Article Electronically published on January 12, 2017

Commun.Fac.Sci.Univ.Ank.Series A1
Volume 66, Number 2, Pages 1–13 (2017)
DOI: 10.1501/Commua1_0000000796
ISSN 1303–5991
http://communications.science.ankara.edu.tr/index.php?series=A1

A NEW SUBCLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE AND FIXED SECOND COEFFICIENTS DEFINED BY THE RAFID-OPERATOR

ARZU AKGÜL

Abstract. The aim of the present paper is to introduce a new subclass of meromorphic functions with positive and fixed second coefficients by means of Rafid-operator by fixing second coefficient. We give a necessary and sufficient condition for a function f to be in this class. Also we obtain coefficient inequality, distortion properties, meromorphically radii of close-to-convexity, starlikeness and convexity, extreme points, convex linear combinations, for the functions f in this class.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad n \in \mathbb{N} = \{1, 2, 3, ...\} \tag{1.1}$$

which are analytic in the punctured unit disc

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} - \{0\}.$$

Analytically a function $f \in \Sigma$ given by (1.1) is said to be meromorphically starlike of order α if it satisfies the following

$$R\left(-\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U})$$

for some $\alpha(0 \leq \alpha < 1)$. We say that f is in the class $\sum^*(\alpha)$ of such functions. Similarly a function $f \in \Sigma$ given by (1.1) is said to be meromorphically convex of

Received by the editors: September 09, 2016; Accepted: October 10, 2016 .
2010 *Mathematics Subject Classification.* Primary 30C45.
Key words and phrases. Meromorphic functions, positive coefficients, coefficient inequality, convex linear combination,meromorphically starlikeness, convexity, close-to-convexity and extreme points .

order α if it satisfies the following:

$$R\left\{-\left(1+\frac{zf^{''}(z)}{f'(z)}\right)\right\}>\alpha,\ (z\in\mathbb{U})$$

for some $\alpha(0\leq\alpha<1)$. We say that f is in the class $\sum_c(\alpha)$ of such functions. For a function $f\in\Sigma$ given by (1.1) is said to be meromorphically close-to-convex of order β and type α if there exists a function $g\in\sum^*(\alpha)$ such that

$$R\left(-\frac{zf^{'}(z)}{g(z)}\right)>\beta,\quad (0\leq\alpha<1,\ 0\leq\beta<1,\ z\in\mathbb{U}).$$

We say that f is in the class $K(\beta,\alpha)$.

The class $\sum^*(\alpha)$ and varius other subclasses of Σ have been studied rather extensively by J.Clunie [7], J. E. Miller [12], Ch. Pommerenke [13], W. C. Royster [15]. See also P. L. Duren [8](pages 29-137) and H. M. Srivastava, S. Owa [17] (pages 86-429), Akgül [1], Akgül [2] and Akgül and Bulut [3]. Recent years, many authors investigated the subclass of meromorphic functions with positive coefficient. In 1985, Junea and Reddy [10] introduced the class of $\sum_p$ functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n,\quad a_n\geq 0 \tag{1.2}$$

which are regular and univalent in $\mathbb{U}$. The functions in this class are said to be meromorphic functions with positive coefficient. In [4], Athsan and Buti introduced Rafid-operator for analytic functions and T. Rosy and S. Sunil Varma [16] modified their operator to meromorphic functions as follows.

Lemma 1 ([16])**.** *For $f\in\sum$ given by(1.1), $0\leq\mu<1$ and $0\leq\gamma\leq1$, if the operator $S_\mu^\gamma:\sum\longrightarrow\sum$ is defined by*

$$S_\mu^\gamma f(z)=\frac{1}{(1-\mu)^{\gamma+1}\Gamma(\gamma+1)}\int_0^\infty t^{\gamma+1}e^{-\left(\frac{t}{1-\mu}\right)}f\left(zt\right)dt, \tag{1.3}$$

then

$$S_\mu^\gamma f(z)=z+\sum_{n=2}^{\infty}L(n,\mu,\gamma)a_nz^n \tag{1.4}$$

where $L(n,\mu,\gamma)=(1-\mu)^{n+1}\frac{\Gamma(n+\gamma+2)}{\Gamma(\gamma+1)}$ and Γ is the familiar Gamma function. Using the equation (1.4), it is easily seen that

$$S_\mu^\gamma\left(zf^{'}(z)\right)=z\left(S_\mu^\gamma f(z)\right)^{'}. \tag{1.5}$$

We defined the subclass $\sum_p S(\alpha,\lambda,\mu,q,\zeta)$ of Σ_p for meromorphic functions with positive coefficient associated with the integral operator $S_\mu^\gamma f(z)$ and investigated the certain properties of this class.

Definition 1. *A function $f \in \sum$ is said to be in the class$\sum S(\alpha,\lambda,\mu,q,\zeta)$ if and only if satisfies the inequality:*

$$\Re\left\{\frac{-z\left(\Phi(z)\right)'}{\Phi(z)}\right\} \geq q\left|\frac{z\left(\Phi(z)\right)'}{\Phi(z)}+1\right|+\zeta \tag{1.6}$$

where $0 \leq \mu < 1$, $0 \leq \zeta < 1$, $0 \leq \alpha \leq \lambda < \frac{1}{2}$, $q \geq 0$ *and*

$$\Phi(z) = \lambda\alpha z^2 (S_\mu^\gamma f(z))'' + (\lambda-\alpha)\, z(S_\mu^\gamma f(z))' + (1-\lambda+\alpha)\, S_\mu^\theta f(z). \tag{1.7}$$

It is easily shown that there is following equality between these subclasses

$$\sum_p S(\alpha,\lambda,\mu,q,\zeta) = \sum S(\alpha,\lambda,\mu,q,\zeta) \cap \sum_p .$$

Theorem 1. *A meromorphic function f defined by the equation (1.2) in the class $\sum_p S(\alpha,\lambda,\mu,\gamma,\zeta,q)$ if and only if*

$$\sum_{n=1}^{\infty}[(n+\zeta)+q(n+1)]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]L(n,\mu,\gamma)a_n \leq (1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right). \tag{1.8}$$

for some $0 \leq \zeta < 1$, $\beta > 0$, $0 \leq \mu < 1$, $0 \leq \gamma \leq 1$, $0 \leq \alpha \leq \lambda < \frac{1}{2}$ *and* $q \geq 0$.

In view of (1.8), we can see that the functions $f(z)$ defined by (1.2) in the class $\sum_p S(\alpha,\lambda,\mu,q,\zeta)$ satisfy the coefficient inequality

$$L(1,\mu,\gamma)a_1 \leq \frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)}{(2q+\zeta+1)}. \tag{1.9}$$

Hence we may take

$$L(1,\mu,\gamma)a_1 = \frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)}{(2q+\zeta+1)}c, \quad 0<c<1. \tag{1.10}$$

Making use of equation (1.10), we now introduce the following class of functions: Let $\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$ denote the subclass of $\sum_p S(\alpha,\lambda,\mu,q,\zeta)$ consisting of functions of the form

$$f(z) = \frac{1}{z} + \frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)c}{(2q+\zeta+1)}z + \sum_{n=1}^{\infty} L(n,\mu,\gamma)a_n z^n, \tag{1.11}$$

where

$$a_n \geq 0 \text{ and } \; 0<c<1.$$

In this paper, coefficient estimates, extreme points, growth and distortion bounds, radii of meromorphically starlikeness, convexity and close-to-convexity are obtained for the class $\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$ by fixing the second coefficient. Further, it is shown that the class $\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$ is closed under convex linear combination. Techniques used are similar to those of Aouf and Darwish [5], Aouf and Josi [6], Ghanim and Darus [9] and Ureagaldi [18].

2. Coefficient Bounds

Theorem 2. *Let the function defined by the equality (1.11). Then the function $f(z)$ is in the class $\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$ if and only if*

$$\begin{aligned} &\sum_{n=2}^{\infty}[(n+\zeta)+q(n+1)]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]L(n,\mu,\gamma)a_n \\ \leq\ & (1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)(1-c). \end{aligned} \tag{2.1}$$

The result is sharp.

Proof. By putting in the inequality(1.8)

$$L(1,\mu,\gamma)a_1=\frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)c}{(2q+\zeta+1)},\quad 0<c<1$$

the result is easily obtained. The result is sharp for the function

$$\begin{aligned} f(z) &= \frac{1}{z}+\frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)c}{(2q+\zeta+1)}z \\ &+ \frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)(1-c)}{[(n+\zeta)+q(n+1)]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)+1\right]}z^n,\quad n\geq 2. \end{aligned} \tag{2.2}$$

□

Corollary 1. *Let the function f defined by the equation (1.11) be in the class $\sum_p(\alpha,\lambda,\mu,q,\zeta,c)$. Then*

$$a_n\leq\frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)(1-c)}{[(n+\zeta)+q(n+1)]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)+1\right]L(n,\mu,\gamma)},\quad n\geq 2. \tag{2.3}$$

The result is sharp for the function given by the equation (1.2)

Corollary 2. *If $0<c_1<c_2<1$, then*

$$\sum_p S(0,\lambda,\mu,q,\zeta,c_2)\subset\sum_p S(0,\lambda,\mu,q,\zeta,c_1).$$

3. Distortion Bounds

In this section, we obtain growth and distortion bounds for the class $\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$.

Theorem 3. *If the function $f \in \sum_p$ given by the equation (1.11) is in the class $\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$ for $0<|z|=r<1$, then one has*

$$\begin{aligned}\frac{1}{r}-\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}r \\ -\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{(3q+\zeta+1)(2\alpha\lambda-\lambda+\alpha+1)}r^2 \leq |f(z)| \\ \leq \frac{1}{r}+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}r \\ +\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{(3q+\zeta+1)(2\alpha\lambda-\lambda+\alpha+1)}r^2 \qquad (3.1)\end{aligned}$$

and the result is sharp for the function f given by

$$\begin{aligned}f(z) &= \frac{1}{z}+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}z \\ &+ \frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{[(n+\zeta)+q(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)+1]}z^n, \quad n\geq 2. \qquad (3.2)\end{aligned}$$

Proof. Since $f \in \sum_p(\alpha,\lambda,\mu,q,\zeta,c)$ in view of Theorem 2, yields

$$L(n,\mu,\gamma)a_n \leq \frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{[(n+\zeta)+q(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)+1]}, \quad n\geq 2$$

and we have

$$\begin{aligned}(3q+\zeta+1)(2\alpha\lambda-\lambda+\alpha+1)\sum_{n=2}^{\infty}L(n,\mu,\gamma)a_n \\ \leq \sum_{n=1}^{\infty}[(n+\zeta)+q(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)+1]L(n,\mu,\gamma)a_n \\ \leq (1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)\end{aligned}$$

which gives

$$\sum_{n=2}^{\infty}L(n,\mu,\gamma)a_n \leq \frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{(3q+\zeta+1)(2\alpha\lambda-\lambda+\alpha+1)} \qquad (3.3)$$

Thus, for $0<|z|=r<1$,

$$\begin{aligned}|f(z)| &\leq \left|\frac{1}{z}\right|+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}|z|+\sum_{n=2}^{\infty}L(n,\mu,\gamma)a_n|z|^n\\ &\leq \frac{1}{r}+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}r+r^2\sum_{n=2}^{\infty}L(n,\mu,\gamma)a_n\\ &\leq \frac{1}{r}+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}r\\ &+ \frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{(3q+\zeta+1)(2\alpha\lambda-\lambda+\alpha+1)}r^2. \end{aligned}\tag{3.4}$$

Similarly, we obtain

$$\begin{aligned}|f(z)| &\geq \frac{1}{r}-\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}r\\ &- \frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{(3q+\zeta+1)(2\alpha\lambda-\lambda+\alpha+1)}r^2\end{aligned}\tag{3.5}$$

Combining the inequalities (3.4) and (3.5) we get desired result and the result is sharp for the function given by the equation (3.2). □

Theorem 4. *If the function $f\in\sum_p$ given by the equation(1.11) is in the class $\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$ for $0<|z|=r<1$, then one has*

$$\begin{aligned}&\frac{1}{r^2}-\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}\\ &\quad-\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{(3q+\zeta+1)(2\alpha\lambda-\lambda+\alpha+1)}r\leq\left|f'(z)\right|\\ &\quad\leq\frac{1}{r^2}+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}\\ &\quad\quad+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{(3q+\zeta+1)(2\alpha\lambda-\lambda+\alpha+1)}r\end{aligned}\tag{3.6}$$

for $0<|z|=r<1$ and the result is sharp for the function f given by

$$f(z)=\frac{1}{z}+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}z+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{(3q+\zeta+1)(2\alpha\lambda-\lambda+\alpha+1)}z^2.$$

Proof. In view of Theorem 2, it follows that

$$nL(n,\mu,\gamma)a_n\leq\frac{n(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{[(n+\zeta)+q(n+1)][(n-1)(n\lambda\alpha+\lambda-\alpha)+1]},\quad n\geq 2.\tag{3.7}$$

Thus, for $0 < |z| = r < 1$ and making use of (3.7), we obtain

$$\begin{aligned}\left|f^{'}(z)\right| &\leq \left|-\frac{1}{z^2}\right| + \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)\,c}{(2q+\zeta+1)} + \sum_{n=2}^{\infty} nL(n,\mu,\gamma)a_n |z|^{n-1} \\ &\leq \frac{1}{r^2} + \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)\,c}{(2q+\zeta+1)} + r\sum_{n=2}^{\infty} nL(n,\mu,\gamma)a_n \\ &\leq \frac{1}{r^2} + \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)\,c}{(2q+\zeta+1)} \\ &+ \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)(1-c)}{(3q+\zeta+1)(2\alpha\lambda - \lambda + \alpha + 1)} r \end{aligned} \tag{3.8}$$

and similarly,

$$\begin{aligned}\left|f^{'}(z)\right| &\geq \left|-\frac{1}{z^2}\right| - \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)\,c}{(2q+\zeta+1)} - \sum_{n=2}^{\infty} nL(n,\mu,\gamma)a_n |z|^{n-1} \\ &\geq \frac{1}{r^2} - \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)\,c}{(2q+\zeta+1)} - r\sum_{n=2}^{\infty} nL(n,\mu,\gamma)a_n \\ &\geq \frac{1}{r^2} - \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)\,c}{(2q+\zeta+1)} \\ &- \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)(1-c)}{(3q+\zeta+1)(2\alpha\lambda - \lambda + \alpha + 1)} r \end{aligned} \tag{3.9}$$

Combining the inequalities (3.8)and (3.9), we get desired result and the result is sharp. □

4. Convex Linear Combination

In this section, we shall prove the class $\sum_p S(\alpha, \lambda, \mu, q, \zeta, c)$ is closed under convex linear combination.

Theorem 5. *The class $\sum_p S(\alpha, \lambda, \mu, q, \zeta, c)$ is closed under convex linear combination.*

Proof. Let the functions f is given by(1.11) and the function g be given by

$$g(z) = \frac{1}{z} + \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)\,c}{(2q+\zeta+1)} z + \sum_{n=2}^{\infty} L(n,\mu,\gamma)\,|b_n|\, z^n,$$

where $b \geq 0$, $n \geq 2$, $0 < c < 1$ are in the class $\sum_p S(\alpha, \lambda, \mu, q, \zeta, c)$. Then by Theorem 2, we have

$$\sum_{n=2}^{\infty} [(n+\zeta) + q(n+1)]\,[(n-1)(n\lambda\alpha + \lambda - \alpha)]\, L(n,\mu,\gamma)a_n \\ \leq (1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)(1-c)$$

and

$$\sum_{n=2}^{\infty}[(n+\zeta)+q(n+1)]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]L(n,\mu,\gamma)\left|b_n\right|$$
$$\leq (1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)(1-c).$$

Assuming that f and g in the class$\sum_p S(\alpha,\lambda,q,\zeta,c)$, it is enough to prove that the function h defined by

$$h(z)=\tau f(z)+(1-\tau)g(z), \quad 0\leq\tau\leq 1, \tag{4.1}$$

is also in the class $\sum_p S(\alpha,\lambda,q,\zeta,c)$. Since

$$\begin{aligned} h(z) &= \frac{1}{z}+\frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)c}{(2q+\zeta+1)}z \\ &+ \sum_{n=1}^{\infty}L(n,\mu,\gamma)\left|\tau a_n+(1-\tau)b_n z^n\right|, \end{aligned} \tag{4.2}$$

we observe that

$$\begin{aligned} &\sum_{n=1}^{\infty}[(n+\zeta)+q(n+1)]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]L(n,\mu,\gamma)\left|\tau a_n+(1-\tau)b_n z^n\right| \\ &\leq (1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)(1-c), \end{aligned} \tag{4.3}$$

So, $h(z)\in\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$. □

5. Extreme Point

Theorem 6. *If*

$$f_1(z)=\frac{1}{z}+\frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)c}{(2q+\zeta+1)}z \tag{5.1}$$

and

$$\begin{aligned} f_n(z) &= \frac{1}{z}+\frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)c}{(2q+\zeta+1)}z \\ &+ \sum_{n=2}^{\infty}\frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)(1-c)}{[(n+\zeta)+q(n+1)]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)+1\right]L(n,\mu,\gamma)}z^n, \end{aligned} \tag{5.2}$$

then $f\in\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$ if and only if it can be represented in the form

$$f(z)=\sum_{n=1}^{\infty}\mu_n f_n(z), \tag{5.3}$$

where $\mu_n\geq 0$ and $\sum_{n=1}^{\infty}\mu_n=1$

Proof. Assume that $f(z)=\sum_{n=1}^{\infty}\mu_n f_n(z)$, $(\mu_n\geq 0,\ \sum_{n=1}^{\infty}\mu_n=1)$. Then, from equalities (5.1),(5.2) and (5.3), we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}\mu_n f_n(z) \\ &= \mu_1 f_1(z)+\sum_{n=2}^{\infty}\mu_n f_n(z) \\ &= \frac{1}{z}+\frac{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,c}{(2q+\zeta+1)}z \\ &\quad +\sum_{n=2}^{\infty}\frac{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,(1-c)}{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\mu,\gamma)}\mu_n z^n. \end{aligned}$$

Since

$$\sum_{n=2}^{\infty}\frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\mu,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,(1-c)} \times\frac{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,(1-c)}{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\mu,\gamma)}\mu_n$$
$$\sum_{n=2}^{\infty}\mu_n=1-\mu_1\leq 1,$$

it follows from Theorem 2 that $f\in\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$.
Conversely, suppose that $f\in\sum_p S(\alpha,\lambda,\mu,q,\zeta,c)$. Since

$$a_n\leq\frac{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,(1-c)}{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)+1]\,L(n,\mu,\gamma)},\quad n\geq 2,$$

if we set

$$\mu_n=\frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\mu,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,(1-c)}a_n,$$

and

$$\mu_1=1-\sum_{n=2}^{\infty}\mu_n,$$

then we obtain

$$f(z)=\sum_{n=1}^{\infty}\mu_n f_n(z).$$

This completes the proof of the theorem. □

6. Radii of Starlikeness and Convexity

In this section, we find the radii of meromorphically close-to-convexity, starlikeness and convexity for functions f in the class $\sum_p S(\alpha,\lambda,\mu,q,\zeta)$.

Theorem 7. *Let the function defined by (1.11) be in the class* $\sum_p S(\alpha,\lambda,q,\zeta,c)$. *Then* f *is meromorphically starlike of order* β $(0\leq\beta<1)$ *in the disk* $|z|<r_1(\alpha,\lambda,q,\zeta,c,\beta)$, *where* $r_1(\alpha,\lambda,q,\zeta,c,\beta)$ *is the largest value for which*

$$\frac{(3-\beta)(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)\,c}{(2q+\zeta+1)}r^2 + \frac{(n+2-\beta)(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{[(n+\zeta)+q(n+1)]\,[(n-1)(n\lambda\alpha+\lambda-\alpha)+1]}r^{n+1}\leq(1-\beta),\qquad(n\geq 2).$$

The result is sharp for the extremal function f *given by the equation(2.2)*

Proof. It is sufficient to prove that

$$\left|z\frac{f^{'}(z)}{f(z)}+1\right|\leq 1-\beta,\quad |z|<r_1. \tag{6.1}$$

Note that

$$\begin{aligned}\left|z\frac{f^{'}(z)}{f(z)}+1\right| &= \left|\frac{\frac{2(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}z+\sum_{n=2}^{\infty}L(n,\mu,\gamma)a_n(n+1)z^n}{\frac{1}{z}+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}z+\sum_{n=2}^{\infty}L(n,\mu,\gamma)a_nz^n}\right| \\ &= \left|\frac{\frac{2(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}z^2+\sum_{n=2}^{\infty}L(n,\mu,\gamma)a_n(n+1)z^{n+1}}{1+\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)c}{(2q+\zeta+1)}z^2+\sum_{n=2}^{\infty}L(n,\mu,\gamma)a_nz^{n+1}}\right| \\ &\leq 1-\beta\end{aligned}$$

for $|z|<r_1(\alpha,\lambda,q,\zeta,c)$ if and only if

$$\frac{(3-\beta)(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)\,c}{(2q+\zeta+1)}r^2 + \sum_{n=2}^{\infty}L(n,\mu,\gamma)a_n(n+2-\beta)\,r^{n+1}\leq 1-\beta \tag{6.2}$$

from the inequality (2.3) we may take

$$a_n=\frac{(1-\zeta)(2\alpha\lambda-2\lambda+2\alpha+1)(1-c)}{[(n+\zeta)+q(n+1)]\,[(n-1)(n\lambda\alpha+\lambda-\alpha)+1]\,L(n,\mu,\gamma)}\lambda_n,\quad n\geq 2 \tag{6.3}$$

where $\lambda_n \geq 0$ $(n \geq 2)$ and

$$\sum_{n=2}^{\infty} \lambda_n \leq 1.$$

For each fixed r, we choose the positive integer $n_0 = n_0(r)$ for which

$$\frac{(n+2-\beta)}{[(n+\zeta)+q(n+1)]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)+1\right]} L(n,\mu,\gamma) r^{n+1}$$

is maximal. Then it follows that

$$\sum_{n=2}^{\infty} (n+2-\beta)\, L(n,\mu,\gamma) a_n r^{n+1}$$
$$\leq \frac{(n_0+2-\beta)\,(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,(1-c)}{[(n_0+\zeta)+q(n+1)]\left[(n_0-1)\left(n_0\lambda\alpha+\lambda-\alpha\right)+1\right]} r^{n_0+1}. \tag{6.4}$$

Then f is starlike of order β in $|z| < r_1(\alpha,\lambda,q,\zeta,c)$ provided that

$$\frac{(3-\beta)\,(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,c}{(2q+\zeta+1)} r^2$$
$$+\frac{(n_0+2-\beta)\,(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,(1-c)}{[(n_0+\zeta)+q(n_0+1)]\left[(n_0-1)\left(n_0\lambda\alpha+\lambda-\alpha\right)+1\right]} r^{n_0+1} \leq \ (1-\beta)\,. \tag{6.5}$$

We find the value $r_0 = r_0(\alpha,\lambda,q,\zeta,c)$ and the corresponding integer $n_0(r_0)$ so that

$$\frac{(3-\beta)\,(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,c}{(2q+\zeta+1)} r^2$$
$$+\frac{(n_0+2-\beta)\,(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,(1-c)}{[(n_0+\zeta)+q(n_0+1)]\left[(n_0-1)\left(n_0\lambda\alpha+\lambda-\alpha\right)+1\right]} r^{n_0+1} = (1-\beta)\,. \tag{6.6}$$

Then this value r_0 is the radius of meromorphically starlike of order β for functions belonging to the class $\sum_p S(\alpha,\lambda,q,\zeta,c)$. □

Theorem 8. *Let the function defined by the equation (1.11) be in the class* $\sum_p S(\alpha,\lambda,q,\zeta,c)$*. Then* f *is meromorphically convex of order* β $(0 \leq \beta < 1)$ *in the disk* $|z| < r_2(\alpha,\lambda,q,\zeta,c,\beta)$*, where* $r_2(\alpha,\lambda,q,\zeta,c,\beta)$ *is the largest value for which*

$$\frac{(3-\beta)\,(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,c}{(2q+\zeta+1)} r^2$$
$$+\frac{n(n+2-\beta)\,(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,(1-c)}{[(n+\zeta)+q(n+1)]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)+1\right]} r^{n+1} \leq \ (1-\beta)\,, \qquad (n \geq 2)\,.$$

The result is sharp for the extremal function f *given by(2.2).*

Proof. By using the technique employed in the proof of Theorem 7 we can show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \leq 1-\beta,$$

for $|z| < r_2(\alpha, \lambda, q, \zeta, c)$, and prove that the assertion of the theorem is true and the result is sharp. □

Theorem 9. *Let the function defined by (1.11) be in the class* $\sum_p S(\alpha, \lambda, q, \zeta, c)$. *Then* f *is meromorphically close-to-convex of order* β $(0 \leq \beta < 1)$ *in the disk* $|z| < r_3(\alpha, \lambda, q, \zeta, c, \beta)$, *where* $r_3(\alpha, \lambda, q, \zeta, c, \beta)$ *is the largest value for which*

$$\frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)\,c}{(2q + \zeta + 1)} r^2 + \frac{n(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)(1-c)}{[(n+\zeta) + q(n+1)]\,[(n-1)(n\lambda\alpha + \lambda - \alpha) + 1]} r^{n+1} \leq (1-\beta), \qquad (n \geq 2)$$

and the result is sharp.

Proof. Let $f \in \sum_p(\alpha, \lambda, q, \zeta)$. By using the technique employed in the proof of Theorem 7, we can show that

$$\left| z^2 f'(z) + 1 \right| \leq 1 - \beta. \tag{6.7}$$

for $|z| < r_3(\alpha, \lambda, q, \zeta, c)$, and prove that the assertion of the theorem is true and the result is sharp. □

References

[1] Akgül, A., *A new subclass of meromorphic functions defined by Hilbert space operator*, Honam Math. J. **38(3)**,(2016), 495-506.

[2] Akgül, A. *A subclass of meromorphic functions defined by a certain integral opetor on Hilbert space* , Creat. Math. Inform. (Accepted),(2016).

[3] Akgül, A. and Bulut, S., *On a certain subclass of meromorphic functions defined by Hilbert space operator*, Acta Univ. Apulensis Math. Inform **45**,(2016), 1-9.

[4] Athsan, W. G. and Buti, R. H., *Fractional Calculus of a class of univalent functions with negative coefficients defined by Hadamard product with Rafid-Operator*, Eur. J. Pure Appl. Math. **4(2)**,(2011), 162-173.

[5] Aouf, M. K. and Darwish, H. E., *On certain subclass of meromorphically starlike functions with positive coefficients*, Turkish J. Math., **21(3)** (1997), 311-316.

[6] Aouf, M. K. and Joshi, S. B., *On certain subclass of meromorphically starlike functions with positive coefficients*, Soochow jornal of Mathematics (1998), **24(2)** (1998), 79-90.

[7] Clunie, J., *On meromorphic schilict functions*, J. London Math.Soc. **34** (1959), 115-216.

[8] Duren,P.L., *Univalent Functions*, Springer, New York,NY, USA,(1983).

[9] Ghanim, F. and Darus, M., *On class of hypergeometric functions with fixed second positive coefficients*, General Mathematics **17(4)** (2009), 13-28.

[10] Junea, O. P. and Reddy, T. R., *Meromorphic starlike univalent functions with positive coefficients*, Ann. Univ. Mariae Curie Sklodowska, Sect. A. **39** (1985), 65-76.

[11] Lashin, A. Y., *On certain subclasses of meromorphic functions associated with certain integral operators*, Comput.Math.Appl. **59(1)**,(2010), 524-531.

[12] Miller, J. E., *Convex meromorphic mapping and related functions*, Proc.Amer.Math.Soc.**25**,(1970),220-228.

[13] Pommerenke, Ch., *On meromorphic starlike functions*, Pac.J.Math. **13**,(1963),221-235.

[14] Ram Reddy, T.,Sharma, R. B. and Saroja, K., *A new subclass of meromorphic functions with positive coefficients*, Indian J. Pure Appl. Math. **44(1)**, (2013), 29-26.

[15] Royster, W. C., *Meromorphic starlike univalent functions*, Trans. Amer. Math. Soc. **107**,(1963), 300-308.
[16] Rosy, T. and Varma, S., Geometry, *On a subclass of meromorphic functions defined by Hilbert space operator*, Vol.(2013), Article ID 671826, 4 pages,
[17] Srivastava, H. M. and Owa, S., *Current topics in analytic function theory*, World Scientific Publishing, New Jersey, (1992).
[18] Ureagaldi, B. A., *Meromorphically starlike functions with positive and fixed second coefficients*, Kyungpook Math. J. **29(1)** (1998),64-68.

Current address, Arzu AKGÜL: Kocaeli University, Department of Mathematics, Faculty of Science and Arts, Umuttepe Campus, 41380, İzmit-Kocaeli, TURKEY.
E-mail address: akgul@kocaeli.edu.tr