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ON MEUSNIER THEOREM FOR PARALLEL SURFACES

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Abstract. In this paper, the geodesic curvature, the normal curvature, the geodesic torsion and the curvature of the image curve on a parallel surface of a given curve on a surface are obtained. Moreover, Meusnier theorem for parallel surfaces are discussed.

1. Introduction

Parallel surfaces as a subject of differential geometry have been intriguing for mathematicians throughout history and so it has been a research field. In theory of surfaces, there are some special surfaces such as ruled surfaces, minimal surfaces and surfaces of constant curvature in which geometricians are interested. Among these surfaces, parallel surfaces are also studied in many papers [4, 6, 7, 9, 12, 13]. Craig had studied to find parallel surface of ellipsoid [3]. Eisenhart gave a chapter for parallel surfaces in his famous a treatise of differential geometry [5]. Nizamoğlu had stated parallel ruled surface as a curve depending on one-parameter and gave some geometric properties of such a surface [11].

A surface M^r whose points are at a constant distance along the normal from another surface M is said to be parallel to M. So, there *is* infinite number of parallel surfaces because we choose the constant distance along the normal arbitrarily. A parallel surface can be regarded as the locus of point which *is* on the normals to M at a non-zero constant distance r from M [16].

In differential geometry, Meusnier's theorem states that all curves on a surface passing through a given point P and having the same tangent line at P also have the same normal curvature at P and their osculating circles form a sphere. The theorem was first announced by Jean Baptiste Meusnier in 1776. He is best known for Meusnier's theorem on the curvature of surfaces, which he formulated while he was at the Royal School of Engineering.

The centre of curvature of all curves on a surface M which pass through an arbitrary point P and whose tangents at P have the same direction, different from

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an asymptotic direction, lie on a circle. The circle of curvature of all plane section of M with common tangent at P whose direction is different from an asymptotic direction lie therefore on a sphere [10].

In this study, the geodesic curvature, the normal curvature, the geodesic torsion and the curvature of the image curve on a parallel surface of a given curve on a surface are obtained. Also Meusnier theorem for parallel surface are discussed.

2. Preliminaries

Definition 2.1. Let M be a surface in $\mathbb{E}^3$ and α be a unit-speed curve on M. Let $\overrightarrow{\mathbf{T}}(t) = \overrightarrow{\alpha}'(t)$, $\overrightarrow{\mathbf{T}}$ is called unit tangent vector to the curve α. Let $\overrightarrow{\mathbf{n}}(t)=\overrightarrow{\mathbf{T}}'(t)/\parallel \overrightarrow{\mathbf{T}}'(t) \parallel$, $\overrightarrow{\mathbf{n}}(t)$ is called principal normal vector. Finally let $\overrightarrow{\mathbf{b}}(t) = \overrightarrow{\mathbf{T}}(t)\wedge\overrightarrow{\mathbf{n}}(t)$, then $\overrightarrow{\mathbf{b}}(t)$ is called binormal vector to the curve α. Note that the vectors $\left\{\overrightarrow{\mathbf{T}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}}\right\}$ are perpendicular each other. They form an orthonormal basis for $\mathbb{R}^3$. It is called the Frenet frame. Let $\overrightarrow{\mathbf{N}}$ be unit normal to the surface M, and let $\overrightarrow{\mathbf{N}}(t) = \overrightarrow{\mathbf{N}}\circ\alpha(t)$ be the restriction of $\overrightarrow{\mathbf{N}}$ to the curve α. The triple $\left\{\overrightarrow{\mathbf{T}}, \overrightarrow{\mathbf{N}}, \overrightarrow{\mathbf{B}} = \overrightarrow{\mathbf{N}}\wedge\overrightarrow{\mathbf{T}}\right\}$ is a new frame and called Darboux frame.

Theorem 2.2. Let α be unit-speed curve and κ be real-valued function such that $\kappa(t) = \left\|\overrightarrow{\mathbf{T}}'(t)\right\|$ is called the curvature function. If $\kappa > 0$, then the Frenet formulas are as follows:

$$\begin{aligned} \overrightarrow{\mathbf{T}}' &= \kappa\overrightarrow{\mathbf{n}} \\ \overrightarrow{\mathbf{n}}' &= -\kappa\overrightarrow{\mathbf{T}} + \tau\overrightarrow{\mathbf{b}} \\ \overrightarrow{\mathbf{b}}' &= -\tau\overrightarrow{\mathbf{n}} \end{aligned} \tag{2.1}$$

Here τ is called the torsion of the curve α [12].

Definition 2.3. The functions $k_g(t)$, $k_n(t)$ are called the geodesic curvature, the normal curvature and also $\tau_g(t)$ is called geodesic torsion of α at the point $P = \alpha(t)$. These functions can be obtained as follows:

$$\begin{aligned} k_g(t) &=< \overrightarrow{\mathbf{T}}'(t), \overrightarrow{\mathbf{B}}(t) > \\ k_n(t) &=< \overrightarrow{\mathbf{T}}'(t), \overrightarrow{\mathbf{N}}(t) > \\ \tau_g(t) &=< \overrightarrow{\mathbf{B}}'(t), \overrightarrow{\mathbf{N}}(t) > \end{aligned}$$

[15].

Theorem 2.4. Let α be a unit-speed curve on surface M. Geodesic and normal curvatures of curve α on surface M are, respectively, denoted by k_g, k_n and geodesic torsion denoted by τ_g. Derivative formulas of Darboux frame in terms of k_g, k_n and τ_g are as follows;

$$\begin{aligned} \overrightarrow{\mathbf{T}}' &= k_g\overrightarrow{\mathbf{B}} + k_n\overrightarrow{\mathbf{N}} \\ \overrightarrow{\mathbf{B}}' &= -k_g\overrightarrow{\mathbf{T}} + \tau_g\overrightarrow{\mathbf{N}} \\ \overrightarrow{\mathbf{N}}' &= -k_n\overrightarrow{\mathbf{T}} - \tau_g\overrightarrow{\mathbf{B}} \end{aligned} \tag{2.3}$$

[15].

Definition 2.5. Curves on a surface along which k_g, k_n, or τ_g vanish are called as follows:

$$\begin{array}{ll} k_g = 0: & \text{geodesic lines or geodesics} \\ k_n = 0: & \text{asymptotic lines} \\ \tau_g = 0: & \text{lines of curvature} \end{array} \tag{2.4}$$

[2].

Theorem 2.6. Let α be a regular curve in $\mathbb{E}^3$. Then

$$\begin{aligned} \overrightarrow{\mathbf{T}} &= \frac{\overrightarrow{\alpha}'}{\|\overrightarrow{\alpha}'\|} \\ \overrightarrow{\mathbf{b}} &= \frac{\overrightarrow{\alpha}' \wedge \overrightarrow{\alpha}''}{\|\overrightarrow{\alpha}' \wedge \overrightarrow{\alpha}''\|} \quad , \ \kappa = \frac{\|\overrightarrow{\alpha}' \wedge \overrightarrow{\alpha}''\|}{\|\overrightarrow{\alpha}'\|^3} \\ \overrightarrow{\mathbf{n}} &= \overrightarrow{\mathbf{b}} \wedge \overrightarrow{\mathbf{T}} \qquad\quad , \ \tau = \frac{< \overrightarrow{\alpha}' \wedge \overrightarrow{\alpha}'', \overrightarrow{\alpha}''' >}{\|\overrightarrow{\alpha}' \wedge \overrightarrow{\alpha}''\|^2} \end{aligned} \tag{2.5}$$

[12].

Theorem 2.7. If α be a regular curve in $\mathbb{E}^3$ and $\|\overrightarrow{\alpha}'\| = v$ with $\kappa > 0$, then

$$\begin{aligned} \overrightarrow{\mathbf{T}}' &= v\kappa\overrightarrow{\mathbf{n}} \\ \overrightarrow{\mathbf{n}}' &= v(-\kappa\overrightarrow{\mathbf{T}} + \tau\overrightarrow{\mathbf{b}}) \\ \overrightarrow{\mathbf{b}}' &= -v\tau\overrightarrow{\mathbf{n}} \end{aligned} \tag{2.6}$$

[11].

Theorem 2.8. Let α be non-unit speed curve on M surface in $\mathbb{E}^3$. The Darboux frame of curve α which is $\|\overrightarrow{\alpha}'\| = v$, is $\left\{\overrightarrow{\mathbf{T}}, \overrightarrow{\mathbf{B}}, \overrightarrow{\mathbf{N}}\right\}$. Geodesic, normal curvatures, and geodesic torsion of this curve-surface pair which is, respectively, denoted by k_g, k_n and τ_g are defined as follows:

$$\begin{aligned} k_g &= \frac{1}{v^2} < \overrightarrow{\alpha}'', \overrightarrow{\mathbf{B}} > \\ k_n &= \frac{1}{v^2} < \overrightarrow{\alpha}'', \overrightarrow{\mathbf{N}} > \\ \tau_g &= -\frac{1}{v} < \overrightarrow{\mathbf{N}}', \overrightarrow{\mathbf{B}} > \end{aligned} \tag{2.7}$$

[15].

Definition 2.9. Let be a curve α be given by allowable parametric representation $\alpha(t)$ of class $r \geqslant 2$ with arc length t as parameter. Differentiating the relation $\mathbf{T} \cdot \mathbf{T} = \mathbf{1}$ we obtain $\mathbf{T} \cdot \mathbf{T}' = 0$. Hence, if the vector

$$\mathbf{T}' = \alpha''$$

it is orthogonal to the unit tangent vector $\mathbf{T}$ and consequently lies in the normal plane to α at the point under consideration; $\mathbf{T}'$ also lies in the osculating plane.

The unit vector

$$\mathbf{N}(t) = \frac{\mathbf{T}^{\shortmid}(t)}{\|\mathbf{T}^{\shortmid}(t)\|}$$

which has direction and sense of $\mathbf{T}^{\shortmid}$ the called the unit principal normal vector to the curve α at the point $\alpha(t)$. The absolute value of the $\mathbf{T}^{\shortmid}$,

$$\kappa(t) = \left\|\mathbf{T}^{\shortmid}(t)\right\| = \sqrt{< \alpha^{\shortmid\shortmid}(t), \alpha^{\shortmid\shortmid}(t) >}\ , \quad (\kappa \geqslant 0),$$

is called the curvature of the curve α at the point $\alpha(t)$. The reciprocal of the curvature,

$$R(t) = \frac{1}{\kappa(t)}\ , \quad (\kappa > 0),$$

is called the radius of curvature of the curve α at he point $\alpha(t)$.

We now mention the following fact: while the sense of the unit tangent vector to a curve depends on the orientation of the curve resulting from the choice of a certain parametric representation, the unit principal normal vector is independent of the orientation of the curve; its sense does not change if the parameter t is replaced by the parameter $t^* = -t$ or any other allowable parameter.

The point C on the positive ray of the principal normal at distance $R(t)$ from the corresponding point P of the curve α is called the centre of the curvature. The circle in the osculating plane whose radius is R and whose centre is C is called osculating circle or circle of curvature of the α at P [10].

Theorem 2.10. (Meusnier Theorem)

If a set of planes be drawn through a tangent to a surface in a nonasymptotic direction, then the osculating circles of the intersections with the surface lie upon a sphere [15].

From the equations 2.1. and 2.3, we obtain the following equation known as Meusnier formula

$$k_n = \kappa \cos \Phi$$

Where Φ $(-\frac{\Pi}{2} \leq \Phi \leq \frac{\Pi}{2})$ is angle between $\overrightarrow{\mathbf{n}}$ and $\overrightarrow{\mathbf{N}}$. This equation can be cast into another form for a direction $\overrightarrow{\mathbf{T}}$ for which $k_n \neq 0$, hence also $\kappa \neq 0$. Such a direction are called nonasymptotic directions. For curves in such directions we can write $R = \kappa^{-1}$, $R_n = k_n^{-1}$. The quantities R and R_n are here positive, R_n represents the radius of curvature of a curve with tangent $\overrightarrow{\mathbf{T}}$ and $\Phi = 0$. One such curve is the intersection of the surface with the plane at P through $\overrightarrow{\mathbf{T}}$ and the surface normal; this curve is called the normal section of the surface at P in the direction of α. Equation 2.8 now takes the form

$$R_n \cos \Phi = R.$$

The center of curvature C_1 of a curve α in a nonasymptotic direction at P is the projection on the principal normal of the center of curvature C_0 of the normal section which is tangent to α at P.

If a set of curve planes be drawn through a tangent to a surface in a nonasymptotic direction, then the osculating circles of the intersections with the surface lie upon a sphere whose radius and center are, respectively, R_n and $C = P + R_n\overrightarrow{\mathbf{N}}_P$. Here, $R_n = \dfrac{1}{k_n}$.

Theorem 2.11 At a given point of a surface, the normal curvature and geodesic torsion are the same for all surface curves having a common tangent there [2].

Let be α be a curve in M that has initial velocity $\overrightarrow{\alpha}'(t) = \overrightarrow{v}$. Let $\overrightarrow{\mathbf{N}}_\alpha$ be the restriction of $\overrightarrow{\mathbf{N}}$ to α, that is, the vector field $t \to \overrightarrow{\mathbf{N}}(\alpha(t))$ on α. Then

$$D_{\overrightarrow{v}}\ \overrightarrow{\mathbf{N}} = (\overrightarrow{\mathbf{N}}_\alpha)'(0) \tag{2.9}$$

above the equation is the derivative of $\overrightarrow{\mathbf{N}}$ in the direction $\overrightarrow{v}$.

Definition 2.12. If P is a point of M , then for each tangent vector $\overrightarrow{v}$ to M at P, let

$$S_P(\overrightarrow{v}) = -D_{\overrightarrow{v}}\overrightarrow{\mathbf{N}} \tag{2.10}$$

where $\overrightarrow{\mathbf{N}}$ is unit normal vector field on a neighborhood of P in M. S_P is called the shape operator of M at P (derived from $\overrightarrow{\mathbf{N}}$) [12].

Definition 2.13. Let M and M^r be two surfaces in Euclidean space. The function

$$\begin{array}{llll} f: & M & \to & M^r \\ & P & \to & f(P) = P + r\overrightarrow{\mathbf{N}}_P \end{array} \tag{2.11}$$

is called the parallelization function between M and M^r and furthermore M^r is called parallel surface to M where $\overrightarrow{\mathbf{N}}$ is the unit normal vector field on M and r is a given real number [8].

Theorem 2.14. Let M and M^r be two parallel surfaces in Euclidean space and

$$f: \quad M \quad \to \quad M^r \tag{2.12}$$

be the parallelization function. Then for $X \in \chi(M)$

1. $f_*(X) = X - rS(X)$
2. $S^r\,(f_*(X)) = S(X)$
3. f preserves principal directions of curvature, that is

$$S^r(f_*(X)) = \frac{k}{1-rk} f_*(X) \tag{2.13}$$

where S^r is the shape operator on M^r; and k is a principal curvature of M at P in direction X [15].

3. Darboux frame of an image of a curve on a parallel surface

Definition 3.1. Let M and M^r be two parallel surfaces. Let α be a unit-speed curve on M and image of α stands on M^r which $(f \circ \alpha) = \beta$. For β is non-unit speed curve than $\left\|\overrightarrow{\beta}^{\shortmid}\right\| = \left\|f_*(\overrightarrow{\mathbf{T}})\right\| = v \neq 1$. Darboux frame of curve β on M^r is

$$\left\{\overrightarrow{\mathbf{T}}^r = \frac{\overrightarrow{f_*(\mathbf{T})}}{v},\ \overrightarrow{\mathbf{B}}^r = \overrightarrow{\mathbf{T}}^r \wedge \overrightarrow{\mathbf{N}}^r, \overrightarrow{\mathbf{N}}^r = \overrightarrow{\mathbf{N}}\right\} \tag{3.1}$$

where $\overrightarrow{\mathbf{N}}^r$ is unit normal vector of M^r.

Theorem 3.2. Let Darboux frame of curve β at $f(\alpha(t_0)) = f(P)$ on M^r be $\left\{\overrightarrow{\mathbf{T}}^r, \overrightarrow{\mathbf{B}}^r, \overrightarrow{\mathbf{N}}^r\right\}$, then

$$\begin{aligned} \overrightarrow{\mathbf{T}}^r &= \frac{1}{v}\left[(1 - rk_n)\overrightarrow{\mathbf{T}} - r\tau_g\overrightarrow{\mathbf{B}}\right] \\ \overrightarrow{\mathbf{B}}^r &= \frac{1}{v}\left[(1 - rk_n)\overrightarrow{\mathbf{B}} + r\tau_g\overrightarrow{\mathbf{T}}\right] \\ \overrightarrow{\mathbf{N}}^r &= \overrightarrow{\mathbf{N}} \end{aligned} \tag{3.2}$$

Proof. From the theorems 2.4, 2.14 and the equation 2.10 tangent vector of $(f \circ \alpha) = \beta$ curve at $f(\alpha(t_0))$

$$\begin{aligned} \overrightarrow{\beta}^{\shortmid} = f_*(\overrightarrow{\mathbf{T}}) &= \overrightarrow{\mathbf{T}} - rS(\overrightarrow{\mathbf{T}}) \\ &= (1 - rk_n)\overrightarrow{\mathbf{T}} - r\tau_g\overrightarrow{\mathbf{B}} \end{aligned} \tag{3.3}$$

and the norm of $\overrightarrow{\beta}^{\shortmid}$ is

$$\left\|\overrightarrow{\beta}^{\shortmid}\right\| = \sqrt{(1 - rk_n)^2 + r^2\tau_g^2} = v \tag{3.4}$$

For $\overrightarrow{\mathbf{T}}^r = f_*(\overrightarrow{\mathbf{T}})/\left\|f_*(\overrightarrow{\mathbf{T}})\right\|$, tangent vector of curve $(f \circ \alpha) = \beta$ is $\overrightarrow{\mathbf{T}}^r = \frac{1}{v}\left[(1 - rk_n)\overrightarrow{\mathbf{T}} - r\tau_g\overrightarrow{\mathbf{B}}\right]$. Surface M^r is parallel to surface M. And also there is the equation $\overrightarrow{\mathbf{N}}^r = \overrightarrow{\mathbf{N}}$ between normal vectors of surfaces M and M^r. Finally

$$\overrightarrow{\mathbf{B}}^r = \frac{1}{v}\left[(1 - rk_n)\overrightarrow{\mathbf{B}} + r\tau_g\overrightarrow{\mathbf{T}}\right].$$

Here $\left\|f_*(\overrightarrow{\mathbf{T}})\right\| = \sqrt{(1 - rk_n)^2 + (r\tau_g)^2} = v$.

Theorem 3.3. Let α be a regular curve on the surface M. Then the geodesic curvature, the normal curvature and the geodesic torsion of the curve $(f \circ \alpha) = \beta$ are respectively;

$$\begin{aligned} k_g^r &= \frac{k_g}{v} - \frac{r}{v^3}\left[(\tau_g^{\shortmid} + r(\tau_g k_n^{\shortmid} - \tau_g^{\shortmid} k_n))\right] \\ k_n^r &= \frac{1}{v^2}\left[k_n - r(k_n^2 + \tau_g^2)\right] \\ \tau_g^r &= \frac{\tau_g}{v^2} \end{aligned} \tag{3.5}$$

at the point $f(\alpha(t_0))$ on the parallel surface M^r.

Proof. Because of non-unit speed curve β, we use the theorem 2.8 and the equation (3.3), so the following equation are obtained:

$$\begin{aligned} \overrightarrow{\beta}'' &= -rk'_n\overrightarrow{\mathbf{T}} + (1-rk_n)\overrightarrow{\mathbf{T}}' - r\tau'_g\overrightarrow{\mathbf{B}} - r\tau_g\overrightarrow{\mathbf{B}}' \\ &= (r\tau_g k_g\text{-}rk'_n)\overrightarrow{\mathbf{T}} + (k_g(1\text{-}rk_n)\text{-}r\tau'_g)\overrightarrow{\mathbf{B}} + \left[(1\text{-}rk_n)k_n\text{-}r\tau_g^2\right]\overrightarrow{\mathbf{N}}. \end{aligned} \tag{3.6}$$

Using the equations (2.7), (3.2) and (3.6), the geodesic curvature of curve β on surface M^r is

$$k_g^r = \frac{k_g}{v} - \frac{r}{v^3}\left[(\tau'_g + r(\tau_g k'_n - \tau'_g k_n))\right] \tag{3.7}$$

its normal curvature is

$$k_n^r = \frac{1}{v^2}\left[k_n - r(k_n^2 + \tau_g^2)\right] \tag{3.8}$$

and its geodesic torsion is

$$\tau_g^r = \frac{\tau_g}{v^2}. \tag{3.9}$$

Corollary 3.4. The image of a geodesic curve of M on the parallel surface M^r is also geodesic under the following conditions;

i) The geodesic on M is a line of curvature

ii) The normal curvature and the geodesic torsion of the geodesic curve on M are both constants.

On the other hand, the image of a non-geodesic curve on M is a geodesic on M^r if

$$k_g = \frac{r}{v^2}\left[(\tau'_g + r(\tau_g k'_n - \tau'_g k_n))\right].$$

Proof. By the definition 2.5, $k_g = 0$ for a geodesic curve on M. Similarly, k_g^r is also zero for a geodesic on the parallel surface M^r.

i) From the equation (3,7), for a geodesic on M (i.e. $k_g = 0$) if $\tau_g = 0$ then $k_g^r = 0$. Therefore being a curvature line of a geodesic on M implies being a geodesic on M^r.

ii) If the normal curvature, k_n, and the geodesic torsion, τ_g are constants then from equation (3.7) $k_g^r = 0$, that is the image curve is a geodesic.

It is clear from the equation (3.7) that, a non zero $k_g = \frac{r}{v^2}\left[(\tau'_g + r(\tau_g k'_n - \tau'_g k_n))\right]$ implies $k_g^r = 0$.

Corollary 3.5. If an asymptotic curve on M is a line of curvature then the image curve on M^r is also asymptotic curve.

On the other hand,the image of a non-asymptotic curve with the condition

$$k_n = r(k_n^2 + \tau_g^2)$$

on M is an asymptotic curve on M^r.

Proof. By the definition 2.5, a curve is asymptotic if its normal curvature is zero. By the equation (3.8) the image of an asymptotic curve (i.e. $k_n = 0$) which is also a line of curvature (i.e. $\tau_g = 0$) is an asymptotic curve on M^r (i.e. $k_n^r = 0$).

If the curve is non-asymptotic on M (i.e. $k_n \neq 0$) but we have the equation $k_n = r(k_n^2 + \tau_g^2)$ then image curve is an asymptotic on M^r by the equation (3.8) (i.e. $k_n^r = 0$).

Corollary 3.6. The image curve is line of curvature on M^r if and only if it is a line of curvature on M.

Proof. From the equation (3,7) it is clear that $\tau_g^r = 0$ if and only if $\tau_g = 0$.

4. Frenet frame of image of a curve on a parallel surface

Theorem 4.1. Let Frenet frame of the curve $(f \circ \alpha) = \beta$ at $f(\alpha(t_0)) = f(P)$ on the surface M^r be $\left\{\overrightarrow{\mathbf{T}}^r, \overrightarrow{\mathbf{n}}^r, \overrightarrow{\mathbf{b}}^r\right\}$, then Frenet frame of parallel surface is as follows:

$$\overrightarrow{\mathbf{T}}^r = \frac{1}{v}\left[(1 - rk_n)\overrightarrow{\mathbf{T}} - r\tau_g \overrightarrow{\mathbf{B}}\right] \tag{4.1}$$

$$\overrightarrow{\mathbf{n}}^r = \frac{1}{v\sqrt{\left(k_n^r\right)^2 + \left(k_g^r\right)^2}}\left[r\tau_g k_g^r \overrightarrow{\mathbf{T}} + (1\text{-}rk_n)k_g^r \overrightarrow{\mathbf{B}} + vk_n^r \overrightarrow{\mathbf{N}}\right]$$

$$\overrightarrow{\mathbf{b}}^r = \frac{1}{v^3\sqrt{\left(k_n^r\right)^2 + \left(k_g^r\right)^2}}\left[-r\tau_g v^2 k_n^r \overrightarrow{\mathbf{T}}\text{-}(1\text{-}rk_n)v^2 k_n^r \overrightarrow{\mathbf{B}} + v^3 k_g^r \overrightarrow{\mathbf{N}}\right]$$

Proof. We use the theorem 2.6. because of non-unit speed curve β. If the equations (3.3) and (3.6) are used, the following equation is obtained;

$$\overrightarrow{\beta}' \wedge \overrightarrow{\beta}'' = -r\tau_g v^2 k_n^r \overrightarrow{\mathbf{T}} - (1 - rk_n)v^2 k_n^r \overrightarrow{\mathbf{B}} + v^3 k_g^r \overrightarrow{\mathbf{N}}. \tag{4.2}$$

If the equation (4.2) is normalized the following equation

$$\left\|\overrightarrow{\beta}' \wedge \overrightarrow{\beta}''\right\| = v^3\sqrt{\left(k_n^r\right)^2 + \left(k_g^r\right)^2} \tag{4.3}$$

is obtained. By using the equations (4.2) and (4.3), binormal vector of this curve is obtained as

$$\overrightarrow{\mathbf{b}}^r = \frac{1}{v\sqrt{\left(k_n^r\right)^2 + \left(k_g^r\right)^2}}\left[\text{-}r\tau_g k_n^r \overrightarrow{\mathbf{T}}\text{-}(1 - rk_n)k_n^r \overrightarrow{\mathbf{B}} + vk_g^r \overrightarrow{\mathbf{N}}\right]. \tag{4.4}$$

The normal vector $\overrightarrow{\mathbf{n}}^r$ is found by cross product of $\overrightarrow{\mathbf{b}}^r$ and $\overrightarrow{\mathbf{T}}^r$;

$$\overrightarrow{\mathbf{n}}^r = \frac{1}{v\sqrt{\left(k_n^r\right)^2 + \left(k_g^r\right)^2}}\left[r\tau_g k_g^r \overrightarrow{\mathbf{T}} + (1 - rk_n)k_g^r \overrightarrow{\mathbf{B}} + vk_n^r \overrightarrow{\mathbf{N}}\right]. \tag{4.5}$$

As a result, $\left\{\overrightarrow{\mathbf{T}}^r, \overrightarrow{\mathbf{b}}^r, \overrightarrow{\mathbf{n}}^r\right\}$ is Frenet frame of curve β at $f(P)$ on surface M^r.

Theorem 4.2. Let $\left\{\overrightarrow{\mathbf{T}}^r, \overrightarrow{\mathbf{b}}^r, \overrightarrow{\mathbf{n}}^r\right\}$ be Frenet frame of curve β at $f(P)$ on surface M^r. And let $\overrightarrow{\mathbf{n}}^r$ be principal normal vector of curve β, $\overrightarrow{\mathbf{N}}^r$ be unit normal of surface

M^r, Φ^r be the angle between the vectors $\overrightarrow{\mathbf{n}}^r$ and $\overrightarrow{\mathbf{N}}^r$ and κ^r be curvature of curve β at $f(P)$ on M^r, then

$$(\kappa^r)^2 = (k_n^r)^2 + \left(k_g^r\right)^2, \qquad \cos\Phi^r = \frac{k_n^r}{\sqrt{(k_n^r)^2 + \left(k_g^r\right)^2}}. \tag{4.6}$$

Proof. From the equations 3.4 and 4.3 the values of $\left\|\overrightarrow{\beta}' \wedge \overrightarrow{\beta}''\right\|$ and $\left\|\overrightarrow{\beta}'\right\|$ are substituted in the equation (2.5). Curvature of curve β becomes as follows

$$\kappa^r \;=\; \sqrt{(k_n^r)^2 + \left(k_g^r\right)^2}.$$

From the equation 4.5 by using the vectors $\overrightarrow{\mathbf{n}}^r$and $\overrightarrow{\mathbf{N}}^r = \overrightarrow{\mathbf{N}}$, the following equation is found;

$$\cos\Phi^r =< \overrightarrow{\mathbf{n}}^r, \overrightarrow{\mathbf{N}}^r >= \frac{k_n^r}{\sqrt{(k_n^r)^2 + \left(k_g^r\right)^2}}$$

or

$$\cos\Phi^r = \frac{k_n^r}{\kappa^r}.$$

Theorem 4.3. Images of all curves which have the same tangent vector at the point $P = \alpha(t_0)$ on a surface M have the same $f_*(\overrightarrow{\mathbf{T}})$ tangent vector at the point $f(P) = \beta(s_0)$ on the surface M^r.

Proof. Let the equation $f_*(\overrightarrow{\mathbf{T}}) = (1-rk_n)\overrightarrow{\mathbf{T}} - r\tau_g\overrightarrow{\mathbf{B}}$ be taken into consideration in (3.3). All curves which have the same tangent vector at the point $P = \alpha(t_0)$ on a surface M have the same tangent vector and the same normal vector at the point, hence the vector $\overrightarrow{\mathbf{B}} = \overrightarrow{\mathbf{N}} \wedge \overrightarrow{\mathbf{T}}$ is the same for all these curves. From the theorem 2.11, all components of the vector $f_*(\overrightarrow{\mathbf{T}})$ are the same at this point because normal curvature and geodesic torsion of all these curves are the same at that point.

The equation in the following corollary is Meusnier formula for parallel surfaces.

Corollary 4.4. By the equation (4.6), the following formula is obtained:

$$k_n^r = \kappa_i^r \cos\Phi_i^r$$

The curvature circles of all the curves, which have the same $f_*(\overrightarrow{\mathbf{T}})$ tangent (not asymptotic direction) at the point $f(P) \in M^r$,lie upon a sphere whose radius and center are, respectively, R^r and $C^r = f(P) + R^r\overrightarrow{\mathbf{N}}^r_{f(P)}$. Here, $R^r = \dfrac{1}{k_n^r}$.

Example 4.1. Let sphere surface M be given with the following parameterization

$$\varphi(u,v) = \left(u, v, \sqrt{r^2 - u^2 - v^2}\right).$$

i) Let us show that for $I = (0,\pi)$, $\alpha : I \to \mathbb{R}^3$; the curve $\alpha(t) = (r\cos t, 0, r\sin t)$ is on surface M**.** Points on surface M are like

$$(p_1, p_2, \sqrt{r^2 - (p_1)^2 - (p_2)^2})$$

If $\cos t$ for p_1 and 0 for p_2 are taken, then the point $(r\cos t, 0, r\sin t)$ is obtained. Finally, it is shown that every point $\alpha(t)$ is on surface M.

ii) Let's obtain Darboux triple of the pair (M, α) at the point $\alpha(t)$. Normal vector of surface M is found as

$$\overrightarrow{\mathbf{N}} = \frac{1}{r}(u, v, \sqrt{r^2 - u^2 - v^2}).$$

Also, because of $\alpha'(t) = (-r\sin t, 0, r\cos t)$ and $v = \|\alpha'(t)\| = r$,

$$\overrightarrow{\mathbf{T}} = (-\sin t, 0, \cos t)$$

and

$$\overrightarrow{\mathbf{N}}(\alpha(t)) = \frac{1}{r}\alpha(t) = (\cos t, 0, \sin t).$$

Binormal vector of Darboux triple is found as follows

$$\overrightarrow{\mathbf{B}} = \overrightarrow{\mathbf{N}} \wedge \overrightarrow{\mathbf{T}} = (0, -r, 0).$$

Thus the Darboux triple $\left\{\overrightarrow{\mathbf{T}}, \overrightarrow{\mathbf{N}}, \overrightarrow{\mathbf{B}}\right\}$ is found.

iii) Let's calculate curvatures of the pair (M, α) and curvature radius of the curve α at the point $\alpha(t)$. By using the expression (2.7) and the equations $\overrightarrow{\mathbf{N}}'(\alpha(t)) = (-\sin t, 0, \cos t)$ and $\alpha''(t) = (-r\cos t, 0, -r\sin t)$,

$$\tau_g(t) = 0,\ k_n(t) = -\frac{1}{r} \text{ and } k_g(t) = 0$$

are obtained. Curvature of a curve $\alpha(t)$ on surface M is found as follows

$$\kappa(t) = \sqrt{k_n^2(t) + k_g^2(t)} = \frac{1}{r}.$$

From the definition 2.10, $R = r$ is found because of $R = \dfrac{1}{\kappa}$.

iv) Let surface M^r, parallel to surface M, be a sphere with radius $2r$. Let's calculate curvatures of the pair $(M^r, f\circ\alpha)$ and curvature of the curve $f\circ\alpha$ at the point $f(\alpha(t))$. Let $f\circ\alpha = \beta$ be a curve. The tangent vector of curve β is $\overrightarrow{\beta}' = (1 - rk_n)\overrightarrow{\mathbf{T}} - r\tau_g\overrightarrow{\mathbf{B}}$. The expressions $\tau_g = 0$, $k_n = -\dfrac{1}{r}$ and $k_g = 0$ of original surface are used in the equation (3.3). The norm of the tangent vector $\overrightarrow{\beta}'$ is

$$\left\|\overrightarrow{\beta'}\right\| = \sqrt{< (1 - rk_n)\overrightarrow{\mathbf{T}} - r\tau_g\overrightarrow{\mathbf{B}}, (1 - rk_n)\overrightarrow{\mathbf{T}} - r\tau_g\overrightarrow{\mathbf{B}} >} = 2.$$

Geodesic curvature k_g^r, normal curvature k_n^r and geodesic torsion τ_g^r of β on parallel surface M^r are, respectively, as follows:

$$k_g^r = \frac{1}{v^3}\left[k_g v^2 - r(r\tau_g k_n' + (1 - rk_n)\tau_g')\right] = 0,$$

$$k_n^r = \frac{1}{v^2}(k_n - r(k_n^2 + \tau_g^2)) = -\frac{1}{2r}$$

and

$$\tau_g^r = \frac{\tau_g}{v^2} = 0.$$

Curvature of the curve $f \circ \alpha = \beta$ is found, by using the equation (3.3), as follows:

$$\kappa^r = \sqrt{\left(k_n^r\right)^2 + \left(k_g^r\right)^2} = \frac{1}{2r}.$$

If curvature radius of the curve β is denoted by R^r, $R^r = 2r$ is found.

5. Conclusion

There are many studies related to Meusnier theorem. In this study, the elements of the Darboux and Frenet frames of the curve which is the image of a curve at the original surface upon the parallel surface were obtained in terms of the frame elements belonging to the original surface. Then normal and geodesic curvatures and geodesic torsion of parallel surface were found in respect of those of the original surface. Thereafter, the condition was given for a curve on the original surface which is also an asymptotic curve to be again its image on the parallel surface an asymptotic curve. It is also shown that the condition is preserved for an image curve to be a line of curvature. A relation was given among curvature, normal and geodesic curvatures of parallel surface. Finally, it was shown that Meusnier theorem has been provided for parallel surfaces therefore some results and an example were given.

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