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RELIABILITY PROPERTIES OF THE SYSTEM CONSTRUCTED BY SWITCHING FROM ONE COMPONENT TO TWO-DEPENDENT UNIT REDUNDANT STANDBY SYSTEM

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Abstract. In this work, we consider a system with switching towards to standby redundant system composed of two dependent components. Marginal distributions of component lifetimes are exponential and joint distribution belongs to Farlie-Gumbel-Morgenstern family. We examine reliability properties of switching system such as shape of hazard rate function, mean residual lifetime and some stochastic orders under determined circumstances on parameter spaces.

1. Introduction

Let T_1 and T_2 be the component lifetimes whose joint distribution is the bivariate Farlie-Gumbel-Morgenstern distribution with exponential marginals. The joint survival function of the components is given by

$$S(t_1,t_2) = S_1(t_1)\, S_2(t_2)\, [1+\alpha F_1(t_1)\, F_2(t_2)], \qquad t_i > 0,$$

where $\alpha \in [-1,1]$ denotes association parameter, S_i and F_i $(i = 1, 2)$ are the survival and distribution functions of T_1 and T_2, respectively (Morgenstern 1956,[5] Gumbel 1960, [3]). Throughout this study, it will be assumed that marginal distributions of the lifetimes are exponential with means θ_i. Let D be a binary random variable which determines the status of the switching device. Assume that D is a Bernoulli random variable with probability λ. Operation of this switching device is independent from the functioning of the components. While $D = 1$, $unit1$ conducts a task alone, and while $D = 0$, a parallel system, associatively composed of $unit1$ and $unit2$, carries out the task. So, changeover device switches from one component to two components in parallel. Let T_{sw} denote the lifetime of the system established in this way, then it is clear that

$$T_{sw} = DT_1 + (1-D)\, max\, \{T_1, T_2\}$$

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and hence the survival function of T_{sw} is found as

$$S(t) = Pr(T_1 > t)\, Pr(D = 1) + Pr(max\{T_1, T_2\} > t)\, Pr(D = 0).$$

Based on the above definitions and assumptions, we clearly can rewrite $S(t)$ as follows:

$$S(t) = \lambda S_1(t) + (1-\lambda)[1 - F(t,t)]$$

The following examples can be given to illustrate the use of this system constructed in this manner in practice; waiting times of the customers serving in a multichannel queuing system with two associative servers (secondary server is considered as a cold standby). The data of amount of water in the reservoir to be fed from at least one source. Supply-demand balance data in the production of a factory that has received a request from at least one customer. Lifetimes or recovery times data obtained from patients with two groups; such that, while the specific treatment is applied to first group, other appropriate treatment methods are also applied to second group in addition to the same treatment. Total duration data of movements in gold and dollar prices moving above a certain threshold level.

2. Distributional Properties

Let ϕ stand for the parameter vector $(\theta_1,\ \theta_2,\ \alpha,\ \lambda)$ then the distribution function of the T_{sw} is given by

$$F(t;\phi) = \lambda F_1(t;\theta_1) + (1-\lambda)\{F_1(t;\theta_1)\, F_2(t;\theta_2)\,[1 + \alpha S_1(t;\theta_1)\, S_2(t;\theta_2)]\} \quad (2.1)$$

where $F_i(t;\theta_i) = 1 - e^{-\frac{t}{\theta_i}}$, $(i = 1, 2)$ and $S_i = 1 - F_i$. Hence, by rewriting (2.1), we obtain

$$\begin{aligned} F(t;\phi) &= \lambda\left(1 - e^{-\frac{t}{\theta_1}}\right) + (1-\lambda)\left\{\left(1 - e^{-\frac{t}{\theta_1}}\right)\left(1 - e^{-\frac{t}{\theta_2}}\right)\left[1 + \alpha\left(e^{-\frac{t}{\theta_1}}\right)\left(e^{-\frac{t}{\theta_2}}\right)\right]\right\} \\ &= \left(1 - e^{-\frac{t}{\theta_1}}\right) + (1-\lambda)\left[\left\{e^{-t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)} - e^{-\frac{t}{\theta_2}}\right\}\right. \\ &\qquad \left. + \alpha\left\{e^{-t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)} - e^{-t\left(\frac{1}{\theta_1}+\frac{2}{\theta_2}\right)} - e^{-t\left(\frac{2}{\theta_1}+\frac{1}{\theta_2}\right)} + e^{-2t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)}\right\}\right] \end{aligned} \quad (2.2)$$

By differentiating (2.2) and organizing obtained result, we have the probability density function as follows:

$$\begin{aligned} f(t;\phi) &= \frac{1}{\theta_1}e^{-\frac{t}{\theta_1}} + (1-\lambda)\left\{\frac{1}{\theta_2}e^{-\frac{t}{\theta_2}} - \left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)e^{-t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)}\right\} \\ &\quad + (1-\lambda)\,\alpha\left\{\left(\frac{1}{\theta_1}+\frac{2}{\theta_2}\right)e^{-t\left(\frac{1}{\theta_1}+\frac{2}{\theta_2}\right)} - \left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)e^{-t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)}\right. \\ &\qquad \left. + \left(\frac{2}{\theta_1}+\frac{1}{\theta_2}\right)e^{-t\left(\frac{2}{\theta_1}+\frac{1}{\theta_2}\right)} - 2\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)e^{-2t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)}\right\} \end{aligned} \quad (2.3)$$

We have the different shapes of the probability density function for various values of the switching probability and association parameter.

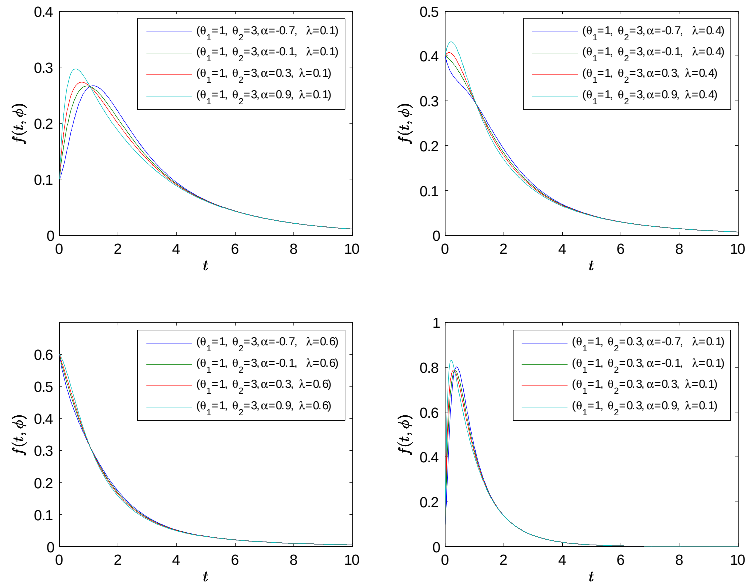

FIGURE 1. Shapes of the probability density function with respect to some values of association parameter.

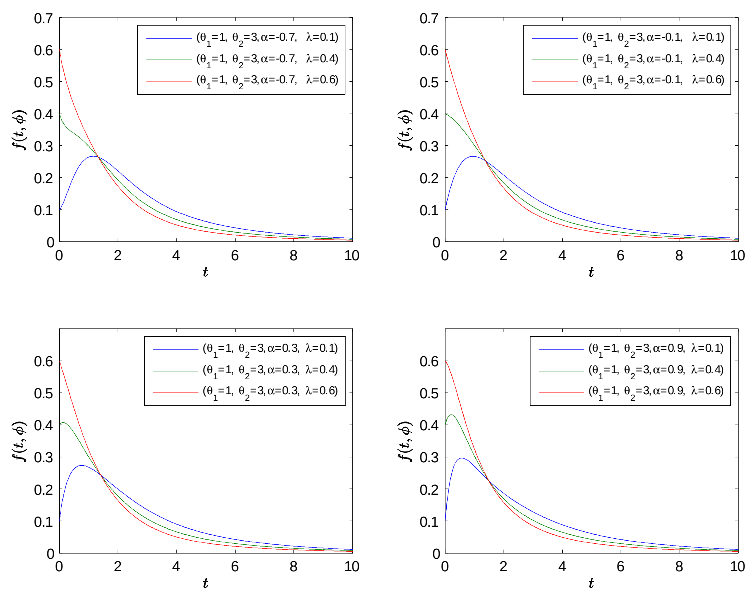

FIGURE 2. Shapes of the probability density function with respect to some values of switching probabilities

The peaks of the probability density functions are flattened while probability of switching decreases (*i.e.* $\lambda\uparrow$). These begin to look like the exponential distribution. In addition, shape of the probability density function is quite different according to whether the mean life of a spare part is also be larger or smaller than the main part.

2.1. **Moment generating function.** The moment generating function of T_{sw} is obtained as

$$\begin{aligned} M_{T_{sw}}(v) &= \frac{1}{1-\theta_1 v} + (1-\lambda)\left\{\left[\frac{1}{1-\theta_2 v} - \frac{1}{1-\frac{\theta_1\theta_2}{\theta_1+\theta_2}v}\right]\right. \\ &\quad \left. + \alpha\left[\frac{1}{1-\frac{\theta_1\theta_2}{\theta_1+2\theta_2}v} - \frac{1}{1-\frac{\theta_1\theta_2}{\theta_1+\theta_2}v} + \frac{1}{1-\frac{\theta_1\theta_2}{2\theta_1+\theta_2}v} - \frac{1}{1-\frac{\theta_1\theta_2}{2\theta_1+2\theta_2}v}\right]\right\}, \end{aligned}$$

where $v < \min\left\{\frac{1}{\theta_1}, \frac{1}{\theta_2}\right\}$. Let $\mu_k = E\left[T_{sw}^k\right]$ denote the kth raw moment then

$$\mu_k = k!\left[\theta_1^k + (1-\lambda)\left\{\begin{array}{c} \left[\theta_2^k - \left(\frac{\theta_1\theta_2}{\theta_1+\theta_2}\right)^k\right] \\ +\alpha\left[\left(\frac{\theta_1\theta_2}{\theta_1+2\theta_2}\right)^k - \left(1+\frac{1}{2^k}\right)\left(\frac{\theta_1\theta_2}{\theta_1+\theta_2}\right)^k + \left(\frac{\theta_1\theta_2}{2\theta_1+\theta_2}\right)^k\right] \end{array}\right\}\right].$$

2.2. **Methods for random number generation.** We will examine two cases according to the sign of the association parameter α.

Case1. $-1 \le \alpha \le 0$

Let's rewrite the distribution function given with (2.1) in the following form;

$$\begin{aligned} F(t) &= \lambda F_1(t) + (1-\lambda)\left\{F_1(t)F_2(t)\left[1+\alpha-\alpha+\alpha\left(1-F_1(t)\right)\left(1-F_2(t)\right)\right]\right\} \\ &= \lambda F_1(t) + (1-\lambda)(1+\alpha)F_1(t)F_2(t) \\ &\quad + (1-\lambda)(-\alpha)F_1(t)F_2(t)\left[1-\left(1-F_1(t)\right)\left(1-F_2(t)\right)\right] \end{aligned}$$

Then $F(t)$ can be represented as a mixture of three distributions. Accordingly, component weights respectively are $\omega_1 = \lambda$, $\omega_2 = (1-\lambda)(1+\alpha)$, $\omega_3 = (1-\lambda)(-\alpha)$ with $\omega_1+\omega_2+\omega_3 = 1$ $(\omega_i \ge 0)$. Consequently, the component distributions are;

$$G_1(t) = F_1(t) = \Pr\left(T_1 \le t\right),$$

$$G_2(t) = F_1(t)F_2(t) = \Pr_{\alpha=0}\left(\max\left\{T_1, T_2\right\} \le t\right),$$

$$\begin{aligned} G_3(t) &= F_1(t)F_2(t)\left[1-S_1(t)S_2(t)\right] \\ &= \Pr_{\alpha=0}\left(\max\left\{T_1,T_2\right\} \le t\right)\Pr_{\alpha=0}\left(\min\left\{T_1,T_2\right\} \le t\right), \end{aligned}$$

where the notation $\Pr_{\alpha=0}(\bullet)$ represents the case of independence of T_1 and T_2.

Case2. $0 < \alpha \le 1$

The distribution function given with (2.1), can be rewritten as

$$\begin{aligned}F(t) &= \lambda F_1(t) + (1-\lambda)\left\{F_1(t)F_2(t)\left[1-\alpha+\alpha+\alpha\left(1-F_1(t)\right)\left(1-F_2(t)\right)\right]\right\}\\ &= \lambda F_1(t) + (1-\lambda)(1-\alpha)F_1(t)F_2(t)\\ &\quad + \alpha(1-\lambda)F_1(t)F_2(t)\left[1+\left(1-F_1(t)\right)\left(1-F_2(t)\right)\right].\end{aligned}$$

Then we can see that $F(t)$ can be represented by a mixture of three distributions such that the component weights are $\omega_1 = \lambda$, $\omega_2 = (1-\lambda)(1-\alpha)$, $\omega_3 = \alpha(1-\lambda)$ with $\omega_1+\omega_2+\omega_3 = 1$ $(\omega_i \geq 0)$. Accordingly, component distribution functions are given by

$$G_1(t) = F_1(t) = \Pr(T_1 \leq t),$$
$$G_2(t) = F_1(t)F_2(t) = \Pr_{\alpha=0}(\max\{T_1, T_2\} \leq t),$$
$$G_3(t) = F_1(t)F_2(t)\left[1+S_1(t)S_2(t)\right] = \Pr_{\alpha=1}(\max\{T_1, T_2\} \leq t).$$

Whereby, the following further steps to generate a random number from lifetime distribution of the system are given.

step1. Input parameter values $\theta_1, \theta_2, \alpha, \lambda$

step2. Generate a random number u from uniform distribution on $(0,1)$

step3. If $\alpha \leq 0$, then go to **step4** otherwise go to **step5**;

step4.

- If $u \leq \lambda$, then $F_1(t) = u \Rightarrow t = -\theta_1 \log(1-u)$,

else

- If $u \leq \lambda + (1-\lambda)(1+\alpha)$, then $G_2(t) = u \Rightarrow t = G_2^{-1}(u)$. The following calculations can be followed to the solution of the equation: Let $\delta = 1 - e^{\frac{-t}{\theta_1}}$ then an appropriate solution for $\delta \in [0,1]$ can be obtained by the equation $\delta\left(1-(1-\delta)^{\frac{\theta_1}{\theta_2}}\right) = u$. Hence $t = -\theta_1 \log(1-\delta)$,

else

- Solve the equation $G_3(t) = u \Rightarrow t = G_3^{-1}(u)$. This equation can be solved with simple additional regulations such that by letting $\delta = 1 - e^{\frac{-t}{\theta_1}}$, then numerically solve the equation $\delta\left(1-(1-\delta)^{\frac{\theta_1}{\theta_2}}\right)\left[1-(1-\delta)(1-\delta)^{\frac{\theta_1}{\theta_2}}\right] = u$. Hence $t = -\theta_1 \log(1-\delta)$.

step5.

- If $u \leq \lambda$, then $F_1(t) = u \Rightarrow t = -\theta_1 \log(1-u)$,

else

- If $u \leq \lambda + (1-\lambda)(1-\alpha)$, then solve $G_2(t) = u \Rightarrow t = G_2^{-1}(u)$
  i.e. solve $\delta\left(1-(1-\delta)^{\frac{\theta_1}{\theta_2}}\right) = u$ then $t = -\theta_1 \log(1-\delta)$,

else

- Solve the equation $G_3(t) = u \Rightarrow t = G_3^{-1}(u)$,
  i.e. solve $\delta\left(1-(1-\delta)^{\frac{\theta_1}{\theta_2}}\right)\left[1+(1-\delta)(1-\delta)^{\frac{\theta_1}{\theta_2}}\right] = u$
  then $t = -\theta_1 \log(1-\delta)$.

Detailed information about a number generation by inverse method, and a number generation from mixed distribution, can be found in Gentle (2004), [2]

2.3. **Parameter estimations by maximum likelihood.** Let $t_1, t_2, ..., t_n$ be the observed lifetimes of size n from the system. Then the log-likelihood function is given by

$$\begin{aligned}\log L(\theta_1,\theta_2,\alpha,\lambda;\underline{\mathbf{t}}) &= \sum_{i=1}^{n}\log\begin{pmatrix}\lambda f_1(t_i;\theta_1)+(1-\lambda) f_1(t_i;\theta_1) f_2(t_i;\theta_2)\\ \times[1+\alpha(2F_1(t_i;\theta_1)-1)(2F_2(t_i;\theta_2)-1)]\end{pmatrix}\\ &= \sum_{i=1}^{n}\log(f_1(t_i;\theta_1))\\ &\quad+\sum_{i=1}^{n}\log\begin{pmatrix}\lambda+(1-\lambda) f_2(t_i;\theta_2)\\ \times[1+\alpha(2F_1(t_i;\theta_1)-1)(2F_2(t_i;\theta_2)-1)]\end{pmatrix}\end{aligned} \tag{2.4}$$

By differentiating (2.4) with respect to $(\theta_1, \theta_2, \alpha, \lambda)$ then we have

$$\begin{aligned}\frac{\partial \log L}{\partial \theta_1} &= \sum_{i=1}^{n}\frac{\partial}{\partial\theta_1}\log(f_1(t_i;\theta_1))\\ &\quad+2(1-\lambda)\alpha\\ &\quad\times\sum_{i=1}^{n}\frac{\frac{\partial F_1(t_i;\theta_1)}{\partial\theta_1} f_2(t_i;\theta_2)(2F_2(t_i;\theta_2)-1)}{(\lambda+(1-\lambda) f_2(t_i;\theta_2)[1+\alpha(2F_1(t_i;\theta_1)-1)(2F_2(t_i;\theta_2)-1)])}\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \theta_2} &= (1-\lambda)\\ &\quad\times\sum_{i=1}^{n}\frac{\frac{\partial f_2(t_i;\theta_2)}{\partial\theta_2}[1+\alpha(2F_1(t_i;\theta_1)-1)(2F_2(t_i;\theta_2)-1)]}{(\lambda+(1-\lambda) f_2(t_i;\theta_2)[1+\alpha(2F_1(t_i;\theta_1)-1)(2F_2(t_i;\theta_2)-1)])}\\ &\quad+(1-\lambda)\\ &\quad\times\sum_{i=1}^{n}\frac{2\alpha f_2(t_i;\theta_2)\frac{\partial F_2(t_i;\theta_2)}{\partial\theta_2}(2F_1(t_i;\theta_1)-1)}{(\lambda+(1-\lambda) f_2(t_i;\theta_2)[1+\alpha(2F_1(t_i;\theta_1)-1)(2F_2(t_i;\theta_2)-1)])}\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha} &= (1-\lambda)\\ &\quad\times\sum_{i=1}^{n}\frac{f_2(t_i;\theta_2)[(2F_1(t_i;\theta_1)-1)(2F_2(t_i;\theta_2)-1)]}{(\lambda+(1-\lambda) f_2(t_i;\theta_2)[1+\alpha(2F_1(t_i;\theta_1)-1)(2F_2(t_i;\theta_2)-1)])}\end{aligned}$$

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n}\frac{1-f_2(t_i;\theta_2)[1+\alpha(2F_1(t_i;\theta_1)-1)(2F_2(t_i;\theta_2)-1)]}{(\lambda+(1-\lambda) f_2(t_i;\theta_2)[1+\alpha(2F_1(t_i;\theta_1)-1)(2F_2(t_i;\theta_2)-1)])}$$

By equating above system of equations to zero, then we obtain the maximum likelihood estimates $\widehat{\phi} = \left(\widehat{\theta_1}, \widehat{\theta_2}, \hat{\alpha}, \hat{\lambda}\right)$ by solving numerically this nonlinear system of equations.

2.4. **Estimating by EM algorithm.** The p.d.f of T_{sw} can be represented by a mixture of two p.d.fs as the following form:

$$f_{T_{sw}}(t;\theta_1,\theta_2,\alpha,w_1,w_2) = w_1 f_1(t;\theta_1) + w_2 f_{12}(t;\theta_1,\theta_2,\alpha), \quad w_1 + w_2 = 1$$

where f_1 stands for p.d.f of $Exp(\theta_1)$and f_{12} stands for p.d.f of (T_1, T_2). We use Lagrange multipliers to solve a constrained maximization problem.

$$\begin{aligned} & \log L\left(\theta_1,\theta_2,\alpha,w_1,w_2;\underline{\mathbf{t}}\right) \\ = & \sum_{i=1}^{n} \log\left(w_1 f_1\left(t_i;\theta_1\right) + w_2 f_{12}\left(t_i;\theta_1,\theta_2,\alpha\right)\right) - \varepsilon\left(w_1 + w_2 - 1\right) \end{aligned} \tag{2.5}$$

Straightforwardly, we get the system of equations, by differentiating (2.5) with respect to (ϕ, w_1, w_2) and equating it to zero, as follows:

$$\begin{aligned} \frac{\partial \log L}{\partial \theta_1} &= \sum_{i=1}^{n} \frac{\left(w_1 \frac{\partial}{\partial \theta_1} f_1(t_i;\theta_1) + w_2 \frac{\partial}{\partial \theta_1} f_{12}(t_i;\theta_1,\theta_2,\alpha)\right)}{(w_1 f_1(t_i;\theta_1) + w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha))} \\ &= \sum_{i=1}^{n} \frac{w_1 f_1(t_i;\theta_1) \frac{\partial}{\partial \theta_1} \log(f_1(t_i;\theta_1)) + w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha) \frac{\partial}{\partial \theta_1} \log(f_{12}(t_i;\theta_1,\theta_2,\alpha))}{(w_1 f_1(t_i;\theta_1) + w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha))} \\ &= \sum_{i=1}^{n} \Pr\left(1|\, t_i\right) \frac{\partial}{\partial \theta_1} \log\left(f_1\left(t_i;\theta_1\right)\right) \\ &+ \sum_{i=1}^{n} \Pr\left(2|\, t_i\right) \frac{\partial}{\partial \theta_1} \log\left(f_{12}\left(t_i;\theta_1,\theta_2,\alpha\right)\right) = 0 \end{aligned} \tag{2.6}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \theta_2} &= \sum_{i=1}^{n} \frac{\left(w_2 \frac{\partial}{\partial \theta_2} f_{12}(t_i;\theta_1,\theta_2,\alpha)\right)}{(w_1 f_1(t_i;\theta_1) + w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha))} \\ &= \sum_{i=1}^{n} \frac{w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha) \frac{\partial}{\partial \theta_2} (\log f_{12}(t_i;\theta_1,\theta_2,\alpha))}{(w_1 f_1(t_i;\theta_1) + w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha))} \\ &= \sum_{i=1}^{n} \Pr\left(2|\, t_i\right) \frac{\partial}{\partial \theta_2} \left(\log f_{12}\left(t_i;\theta_1,\theta_2,\alpha\right)\right) = 0 \end{aligned} \tag{2.7}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha} &= \sum_{i=1}^{n} \frac{\left(w_2 \frac{\partial}{\partial \alpha} f_{12}(t_i;\theta_1,\theta_2,\alpha)\right)}{(w_1 f_1(t_i;\theta_1) + w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha))} \\ &= \sum_{i=1}^{n} \frac{w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha) \frac{\partial}{\partial \alpha} \log(f_{12}(t_i;\theta_1,\theta_2,\alpha))}{(w_1 f_1(t_i;\theta_1) + w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha))} \\ &= \sum_{i=1}^{n} \Pr\left(2|\, t_i\right) \frac{\partial}{\partial \alpha} \log\left(f_{12}\left(t_i;\theta_1,\theta_2,\alpha\right)\right) = 0 \end{aligned} \tag{2.8}$$

$$\frac{\partial \log L}{\partial w_1} = \sum_{i=1}^{n} \frac{f_1\left(t_i;\theta_1\right)}{\left(w_1 f_1\left(t_i;\theta_1\right) + w_2 f_{12}\left(t_i;\theta_1,\theta_2,\alpha\right)\right)} - \varepsilon = 0 \tag{2.9}$$

$$\frac{\partial \log L}{\partial w_2} = \sum_{i=1}^{n} \frac{f_{12}\left(t_i;\theta_1,\theta_2,\alpha\right)}{\left(w_1 f_1\left(t_i;\theta_1\right) + w_2 f_{12}\left(t_i;\theta_1,\theta_2,\alpha\right)\right)} - \varepsilon = 0 \tag{2.10}$$

If both sides of the last two equations are multiplied by w_1 and w_2, respectively and by taking summation of both terms, then we have

$$\sum_{i=1}^{n}\frac{w_1 f_1(t_i;\theta_1)}{(w_1 f_1(t_i;\theta_1)+w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha))}+\sum_{i=1}^{n}\frac{w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha)}{(w_1 f_1(t_i;\theta_1)+w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha))}-\varepsilon\left(w_1+w_2\right)=0$$
$$=\sum_{i=1}^{n}\frac{w_1 f_1(t_i;\theta_1)+w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha)}{(w_1 f_1(t_i;\theta_1)+w_2 f_{12}(t_i;\theta_1,\theta_2,\alpha))}-\varepsilon=0$$

This implies $\varepsilon = n$. Hereby, equation (2.9) or (2.10) yields EM estimates of λ as $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n}\Pr\left(1|\, t_i\right)$. If initial values $\theta_1^0, \theta_2^0, \alpha^0, \lambda^0$ are given, then $\Pr\left(1|\, x_i\right)$ and $\Pr\left(2|\, x_i\right)$ are calculated. Consequently, solving the equations (2.6)-(2.8) numerically gives updated values of parameters. This step is repeated iteratively until convergence is detected.

Everitt and Hand (1981), [1] may be seen as a source for further reading about using EM algorithm for mixed distributions.

3. Reliability Properties

In this subsection we introduce the reliability function, the hazard rate function and the mean residual life function for this switching system.

3.1. **Reliability function.** Let $S_i(t) = 1 - F_i(t)$ stand for the survival function of i.th component lifetime then the survival function of T_{sw} is given by

$$S(t) = \lambda S_1(t) + (1-\lambda)\left\{1 - F_1(t)F_2(t)\left[1 + \alpha S_1(t)S_2(t)\right]\right\}$$

Since the marginal survival functions are $e^{-\frac{t}{\theta_1}}$ and $e^{-\frac{t}{\theta_2}}$ we have

$$\begin{aligned} S(t) &= \lambda e^{-\frac{t}{\theta_1}} + (1-\lambda)\left\{1-\left(1-e^{-\frac{t}{\theta_1}}\right)\left(1-e^{-\frac{t}{\theta_2}}\right)\left[1+\alpha\left(e^{-\frac{t}{\theta_1}}\right)\left(e^{-\frac{t}{\theta_2}}\right)\right]\right\} \\ &= e^{-\frac{t}{\theta_1}} + (1-\lambda)\, e^{-\frac{t}{\theta_2}}\left(1-e^{-\frac{t}{\theta_1}}\right) \\ &\quad -\alpha\left(1-\lambda\right)e^{-t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)}\left\{\left(1-e^{-\frac{t}{\theta_1}}\right)\left(1-e^{-\frac{t}{\theta_2}}\right)\right\} \end{aligned} \tag{3.1}$$

It can be concluded that when two components are negatively associated, high switching probability can raise the survival probability of the system.

Even if mean lifetime of the spare part is short, these still work with higher probability when they are connected together in parallel. According to these conditions, it will also be attractive for us to examine the behavior of the system's failure rate function.

3.2. **Hazard rate function.** The failure or hazard rate function is defined by $r(t) = -\frac{d}{dt}\log(S(t)) = \frac{f(t)}{S(t)}$. Accordingly, from the expressions (2.3) and (3.1), hazard rate function of T_{sw} is given by

$$r(t) = \frac{1}{\theta_1}\Bigg[1 + (1-\lambda)\frac{1}{\theta_2} \times \frac{\left\{(\theta_1-\theta_2)e^{t\left(\frac{1}{\theta_1}\right)} - \theta_1\right\} + \alpha\left\{2\theta_1 e^{-t\left(\frac{1}{\theta_2}\right)} - \theta_1 + (\theta_1+\theta_2)e^{-t\left(\frac{1}{\theta_1}\right)} - (2\theta_1+\theta_2)e^{-t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)}\right\}}{e^{t\left(\frac{1}{\theta_2}\right)} + (1-\lambda)\left[\left(e^{\frac{t}{\theta_1}} - 1\right) + \alpha\left(e^{-t\left(\frac{1}{\theta_2}\right)} - 1 + e^{-t\left(\frac{1}{\theta_1}\right)} - e^{-t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)}\right)\right]}\Bigg] \tag{3.2}$$

The hazard rate of the system exhibits flexibility according to the switch transition probability λ and association parameter α. In this context, our intent to examine the reliability properties such as failure rate and mean residual life of the system, and obtain some orderings.

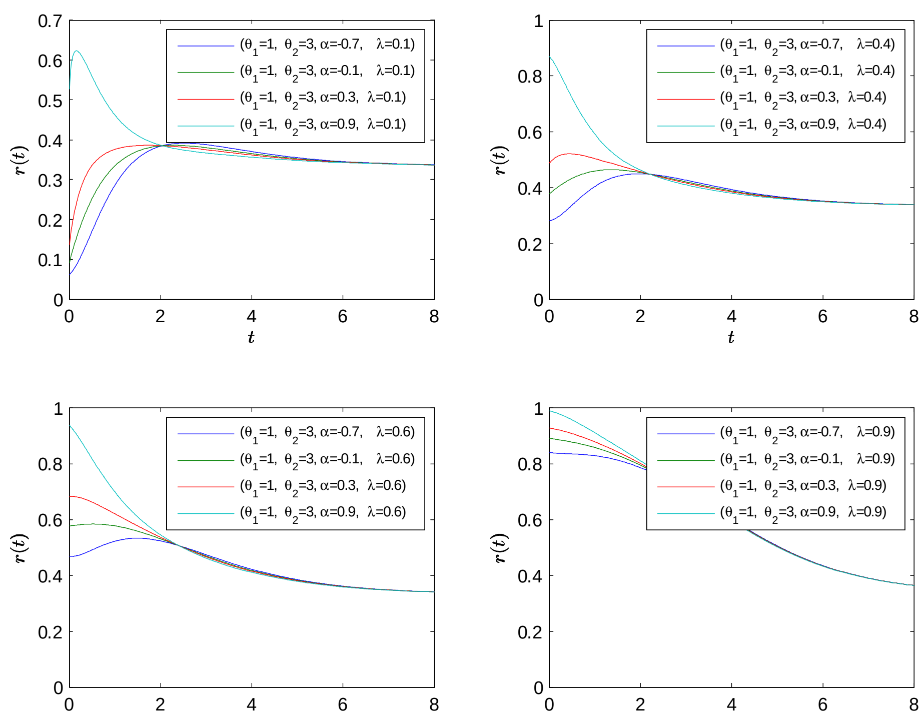

FIGURE 3. Shapes of the hazard rate function with respect to some values of association parameter.

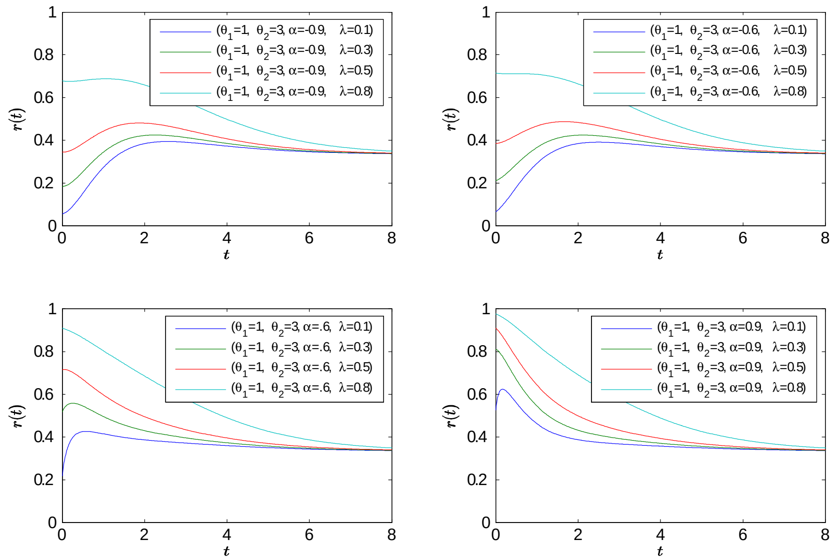

FIGURE 4. Shapes of the hazard rate function with respect to some values of switching probability.

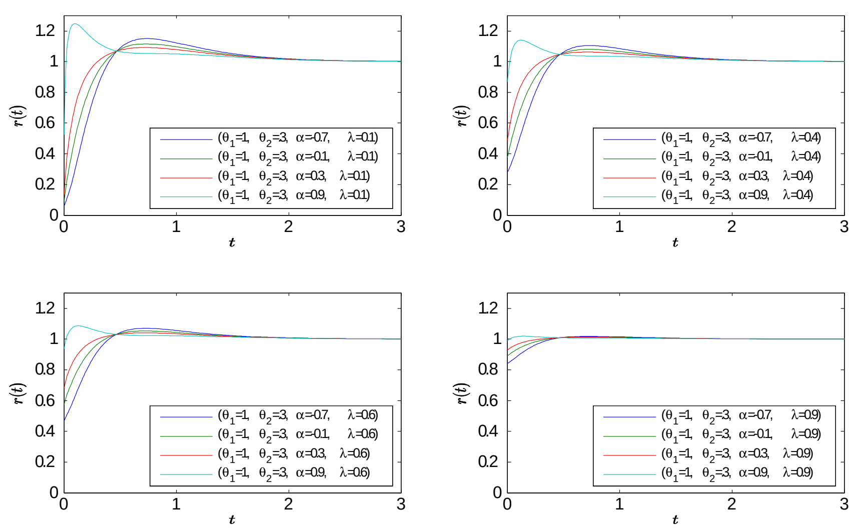

FIGURE 5. Shapes of the hazard rate function with respect to some values of association parameter (mean lifetime of second component is less than main component).

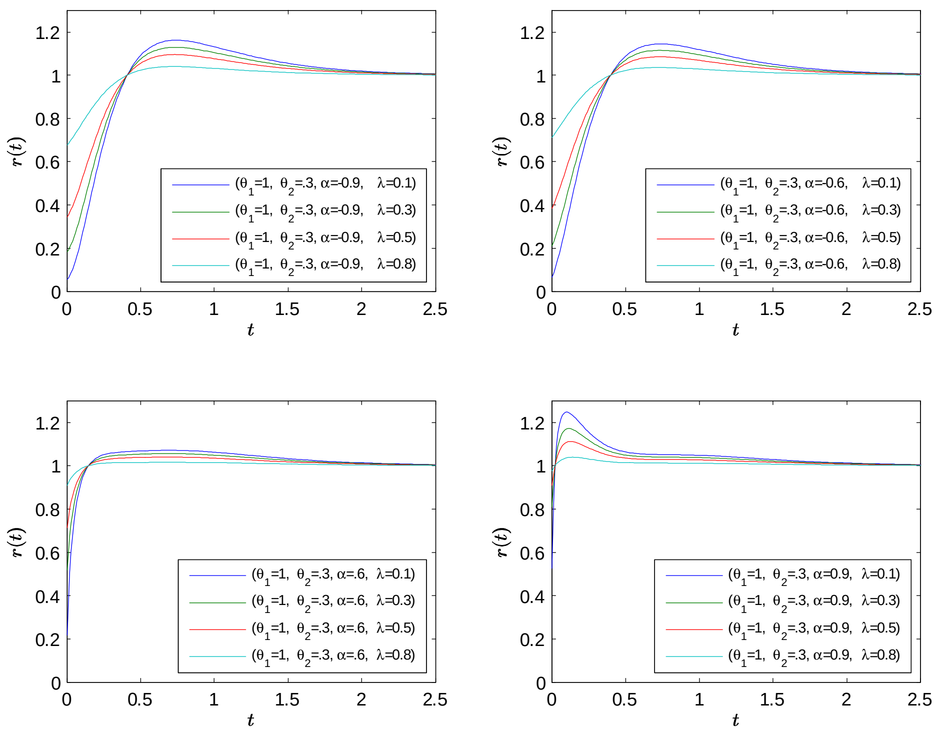

FIGURE 6. Shapes of the hazard rate function with respect to some values of switching probability (mean lifetime of second component is less than main component).

3.3. **Mean residual life function.** The mean residual life of the certain part of age x is defined as the expected value of remaining life of the part (Lai and Xie 2006, [4]). Hence,

$$\mu(x) = E(T_{sw} - x | T_{sw} > x) = \frac{\int_x^\infty S_{T_{sw}}(t)dt}{S_{T_{sw}}(x)}$$

$$= \frac{\theta_1 + (1-\lambda)\frac{\theta_1\theta_2}{\theta_1+\theta_2} e^{\frac{-x}{\theta_2}} \left[\left(\frac{\theta_1+\theta_2}{\theta_1} e^{\frac{x}{\theta_1}} - 1 \right) - \alpha \left\{ 1 - \frac{\theta_1+\theta_2}{2\theta_1+\theta_2} e^{\frac{-x}{\theta_2}} - \frac{\theta_1+\theta_2}{\theta_1+2\theta_2} e^{\frac{-x}{\theta_1}} + \frac{e^{-x\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)}}{2} \right\} \right]}{1 + (1-\lambda) e^{\frac{-x}{\theta_2}} \left[\left(e^{\frac{x}{\theta_1}} - 1 \right) - \alpha \left\{ 1 - e^{\frac{-x}{\theta_2}} - e^{\frac{-x}{\theta_1}} + e^{-x\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)} \right\} \right]}. \tag{3.3}$$

4. Some Orderings

Throughout this section, we will assume that component lifetimes are identical i.e. $\theta_1 = \theta_2$. Stochastic and hazard rate orderings are investigated based on association parameter α and switching probability λ.

4.1. **Stochastic ordering.** Stochastic relationship will be investigated according to the monotonicity of α or λ. First, let's look at the definition of stochastic ordering;

Definition 1: Let X and Y be two random variables defined on the same support, then X is said to be stochastically smaller than Y, denoted by $X \prec_{st} Y)$ if $\Pr(X > x) \leq \Pr(Y > x)$ holds $\forall x \in (-\infty, \infty)$ (Shaked and Shanthikumar 2007, [6]).

According to this definition, survival function of T_{sw} is rewritten below by taking $e^{-\frac{t}{\theta}} = u$ for the simplicity;

$$S(u; \alpha, \lambda) = u\left[1 + (1-\lambda)(1-u)\{1 - \alpha u(1-u)\}\right] \tag{4.1}$$

Firstly, we consider the case of $\alpha < \alpha'$. It can be easily seen that, regardless of the sign of α, $S_{\alpha'}(u) \leq S_\alpha(u)$ holds for all u. Hence a stochastic relationship $T_{sw}^{(\alpha';\lambda)} \prec_{st} T_{sw}^{(\alpha;\lambda)}$ exists for $\alpha < \alpha'$. According to existing relationship, we can say that the lifetime of the system composed of two identical but negatively associated components regardless of the switching probability is longer.

Secondly, we consider the case of $\lambda < \lambda'$. If someone thinks about visual representation of the system, then it will be seen immediately that an increment in switching probability gets a longer lifetime of the system. It is obvious from the statement (4.1) that $S_{\lambda'}(u) \leq S_\lambda(u)$ holds. Hence $T_{sw}^{(\alpha;\lambda')} \prec_{st} T_{sw}^{(\alpha;\lambda)}$ holds. If the components are connected to parallel with regardless of the association parameter, then they extend the lifetime of the system.

Now, we will investigate a relationship when both the association parameter and the switching probability have a simultaneous increment. Namely, we consider the case of $\alpha < \alpha'$ and $\lambda < \lambda'$. Let's consider the ratio $\frac{S_{\alpha',\lambda'}(u) - S_{\alpha,\lambda}(u)}{u(1-u)}$. Then

$$\frac{S_{\alpha',\lambda'}(u) - S_{\alpha,\lambda}(u)}{u(1-u)} = -\left(\lambda' - \lambda\right) + u(1-u)\left[\alpha(1-\lambda) - \alpha'\left(1-\lambda'\right)\right].$$

By noting that $u(1-u) \leq 1$ holds, then we conclude that

$$\begin{aligned} -\left(\lambda' - \lambda\right) + \alpha(1-\lambda) - \alpha'\left(1-\lambda'\right) &= (1-\alpha')\left(1-\lambda'\right) - (1-\alpha)(1-\lambda) \\ &\leq (1-\alpha')(1-\lambda) - (1-\alpha)(1-\lambda) \\ &= -(1-\lambda)[\alpha' - \alpha] \leq 0 \end{aligned}$$

namely, the sign of this ratio is negative. Hence $S_{\alpha',\lambda'}(u) \leq S_{\alpha,\lambda}(u)$ i.e. $T_{sw}^{(\alpha';\lambda')} \prec_{st} T_{sw}^{(\alpha;\lambda)}$ holds.

4.2. **Hazard rate orderings.**

Definition 2: Let X and Y be two random variables defined on the same support of x, respectively r_X and r_Y denote the hazard rates. If $r_Y(x) \leq r_X(x)$ holds for all $x \in (-\infty, \infty)$, then X is said to be smaller than Y in the hazard rate order, and this relationship is denoted by $X \prec_{hr} Y$. (Shaked and Shanthikumar 2007, [6]).
As seen from these figures (3-6), ordering may exist only for switching probability. However, in terms of being misleading we will first be investigated a relationship for $\alpha < \alpha'$. By letting $e^{-\frac{t}{\theta}} = u$ in statement (3.2), then we have

$$r(u) = \frac{1}{\theta} - \frac{1}{\theta}(1-\lambda)u\left[\frac{1+\alpha(1-u)(1-3u)}{(2-\lambda)-(1-\lambda)u\left[1+\alpha(1-u)^2\right]}\right]. \tag{4.2}$$

We decide the monotonicity of (4.2) by taking first derivative of $r(u)$ with respect to α. Hence

$$\begin{aligned}\frac{d}{d\alpha}r(u) &= \frac{-1}{\theta}\frac{(1-\lambda)u(1-u)}{\left[(2-\lambda)-(1-\lambda)u\left[1+\alpha(1-u)^2\right]\right]^2} \\ &\quad \times\left[2(1-\lambda)u^2-3(2-\lambda)u+(2-\lambda)\right].\end{aligned}$$

The last multiplier in the statement above is a convex function of u. Furthermore, its value is $2-\lambda > 0$ for $u = 0$ and -2 for $u = 1$. Thus, one of the roots of a quadratic polynomial should be in the range $(0, 1)$. The roots of this polynomial respectively are

$$u_{1,2} = \frac{3(2-\lambda) \pm \sqrt{(2-\lambda)(10-\lambda)}}{4(1-\lambda)}.$$

Now, $u_1$will be checked whether it is in $[0, 1]$. Positivity of u_1 is obvious from the statement below

$$9(2-\lambda)^2-(2-\lambda)(10-\lambda) = (2-\lambda)[9(2-\lambda)-(10-\lambda)] = 8(2-\lambda)(1-\lambda) \geq 0.$$

It will be checked whether it is less than 1. For this, positivity of the following statement is

$$3(2-\lambda)-4(1-\lambda)-\sqrt{(2-\lambda)(10-\lambda)}$$

which implies

$$(2+\lambda)^2-(2-\lambda)(10-\lambda) = -16(1-\lambda) \leq 0.$$

In this case, the sign of derivative changes its direction at least once. Therefore, the hazard rate ordering is not valid according to association parameter. Now, we will investigate the existence of the relationship for $\lambda < \lambda'$. By rearranging $r(u)$ as below:

$$r(u) = \frac{1}{\theta} - \frac{1}{\theta}u\left[\frac{1+\alpha(1-u)(1-3u)}{\frac{(2-\lambda)}{(1-\lambda)}-u\left[1+\alpha(1-u)^2\right]}\right],$$

then statement $(2-\lambda)/(1-\lambda) = 1 + 1/(1-\lambda)$ in brackets increases in λ. $r(u)$ also increases in λ as long as nominator in brackets is positive. $(1-u)(1-3u)$ is a convex function and it equals 1 when $u = 0$, it is equals 0 when $u = 1$. The minimum value of this convex function is $\frac{-1}{3}$ which is attained at $\frac{2}{3}$. According to this, since $\alpha(1-u)(1-3u) \geq \frac{-\alpha}{3}$ holds for $\alpha \geq 0$, $1 - \frac{\alpha}{3} \geq 0$ is valid. On the other hand, $(1-u)(1-3u) \leq 1$ implies $1 + \alpha(1-u)(1-3u) \geq 1 + \alpha \geq 0$ holds for $\alpha < 0$. In this case, we obtain $r_{\lambda}(u) \leq r_{\lambda'}(u)$ for $\lambda < \lambda'$ which implies that $T_{sw}^{(\alpha;\lambda')} \prec_{hr} T_{sw}^{(\alpha;\lambda)}$ is valid.
Whatever the association parameter is higher switching probability makes the system more preferable in terms of the hazard rate.

5. Applications

In this section, we want to illustrate the usefulness of the model by using two real data sets.
Data Set 1. This data set includes customer waiting times and considered as grouped data by Shanker et.al (2013) [7]. They have proposed two-parameter Lindley distribution to fit waiting times (in minutes) of 100 bank customers in the queue. They have calculated Chi-Square Statistics for both Lindley and two-parameter Lindley distributions, which respectively are 0.09402 and 0.07482. Similarly, we apply goodness-of-fit to this grouped data by considering our model.
As it is thought to be fiction; the system is running with one booth attendant, but occasionally, another one also serves the customer to help.
Chi-Square goodness of fit test results and the expected frequencies can be obtained as follows:

TABLE 1. Frequency table of waiting times of 100 bank customers.

Class intervals	Frequency	Expected Frequency	Parameter Estimates	Chi-Square Statistics
[0,5)	30	30.00	$\hat{\theta}_1 = 7.1089$	$\chi^2 = 0.000878528$
[5,10)	32	31.96	$\hat{\theta}_2 = 6.2398$	$s.d(d.f) = 1$
[10,15)	19	19.09	$\hat{\alpha} = 0.0000$	
[15, 20)	10	9.94	$\hat{\lambda} = 0.0886$	
[20,25)	5	4.92		
[25,30)	1	2.39		
[30, 35)	2	1.15		
[35,40)	1	0.55		

The main booth attendant serves the customers with an average of 7 minutes, and the other one serves with an-average of 6 minutes. Also, it said that, they were working together in general along the days but independently. According to Lindley types, quite small chi-square statistics were obtained with this model.
Data Set 2. The two data sets in Table 2 represent the survival times (in days) of two groups of patients suffering from head and neck cancer disease. Group 1 was treated using radiotherapy (RT), whereas the patients belonging to Group 2 were

treated using a combined radiotherapy and chemotherapy (CT). These data set are taken from Sharma et. al (2015), [8].

TABLE 2. Survival times of patients (RT,RT+CT)

Group I (RT)														
6.53	7	10.42	14.48	16.1	22.7	34	41.55	42	45.28	49.4	53.62	63	64	83
84	91	108	112	129	133	133	139	140	140	146	149	154	157	160
160	165	146	149	154	157	160	160	165	173	176	218	225	241	248
273	277	297	405	417	420	440	523	583	594	1101	1146	1417		
Group II (RT+CT)														
23.74	25.87	31.98	37	41.35	47.38	55.46	68.46	78.26	74.47	81.43	84	92	94	119
127	130	133	140	146	155	179	194	195	209	249	281	319	469	519
633	725	817	1776											

We merge the survival times of the patients belonging to Group 1 and 2. We apply the model to fit single data set. Maximum Likelihood Estimates and Kolmogorov-Smirnov statistic for the suggested model parameters are tabulated as follows:

TABLE 3. Estimates of the parameters and Kolmogorov-Smirnov Statistic (K-S)

$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	K-S
181	0.93	0.79	0.0951

Most of patients had only been applied only treatment, the effect of the combined treatment is particularly effective in the positive direction.

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