INVERSE NODAL PROBLEM FOR A STURM-LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENT

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ABSTRACT. Inverse nodal problem for Sturm–Liouville equation with discontinuity coefficient is studied. A uniqueness theorem and an algorithm for recovering the coefficients of the problem from a known sequence related to the nodal points are given.

1. INTRODUCTION

Inverse nodal problems consist in recovering the coefficients of operators from the zeros (nodes) of the eigenfunctions. McLaughlin (1988) seems to have been the first to consider this kind of inverse problem for the regular Sturm–Liouville equations with Dirichlet boundary conditions[17]. She showed that the potential of the problem can be determined by a given dense subset of nodal points. In 1989, Hald and McLaughlin generalized this result to more general boundary conditions and provide some numerical schemes for the reconstruction of the potential [13]. From then on, their results have been generalized to various problems. Inverse nodal problems for Sturm–Liouville operators without discontinuities have been studied in the several papers ([8], [10], [12], [13], [14], [19], [21], [22] and [24]). The first result on inverse nodal problems for the Sturm-Liouville operators with a discontinuity condition was obtained by Shieh and Yurko[20]. This study includes discontinuity conditions at the middle of interval. Inverse nodal problem for Sturm-Liouville operator with boundary conditions dependent on the spectral parameter were investigated in [4], [23] and [18]. Additionally, the studies [5] and [6] include inverse nodal problems for differential pencils.

In the present paper, we consider the boundary value problem $L = L\left(q, h, H\right)$ generated by the Sturm–Liouville equation

$$\ell y := -y^{[2]} + q(x)y = \lambda y, \quad x \in (0,1) \tag{1}$$

subject to the boundary conditions

$$U(y) := y^{[1]}(0) - hy(0) = 0 \tag{2}$$

$$V(y) := y^{[1]}(1) + Hy(1) = 0 \tag{3}$$

and transfer conditions

$$\begin{cases} y(d+0) = y(d-0) \\ y^{[1]}(d+0) = y^{[1]}(d-0) - \beta y(d-0) \end{cases} \tag{4}$$

where $y^{[1]} = py'$, $y^{[2]} = p\left(py'\right)'$, $q(x)$ and $p(x)$ are real valued functions in $L_2(0,1)$; h, H and β are real numbers and λ is the spectral parameter. We assume that $p(x) > 0$ and $\frac{\gamma(d)}{\gamma(1)}$ is a rational number in $(0,1)$

The equation (1) appears in some physical applications. A Sturm–Liouville equation with the coefficients which are piecewise constant functions can be regard as special form of (1). Spectral problems for differential equations with discontinuous coefficients were investigated in several works (see [1], [2], [3], [7], [9], [11], [15] and [16]). These works contain inverse problems according to the various spectral data.

2. Preliminaries

Let a function $\varphi(x,\lambda)$ be the solution of (1) under the initial conditions

$$\varphi(0,\lambda) = 1, \quad \varphi^{[1]}(0,\lambda) = h \tag{5}$$

and the jump conditions (4). It can be calculated that $\varphi(x,\lambda) = \begin{cases} \varphi_1(x,\lambda), & x < d \\ \varphi_2(x,\lambda), & x > d \end{cases}$ satisfies the following integral equations:

$$\varphi_1(x,\lambda) = \cos\rho\gamma(x) + h\frac{\sin\rho\gamma(x)}{\rho} \tag{6}$$
$$+ \frac{1}{\rho}\int_0^x \sin\rho\left[\gamma(x) - \gamma(t)\right]\frac{q}{p}(t)\varphi(t,\lambda)dt$$

$$\varphi_2(x,\lambda) = \cos\rho\gamma(x) + h\frac{\sin\rho\gamma(x)}{\rho}$$
$$- \frac{\beta}{2\rho}\left[\sin\rho\gamma(x) - \sin\rho\left(2\gamma(d) - \gamma(x)\right)\right] \tag{7}$$
$$+ \frac{\beta h}{2\rho^2}\left[\cos\rho\gamma(x) - \cos\rho\left(2\gamma(d) - \gamma(x)\right)\right]$$
$$+ \frac{\beta}{2\rho^2}\int_0^d \left[\cos\rho\left(\gamma(x) - \gamma(t)\right) - \cos\rho\left(2\gamma(d) - \gamma(x) - \gamma(t)\right)\right]\frac{q}{p}(t)\varphi(t,\lambda)dt$$
$$+ \frac{1}{\rho}\int_0^x \sin\rho\left[\gamma(x) - \gamma(t)\right]\frac{q}{p}(t)\varphi(t,\lambda)dt \tag{8}$$

where $\gamma(x) = \int_0^x \frac{dt}{p(t)}$ and $\rho = \sqrt{\lambda}$.

Using above integral equation we can obtain the following asymptotic relations for $|\rho| \to \infty$.

$$\varphi_1(x, \lambda) = \cos \rho\gamma(x) + \left(h + \frac{1}{2} \int_0^x \frac{q(u)}{p(u)} du \right) \frac{\sin \rho\gamma(x)}{\rho} + o\left(\frac{1}{\rho} \exp \tau x \right), \quad (9)$$

$$\varphi_2(x, \lambda) = \cos \rho\gamma(x) + \left(h - \frac{\beta}{2} + \frac{1}{2} \int_0^x \frac{q(u)}{p(u)} du \right) \frac{\sin \rho\gamma(x)}{\rho} \quad (10)$$

$$+ \frac{\beta}{2} \frac{\sin \rho \left(2\gamma(d) - \gamma(x) \right)}{\rho} + o\left(\frac{1}{\rho} \exp \tau x \right),$$

where $\tau = |Im\rho|$. By substituting $\gamma(u) = \gamma(1)t$ to the integrals in (9) and (10) we obtain

$$\varphi_1(x, \lambda) = \cos \rho\gamma(x) + f(x) \frac{\sin \rho\gamma(x)}{\rho} + o\left(\frac{1}{\rho} \exp \tau x \right), \quad (11)$$

and

$$\varphi_2(x, \lambda) = \cos \rho\gamma(x) + \left(f(x) - \frac{\beta}{2} \right) \frac{\sin \rho\gamma(x)}{\rho} + \quad (12)$$

$$+ \frac{\beta}{2} \frac{\sin \rho \left(2\gamma(d) - \gamma(x) \right)}{\rho} + o\left(\frac{1}{\rho} \exp \tau x \right),$$

where $f(x) = h + \frac{\gamma(1)}{2} \int_0^{\gamma(x)/\gamma(1)} q_1(t)dt$, $q_1(t) = \left(q o \gamma^{-1} \right) (\gamma(1)t)$

Let $\{\lambda_n\}_{n \geq 0}$ be the set of eigenvalues of (1)-(4) and $\varphi(x, \lambda_n)$ be the eigenfunction corresponding to the eigenvalue λ_n. It can be proven easily that the numbers λ_n are real, simple and satisfy the following asymptotic relation for $n \to \infty$:

$$\rho_n = \sqrt{\lambda_n} = \frac{n\pi}{\gamma(1)} + \frac{A}{n\pi} - \frac{\beta}{2n\pi} \cos \frac{2n\pi\gamma(d)}{\gamma(1)} + o\left(\frac{1}{n} \right) \quad (13)$$

$$\frac{1}{\sqrt{\lambda_n}} = \frac{\gamma(1)}{n\pi} \left(1 - \frac{A\gamma(1)}{n^2\pi^2} - \frac{\beta\gamma(1)}{2n^2\pi^2} \cos \frac{2n\pi\gamma(d)}{\gamma(1)} \right) + o\left(\frac{1}{n^2} \right) \quad (14)$$

where $A = h + H - \frac{\beta}{2} + \frac{\gamma(1)}{2} \int_0^1 q_1(t)dt$.

3. MAIN RESULTS

Let $X = \left\{ x_n^j : n \in \mathbb{N} \right\}$ be the set of nodal points of the eigenfunctions. Consider the set $Y = \left\{ y_n^j : y_n^j = \frac{\gamma\left(x_n^j \right)}{\gamma(1)}, \ x_n^j \in X \right\}$ and the problem $\widetilde{L}$ together with L. It is assumed in what follows that if a certain symbol s denotes an object related to the problem L then $\widetilde{s}$ denotes the corresponding object related to the problem $\widetilde{L}$.

Theorem 1. *Let $Y_0 \subset Y$ be dense set on $(0,1)$. If $\int_0^1 \frac{q(u)}{p(u)} du = 0$, $p(x) = \widetilde{p}(x)$ and $Y_0 = \widetilde{Y}_0$ then $q(x) = \widetilde{q}(x)$ a.e. in $(0,1)$, $h = \widetilde{h}$, $H = \widetilde{H}$ and $\beta = \widetilde{\beta}$. Thus, the coefficients $q(x)$, h, H and β are uniquely determined by Y_0.*

First, it must be given the following lemma, related to the asymptotic formulae for the elements of Y.

Lemma 1. *The elements of Y satisfy the following asymptotic formulae for suffi-ciently large n,*

$$y_n^j = \frac{j + \frac{1}{2}}{n} - \frac{\gamma(1)}{n^2\pi^2}\left[A + \frac{\beta}{2}\cos\frac{2n\pi\gamma(d)}{\gamma(1)}\right]\left(\frac{j + \frac{1}{2}}{n}\right) +$$
$$+ \frac{\gamma(1)}{n^2\pi^2}f(x_n^j) + o\left(\frac{1}{n^2}\right), \quad x_n^j \in (0,d) \tag{15}$$

$$y_n^j = \frac{j + \frac{1}{2}}{n} - \frac{\gamma(1)}{n^2\pi^2}\left[A + \frac{\beta}{2}\cos\frac{2n\pi\gamma(d)}{\gamma(1)}\right]\left(\frac{j + \frac{1}{2}}{n}\right) +$$
$$+ \frac{\gamma(1)}{n^2\pi^2}\left(f(x_n^j) - \frac{\beta}{2} - \frac{\beta}{2}\cos\frac{2n\pi\gamma(d)}{\gamma(1)}\right) \tag{16}$$
$$+ o\left(\frac{1}{n^2}\right), \quad x_n^j \in (d,1)$$

Proof. Use the asymptotic formulae (11) and (12) to get

$$\varphi_1(x_n^j, \lambda_n) = \cos\rho_n\gamma(x_n^j) + f(x_n^j)\frac{\sin\rho_n\gamma(x_n^j)}{\rho_n} + o\left(\frac{1}{\rho_n}\right), \tag{17}$$

and

$$\varphi_2(x_n^j, \lambda_n) = \cos\rho_n\gamma(x_n^j) + \left(f(x_n^j) - \frac{\beta}{2}\right)\frac{\sin\rho_n\gamma(x_n^j)}{\rho_n} + \tag{18}$$
$$+ \frac{\beta}{2}\frac{\sin\rho_n\left(2\gamma(d) - \gamma(x_n^j)\right)}{\rho_n} + o\left(\frac{1}{\rho_n}\right),$$

Let us consider the second case: $\varphi_2(x_n^j, \lambda_n) = 0$. The first case is similar. It is calculated that,

$$\tan\left(\rho_n\gamma\left(x_n^j\right) - \frac{\pi}{2}\right) = \frac{\left(f(x_n^j) - \frac{\beta}{2}\right)}{\rho_n} - \frac{\beta}{2\rho_n}\cos 2\rho_n\gamma(d) + o\left(\frac{1}{\rho_n}\right)$$

This yields

$$\gamma\left(x_n^j\right) \quad = \quad \frac{1}{\rho_n}\left(j+\frac{1}{2}\right)\pi - \frac{\left(f(x_n^j)-\frac{\beta}{2}\right)}{\rho_n^2} +$$
$$-\frac{\beta}{2\rho_n^2}\cos 2\rho_n\gamma(d) + o\left(\frac{1}{\rho_n^2}\right).$$

We can complete the proof using (13) and (14) . □

Proof of Theorem1. Since the set $X_0 := \left\{\dfrac{j+\frac{1}{2}}{n},\ j=\overline{0,n-1},\ n>0\right\}$ is dense on $(0,1)$, for each fixed x in $(0,1)$, there exist a sequence $(j(n))$ such that $\dfrac{j(n)+\frac{1}{2}}{n}$ converges to x. Thus the set Y is also dense on $(0,1)$.

Denote $K_n^j := \dfrac{n^2\pi^2}{\gamma(1)}\left(y_n^j - \dfrac{j(n)+\frac{1}{2}}{n}\right)$ and $m := s.n$, with s is denominator of $\dfrac{\gamma(d)}{\gamma(1)}$. Therefore, we can show from Lemma 1 that the following limits are exist and finite:

$$\lim_{m\to\infty} K_m^{j(m)} = F(x) \tag{19}$$

where

$$F(x) = \begin{cases} -(h+H)x + h + \frac{\gamma(1)}{2}\int\limits_0^x q_1(t)dt,\ x\in[0,d) \\[2mm] -(h+H)x + h - \beta + \frac{\gamma(1)}{2}\int\limits_0^x q_1(t)dt,\ x\in(d,1] \end{cases} \tag{20}$$

Direct calculation yields

$$q_1(x) \quad = \quad 2\left\{F'(x) + F(0) - F(1) + \gamma(1)\left[F(d+0) - F(d-0)\right]\right\}, \tag{21}$$
$$q(x) \quad = \quad q_1\left(\frac{\gamma(x)}{\gamma(1)}\right), \tag{22}$$
$$h \quad = \quad F(0),\quad H = F(d+0) - F(d-0) - F(1)\text{ and} \tag{23}$$
$$\beta \quad = \quad F(d-0) - F(d+0). \tag{24}$$

It is clear that, if $p(x) = \widetilde{p}(x)$ and $Y_0 = \widetilde{Y}_0$ then $F(x) = \widetilde{F}(x)$ and so $q(x) = \widetilde{q}(x)$ a.e. in $(0,1)$, $h = \widetilde{h}$, $H = \widetilde{H}$ and $\beta = \widetilde{\beta}$. □

Corollary 1. *If Y_0 is given by (15) and (16), $q(x)$, h, H and β can be reconstructed by the formulae (21)-(24).*

Corollary 2. *If $\int\limits_0^1 \dfrac{q(u)}{p(u)}du = 0$, $p(x) = \widetilde{p}(x)$ and $X = \widetilde{X}$ then $q(x) = \widetilde{q}(x)$ a.e. in $(0,1)$, $h = \widetilde{h}$, $H = \widetilde{H}$ and $\beta = \widetilde{\beta}$. Thus, the coefficients $q(x)$, h, H and β are uniquely determined by the nodal points.*

Example 1. *Let $p(x) = 1$, d be a rational number in $(0,1)$ and Y_0 be given by the following asymptotics*

$$y_n^j = \frac{j + \frac{1}{2}}{n} + \frac{1}{n^2\pi^2}\left[1 - \cos 2n\pi d\right]\left(\frac{j + \frac{1}{2}}{n}\right) +$$
$$+ \frac{1}{n^2\pi^2}\left(1 + \frac{1}{2\pi}\sin^2 \pi y_n^j\right) + o\left(\frac{1}{n^2}\right), \quad x_n^j \in (0, d) \qquad (25)$$

$$y_n^j = \frac{j + \frac{1}{2}}{n} + \frac{1}{n^2\pi^2}\left[1 - \cos 2n\pi d\right]\left(\frac{j + \frac{1}{2}}{n}\right) +$$
$$+ \frac{1}{n^2\pi^2}\left(\frac{1}{2\pi}\sin^2 \pi y_n^j - \cos 2n\pi d\right) \qquad (26)$$
$$+ o\left(\frac{1}{n^2}\right), \qquad\qquad x_n^j \in (d, 1)$$

It can be calculated from (19) and (20) that,

$$F(x) = \begin{cases} 1 + \frac{1}{2\pi}\sin^2 \pi x, \ x \in [0, d) \\ \frac{1}{2\pi}\sin^2 \pi x - 1, \ x \in (d, 1] \end{cases}$$

By (21)-(24), it is obtained that

$$q(x) = \sin 2\pi x, \ h = 1, \ H = -1 \ and \ \beta = 2.$$

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