Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.
Volume 66, Number 1, Pages 172–178 (2017)
DOI: 10.1501/Commua1_0000000786
ISSN 1303–5991

RESULTS ON $\alpha-^{*}$CENTRALIZERS OF PRIME AND SEMIPRIME RINGS WITH INVOLUTION

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Abstract. Let R be a prime or semiprime ring equipped with an involution $*$ and α be an automorphism of R. An additive mapping $T : R \to R$ is called a left (resp. right) $\alpha-^{*}$centralizer of R if $T(xy) = T(x)\alpha(y^*)$ (resp. $T(xy) = \alpha(x^*)T(y)$) holds for all $x, y \in R$, where α is an endomorphism of R. A left (resp. right) Jordan $\alpha-^{*}$centralizer $T : R \to R$ is an additive mapping such that $T(x^2) = T(x)\alpha(x^*)$ (resp. $T(x^2) = \alpha(x^*)T(x)$) holds for all $x \in R$. In this paper, we obtain some results about Jordan $\alpha-^{*}$centralizer of R with involution.

1. Introduction

This paper deals with the study of $\alpha-^{*}$centralizers of prime and semiprime rings with involution $*$ and was motivated by work of [8] and [6].Throughout, R will represent an associative ring with center Z. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$, and semiprime if $xRx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$ is called an involution and R is called a $*$-ring.

According B. Zalar [10], an additive mapping $T : R \to R$ is called a left (resp. right) centralizer of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. If T is both left as well right centralizer, then it is called a centralizer. This concept appears naturally C^*-algebras. In ring theory it is more common to work with module homorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of a ring module R into itself instead of a left centralizer. In case $T : R \to R$ is a centralizer, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ and $\lambda \in C$, where C is the extended centroid of R. A left (resp. right) Jordan centralizer $T : R \to R$ is an additive mapping such that $T(x^2) = T(x)x$ (resp. $T(x^2) = xT(x)$) holds for all $x \in R$. Zalar proved that any left (right) Jordan centralizer on a $2-$torsion free semiprime ring is a left (right) centralizer. Recently, in [1], E. Albaş introduced the definition of $\alpha-$centralizer of R, i. e. an

Received by the editors: Received: May 10, 2016 , Accepted: July 17, 2016.
2010 *Mathematics Subject Classification.* Primary 16W10 ; Secondary 16N60.
Key words and phrases. Semiprime ring, prime ring, centralizer, $\alpha-^{*}$centralizer.

additive mapping $T : R \to R$ is called a left (resp. right) $\alpha-$centralizer of R if $T(xy) = T(x)\alpha(y)$ (resp. $T(xy) = \alpha(x)T(y)$) holds for all $x, y \in R$, where α is an endomorphism of R. If T is left and right $\alpha-$centralizer then it is natural to call $\alpha-$centralizer. Clearly every centralizer is a special case of a $\alpha-$centralizer with $\alpha = id_R$. Also, an additive mapping $T : R \to R$ associated with a homomorphism $\alpha : R \to R$, if $L_a(x) = a\alpha(x)$ and $R_a(x) = \alpha(x)a$ for a fixed element $a \in R$ and for all $x \in R$, then L_a is a left $\alpha-$centralizer and R_a is a right $\alpha-$centralizer. Albaş showed Zalar's result holds for $\alpha-$centralizer.

On the other hand, in [3], J. Vukman and M. Fosner proved that an additive mapping $T : R \to R$, where R is a prime ring with characteristic different from two into, satisfying $T(x^3) = xT(x)x$ for all $x \in R$, is a two sided centralizer. In [5], the authors investigated this result for a $\alpha-$centralizer of R.

Inspired by the definition of centralizer, the notion of $^*-$centralizer was extended as follow:

Let R be a ring with involution $*$. An additive mapping $T : R \to R$ is called a left (resp. right) $^*-$centralizer of R if $T(xy) = T(x)y^*$ (resp. $T(xy) = x^*T(y)$) holds for all $x, y \in R$. An additive mapping $T : R \to R$ is said to be a left (resp. right) Jordan $^*-$centralizer if $T\left(x^2\right) = T(x)x^*$ (resp. $T\left(x^2\right) = x^*T(x)$) holds for all $x \in R$. For some fixed $a \in R$, the map $x \to ax^*$ is a Jordan left $^*-$centralizer. Every left $*-$centralizer on a ring R is a Jordan left $*-$centralizer. It is natural to question whether the converse of above statement is true and it was be shown that the answer to this question is affirmative if underlying $*-$ring is semiprime in [8]. In [2], the authors introduced the definition of $\alpha-^*$centralizer of R, i. e. an additive mapping $T : R \to R$ is called a left (resp. right) $\alpha-^*$centralizer of R if $T(xy) = T(x)\alpha(y^*)$ (resp. $T(xy) = \alpha(x^*)T(y)$) holds for all $x, y \in R$, where α is an endomorphism of R. They investigeted that T is a Jordan $\alpha-^*$centralizer under some conditions. Considerable work has been done on this topic during the last couple of decades (see [1-8], where further references can be found).

The main aim of the present article is a generalization of above results to the case $\alpha-^*$centralizer of R with involution.

2. Results

Lemma 1. [9, Lemma 1] *Let R be a prime ring, the elements a_i, b_i in the central closure of R satisfy $\sum a_i x b_i = 0$ for all $x \in R$. If $b_i \neq 0$ for some i, then a_i' s are $C-$independent.*

Lemma 2. [5, Theorem 2.1] *Let R be a $2-$torsion free semiprime ring with an identity element, α is a nonzero surjective homomorphism of R and $T : R \to R$ be an additive mapping such that $T(x^3) = \alpha(x)T(x)\alpha(x)$ holds for all $x \in R$. Then T is a $\alpha-$centralizer of R.*

Lemma 3. [3, Theorem 2.1] *Let R be a $2-$torsion free ring, U a square closed Lie ideal of R which has a commutator right (resp. left) nonzero divisor, α is* *an automorphism of R and $T : R \to R$ a left (resp. right) Jordan $\alpha-$centralizer mapping of U into R. Then T is a left (resp. right) $\alpha-$centralizer mapping of U into R.*

Example 1. [4, Example] *A semiprime ring may not contain a commutator nonzero divisor (after all,take commutative semiprime rings, or more generally, semiprime rings R containing a nonzero central idempotent element $e \in R$ such that eR is commutative). Conversely, a ring may contain a commutator nonzero divisor, but is not semiprime. For example, let $R = T_2(A_1)$ be the ring of the 2×2 upper triangular matrices whose entries are elements from the Weyl algebra A_1 (polynomials in x, y such that $xy - yx = 1$). Then R is not semiprime, but the commutator of scalar matrices generated by x and y is the identity matrix.*

Theorem 1. *Let R be a $2-$torsion free semiprime ring, U a square closed Lie ideal of R, α is an automorphism of R and $T : R \to R$ a left (resp. right) Jordan $\alpha-$centralizer mapping of U into R. Then T is a left (resp. right) $\alpha-$centralizer mapping of U into R.*

Proof. The proof is obvious from Lemma 3 and the well known fact that a semiprime ring may not contain a commutator nonzero divisor by above example. □

Theorem 2. *Let R be a non-commutative prime $^*-$ring, α is an automorphism of R and $T : R \to R$ be a Jordan left $\alpha-^*$centralizer. If $T(x) \in Z$ for all $x \in R$, then $T = 0$.*

Proof. By the hyphotesis, we have

$$[T(x), y] = 0 \text{ for all } x, y \in R. \tag{2.1}$$

Replacing x by x^2 in (2.1) and using this, we obtain that

$$T(x)[\alpha(x^*), y] = 0 \text{ for all } x, y \in R.$$

In the view of $T(x) \in Z$ and centre of prime ring is free from zero divisors, we get

$$T(x) = 0 \quad \text{or} \quad [\alpha(x^*), y] = 0 \text{ for all } x, y \in R.$$

We obtain R is union of its two additive subgroups such that

$$K = \{x \in R \mid T(x) = 0\}$$

and

$$L = \{x \in R \mid \alpha(x^*) \in Z\}.$$

Clearly each of K and L is additive subgroup of R. Morever, R is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $K = R$ or $L = R$. In the former case, we have $T = 0$ and the second case, R is commutative, a contradiction. This finishes the proof. □

Theorem 3. *Let R be a $2-$torsion free semiprime $^{*}-$ring, α is an automorphism of R such that $*\alpha = \alpha *$ and $T : R \to R$ be a Jordan left $\alpha-^{*}$centralizer. Then T is a reverse left $\alpha-^{*}$centralizer, that is $T(xy) = T(y)\alpha(x^{*})$ for all $x, y \in R$.*

Proof. By the hyphotesis, we have

$$T\left(x^2\right) = T(x)\,\alpha(x^{*}) \quad \text{for all } x \in R. \tag{2.2}$$

Applying involution both sides to (2.2), we conclude that

$$(T\left(x^2\right))^{*} = \alpha(x^{*})^{*}T(x)^{*} \quad \text{for all } x \in R.$$

Using $*\alpha = \alpha *$, we get

$$(T\left(x^2\right))^{*} = \alpha(x)T(x)^{*} \quad \text{for all } x \in R.$$

Define $S : R \to R, S(x) = T(x)^{*}$ for all $x \in R$. Hence we have

$$\begin{aligned} S(x^2) &= T(x^2)^{*} \\ &= (T(x)\alpha(x^{*}))^{*} \\ &= \alpha(x)T(x)^{*} = \alpha(x)S(x) \end{aligned}$$

for all $x \in R$. This means S is a Jordan right $\alpha-$centralizer on R. By Theorem 1, S is a right $\alpha-$centralizer that is, $S(xy) = \alpha(x)S(y)$ for all $x, y \in R$. This implies that

$$\begin{aligned} T(xy)^{*} &= S(xy) \\ &= \alpha(x)S(y) = \alpha(x)T(y)^{*}, \end{aligned} \tag{2.3}$$

and so

$$T(xy)^{*} = \alpha(x)T(y)^{*} \quad \text{for all } x, y \in R.$$

Applying involution both sides the last equation, we get

$$T(xy) = T(y)\,\alpha(x^{*}) \text{ for all } x, y \in R.$$

Hence T is a reverse left $\alpha-^{*}$centralizer. □

Theorem 4. *Let R be a $2-$torsion free semiprime $^{*}-$ring with an identity element, α is an automorphism of R such that $*\alpha = \alpha *$ and $T : R \to R$ be an additive mapping such that $T(x^3) = \alpha(x^{*})T(x)\alpha(x^{*})$ holds for all $x \in R$. Then T is a reverse $\alpha-^{*}$centralizer, that is $T(xy) = T(y)\alpha(x^{*}) = \alpha(y^{*})T(x)$ for all $x, y \in R$.*

Proof. By the hyphotesis, we have

$$T(x^3) = \alpha(x^{*})T(x)\alpha(x^{*}) \quad \text{for all } x \in R. \tag{2.4}$$

Applying involution both sides to (2.4) and using $*\alpha = \alpha *$, we obtain that

$$T(x^3)^{*} = (\alpha(x^{*})T(x)\alpha(x^{*}))^{*} = \alpha(x)T(x)^{*}\alpha(x) \quad \text{for all } x \in R.$$

Define $S : R \to R, S(x) = T(x)^*$ for all $x \in R$. Hence we have

$$\begin{aligned} S(x^3) &= T(x^3)^* \\ &= \alpha(x)T(x)^* \alpha(x) = \alpha(x)S(x)\alpha(x) \end{aligned}$$

for all $x \in R$. Hence we obtain that

$$S(x^3) = \alpha(x)S(x)\alpha(x) \text{ for all } x \in R.$$

Using Lemma 2, we conclude that S is a two sided $\alpha-$centralizer that is, $S(xy) = \alpha(x)S(y) = S(x)\alpha(y)$ for all $x, y \in R$. This implies for all $x, y \in R$

$$\begin{aligned} T(xy)^* &= S(xy) \\ &= \alpha(x)S(y) = \alpha(x)T(y)^* \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} T(xy)^* &= S(xy) \\ &= S(x)\alpha(y) = T(x)^* \alpha(y). \end{aligned}$$

Applying involution both sides the two last equations and using $*\alpha = \alpha*$, we get

$$T(xy) = T(y)\alpha(x^*) = \alpha(y^*)T(x) \text{ for all } x, y \in R.$$

□

Theorem 5. *Let R be a $2-$torsion free non-commutative prime $^*-$ring, α is an automorphism of R such that $*\alpha = \alpha*$ and $T, S : R \to R$ be two Jordan left $\alpha-^*$centralizer. If $[S(x), T(x)] = 0$ holds for all $x \in R$ and $T \neq 0$, then there exists $\lambda \in C$ such that $S = \lambda T$.*

Proof. We know that S and T are reverse left $\alpha-^*$centralizers by Theorem 3. Now we assume that

$$[S(x), T(x)] = 0 \text{ for all } x \in R. \tag{2.6}$$

Lineerizing (2.6) and using this, we have

$$[S(x), T(y)] + [S(y), T(x)] = 0 \text{ for all } x, y \in R. \tag{2.7}$$

Replacing x by zx in (2.7) and using this, we arrive at

$$S(x)[\alpha(z^*), T(y)] + T(x)[S(y), \alpha(z^*)] = 0 \text{ for all } x, y, z \in R. \tag{2.8}$$

Writing z^* instead of z in (2.8) and using α is an automorphism of R, we get

$$S(x)[z, T(y)] + T(x)[S(y), z] = 0 \text{ for all } x, y, z \in R. \tag{2.9}$$

Taking wx instead of x in (2.9), we find that

$$S(x)\alpha(w^*)[z, T(y)] + T(x)\alpha(w^*)[S(y), z] = 0 \text{ for all } x, y, z, w \in R.$$

Again replacing w^* instead of w and using α is an automorphism of R, we obtain that

$$S(x)w[z, T(y)] + T(x)w[S(y), z] = 0 \text{ for all } x, y, z, w \in R. \tag{2.10}$$

Using Lemma 1, we have $[z,T(y)]=0$ for all $y,z\in R$ or $S(x)=\lambda(x)T(x)$ where $\lambda(x)\in C$. But $[z,T(y)]\neq 0$ for some $z,y\in R$ because of $T\neq 0$ (see Theorem 2). Hence we get $S(x)=\lambda(x)T(x)$ where $\lambda(x)\in C$.

Returning (2.10), we can write

$$\begin{aligned}0&=S(x)w[z,T(y)]+T(x)w[S(y),z]\\&=\lambda(x)T(x)w[z,T(y)]+T(x)w[\lambda(y)T(y),z]\\&=(\lambda(x)-\lambda(y))T(x)w[z,T(y)]\end{aligned}$$

for all $z,y\in R$. By the primeness of R, the last equation yields that either $(\lambda(x)-\lambda(y))T(x)=0$ or $[z,T(y)]=0$. Again using $[z,T(y)]\neq 0$ some $z,y\in R$, we have $(\lambda(x)-\lambda(y))T(x)=0$ for all $x,y\in R$. This implies $\lambda(x)T(x)=\lambda(y)T(x)$, and so, $S(x)=\lambda(y)T(x)$ for all $x,y\in R$. This completes the proof. □

Theorem 6. *Let R be a semiprime $^{*}-$ring, α is an automorphism of R such that $*\alpha=\alpha*$ and $T:R\to R$ be a mapping (not necessary additive mapping) such that $T(x)\alpha(y^{*})=\alpha(x^{*})T(y)$ holds for all $x,y\in R$. Then T is a reverse left $\alpha-^{*}$centralizer of R.*

Proof. By the hypothesis, we get

$$T(x)\alpha(y^{*})=\alpha(x^{*})T(y)\ \text{ for all } x,y\in R. \tag{2.11}$$

We calculate the following equation using (2.11) and α is an automorphism of $R:$

$$\begin{aligned}(T(x+y)-T(x)-T(y))\alpha(z^{*})&=T(x+y)\alpha(z^{*})-T(x)\alpha(z^{*})-T(y)\alpha(z^{*})\\&=\alpha((x+y)^{*})T(z)-\alpha(x^{*})T(z)-\alpha(y^{*})T(z)\\&=(\alpha((x+y)^{*})-\alpha(x^{*})-\alpha(y^{*}))T(z)\\&=\alpha((x+y)^{*}-x^{*}-y^{*})T(z)\\&=\alpha(x^{*}+y^{*}-x^{*}-y^{*})T(z)=0\end{aligned}$$

Hence we have

$$(T(x+y)-T(x)-T(y))\alpha(z^{*})=0.$$

Writing z^{*} instead of z and using α is an automorphism of R in this equation, we arrive at

$$(T(x+y)-T(x)-T(y))z=0\ \text{ for all } x,y,z\in R.$$

Since R is semiprime ring, we obtain that

$$T(x+y)=T(x)+T(y)\ \text{ for all } x,y\in R.$$

Similarly, we calculate the relation $(T(yx)-T(x)\alpha(y^{*}))\alpha(z^{*})$ using (2.11), we find that $T(yx)=T(x)\alpha(y^{*})$ for all $x,y\in R$. Hence T is a reverse left $\alpha-^{*}$centralizer of R. □

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