in \mathbb{Z}.$$

Then R is not semiprime and it is easy to show that, F is a multiplicative (generalized)-derivation associated with f and $H(xy) = H(x)y$, $F(x)F(y) - H(xy) = 0$ holds for all $x, y \in R$. But, we observe that $f(R) \neq 0$ and $F(xy) \neq F(x)y$ for $x, y \in R$. Hence the semiprimeness hypothesis in the Theorem 3 is essential.

Example 3. *Let* $R = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$, *where* $\mathbb{Z}$ *is the set of all integers and the maps* $F, f, H : R \longrightarrow R$ *defined by*

$$F\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a^2 & 0 & 0 \\ b+c & 0 & 0 \end{pmatrix}, \ f\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b^2 & 0 & 0 \end{pmatrix}$$

$$H\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ab & 0 & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = (0)$, *$R$ is not a semiprime ring. It yields that F is a multiplicative (generalized)-derivation associated with f and $H(xy) = H(x)y$, $F(x)F(y) - H(xy) = 0$ holds for all $x, y \in R$. But, we see that $f(R) \neq 0$ and $F(xy) \neq F(x)y$ for $x, y \in R$. Hence the semiprimeness hypothesis in the Theorem 3 is essential.*

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