ON THE INVARIANTS OF RULED SURFACES GENERATED BY THE DUAL INVOLUTE FRENET TRIHEDRON

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ABSTRACT. The purpose of this paper is to describe ruled surfaces generated by a Frenet trihedron of closed dual involute for a given dual curve. We identify relations between the pitch, the angle of the pitch, and the drall of these surfaces. Some new results related to the developability of these surfaces are also obtained. Finally, we illustrate these surfaces by presenting one example.

1. INTRODUCTION

Dual numbers were originally conceived by Clifford in 1873 [13] as a tool for his geometrical investigations. However, their first application to mechanics was conceived by Study in 1901 [4]. Study used dual numbers and vectors in his research on the geometry of lines and kinematics. Since that time, there has been considerable research on dual numbers [2, 5, 8]. In recent years, dual numbers have been used to study the motion of a line in space. The pitches and the angle of pitches of closed ruled surfaces are very important in studying the geometry of lines. In the literature, the integral invariants of closed ruled surfaces corresponding to the parameter of dual spherical curves and oriented lines have been studied by several authors [1, 3, 6, 7, 9 − 12, 14].

In this paper, we investigate the ruled surfaces generated by a Frenet trihedron of closed dual involute for a given dual curve by a firmly connected dual angle between the dual binormal vector and dual Darboux vector of this dual base curve. We then identify the relations between the pitch, the angle of the pitch, and the drall of these surfaces. Some new results related to the developability of these surfaces are also obtained.

2. PRELIMINARIES

A dual number has the form $a + \varepsilon a^*$, where a and a^* are real numbers and $\varepsilon^2 = 0$. The set of all dual numbers, denoted by ID, is an associative ring with the unit

element 1. A dual vector is a triple of dual numbers. Hence, if $\overrightarrow{A}$ is a dual vector, we may write $\overrightarrow{A} = \overrightarrow{a} + \varepsilon\vec{a}^*$, where $\vec{a}$ and $\vec{a}^* \in E^3$ and ε is the dual unit introduced above. The set of all dual vectors is called the dual space, and is denoted by ID^3. The set ID^3 is a $ID - module$ over the ring ID.

The inner product of two dual vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is defined as

$$\left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \varepsilon \left(\left\langle \overrightarrow{a}, \overrightarrow{b^*} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b} \right\rangle \right). \tag{2.1}$$

The cross-product of two dual vectors $\overrightarrow{A}, \overrightarrow{B} \in ID^3$ is given by

$$\overrightarrow{A} \wedge \overrightarrow{B} = \overrightarrow{a} \wedge \overrightarrow{b} + \varepsilon \left(a \wedge b^* + a^* \wedge b \right). \tag{2.2}$$

For $\overrightarrow{A} \neq (0, a^*) \in ID - module$, the norm $\left\| \overrightarrow{A} \right\|$ of $\overrightarrow{A}$ is defined by

$$\left\| \overrightarrow{A} \right\| = \sqrt{\left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle} = \left\| \overrightarrow{a} \right\| + \varepsilon \frac{\left\langle \overrightarrow{a}, \overrightarrow{a^*} \right\rangle}{\left\| \overrightarrow{a} \right\|}. \tag{2.3}$$

Let Φ be the dual angle between the unit dual vectors $\overrightarrow{A}$ and $\overrightarrow{B}$. Then,

$$\left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle = \cos\Phi = \cos\varphi - \varepsilon\varphi^* \cos\varphi, \tag{2.4}$$

where $\Phi = \varphi + \varepsilon\varphi^*$, $0 \leq \varphi \leq \pi$, and $\varphi^* \in \mathbb{R}$ is a dual number. Here, the real numbers φ and φ^* are the angle and the minimal distance, respectively, between the two oriented lines $\overrightarrow{A}$ and $\overrightarrow{B}$. The geometric region satisfying the equality $\left\| \overrightarrow{A} \right\| = (1, 0)$, where $\overrightarrow{A} \neq (0, a^*)$, is called a dual unit sphere in the $ID - module$.

Study [4] established a theorem that states "there is a one-to-one mapping between the dual points of a dual unit sphere and the oriented lines in $\mathbb{R}^3$". According to this theorem, the unit dual vector $\overrightarrow{A} = \overrightarrow{a} + \varepsilon\overrightarrow{a^*}$ corresponds to only one oriented line in $\mathbb{R}^3$, where the real part $\overrightarrow{a}$ shows the direction of this line and the dual part $\overrightarrow{a^*}$ shows the vectorial moment of the unit vector $\overrightarrow{a}$ with respect to the origin. A differentiable curve on the dual unit sphere represents a differentiable family of straight lines in $\mathbb{R}^3$, which is called the ruled surface. This correspondence allows us to study the properties of ruled surfaces within the geometry of dual spherical curves on a dual unit sphere. A differentiable closed curve on the dual unit sphere represents a closed ruled surface in $\mathbb{R}^3$. Thus, in this paper, all of the curves we discuss are closed.

Now, suppose that $V : I \subset \mathbb{R} \longrightarrow ID^3$, $s \to V(s) = V_1(s)$ is a differentiable unit speed curve in the dual unit sphere. We denote the closed ruled surfaces generated by this curve as $[V_1]$. Let $\{V_1, V_2, V_3\}$ be the moving dual Frenet frame of the curve $V = V_1$, with

$$V_1 = V, \ V_2 = \frac{V'}{\|V'\|}, \ V_3 = V_1 \times V_2.$$

Now, take $V(s)$ as a closed curve with curvature $\kappa(s) = k_1 + \varepsilon k_1^*$ and torsion $\tau(s) = k_2 + \varepsilon k_2^*$. Then, the dual Frenet formulas may be expressed as [12] :

$$\begin{bmatrix} V_1'(s) \\ V_2'(s) \\ V_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix}. \tag{2.5}$$

In the unit dual spherical motion K/K', the dual Frenet frame $\{V_1, V_2, V_3\}$ performs a dual rotational motion around the instantaneous dual Pfaff vector. This vector is determined with the equation

$$\Psi = \tau V_1 + \kappa V_3. \tag{2.6}$$

Definition 1. *Let* $\Psi = \psi + \varepsilon \psi^*$ *be the dual rotation vector (instantaneous dual Pfaff vector) of the dual spherical motion. The vector*

$$D = d + \varepsilon d^* = \oint \Psi \tag{2.7}$$

is called the dual Steiner vector of this dual spherical motion [6].

Theorem 1. *The dual angle of a closed ruled surface constructed by the dual unit vector* V_1 *is given by* [10] :

$$\Lambda_{V_1} = -\langle D, V_1 \rangle. \tag{2.8}$$

Theorem 2. *The dual angle of the pitch of a closed ruled surface can be given by:*

$$\Lambda_{V_1} = \lambda_{V_1} - \varepsilon L_{V_1}, \tag{2.9}$$

where L_{V_1} *and* λ_{V_1} *are the pitch and the angle, respectively, of a closed ruled surface* [10].

3. DUAL INVOLUTE FRENET TRIHEDRON

Definition 2. *Let* $\hat{\alpha} : I \to ID^3$ *and* $\hat{\beta} : I \to ID^3$ *be dual unit speed curves. If the tangent lines of the dual curve* $\hat{\alpha}$ *are orthogonal to the tangent lines of the dual curve* $\hat{\beta}$, *then the dual curve* $\hat{\beta}$ *is said to be involute of the dual curve* $\hat{\alpha}$; *equivalently, the dual curve* $\hat{\alpha}$ *is said to be evolute of the dual curve* $\hat{\beta}$. *According to this definition,*

$$\langle V_1, R_1 \rangle = 0, \tag{3.1}$$

where V_1 *is the tangent of the dual curve* $\hat{\alpha}$ *and* R_1 *is the tangent of the dual curve* $\hat{\beta}$.

Theorem 3. *Let* $\hat{\alpha}$, $\hat{\beta}$ *be two dual curves. If* $\hat{\beta}$ *is involute of* $\hat{\alpha}$, *we can write* $\hat{\beta}(s) = \hat{\alpha}(s) + [(c - s) + \varepsilon d] V_1(s), \quad c, d \in \mathbb{R}.$

Corollary 1. *The distance between corresponding dual points of the dual curves* $\hat{\beta}$ *and* $\hat{\alpha}$ *is* $\mu = |c - s| + \varepsilon d$.

Theorem 4. *Let $\hat{\alpha}$, $\hat{\beta}$ be two dual curves. If $\hat{\beta}$ is involute of $\hat{\alpha}$, then the relationship between the dual Frenet vectors of the dual curves $\hat{\alpha}$ and $\hat{\beta}$ can be given by*

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -cos\Phi & 0 & sin\Phi \\ sin\Phi & 0 & cos\Phi \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{3.2}$$

The real and dual parts of R_1, R_2, R_3 are

$$\begin{cases} r_1 = v_2 \\ r_2 = -\cos\varphi v_1 + \sin\varphi v_3 \\ r_3 = \sin\varphi v_1 + \cos\varphi v_3 \\ r_1^* = v_2^* \\ r_2^* = -\cos\varphi v_1^* + \sin\varphi v_3^* + \varphi^* \left(\sin\varphi v_1 + \cos\varphi v_3 \right) \\ r_3^* = \sin\varphi v_1^* + \cos\varphi v_3^* + \varphi^* \left(\cos\varphi v_1 - \sin\varphi v_3 \right) \end{cases}, \tag{3.3}$$

where Φ $\left(\Phi = \varphi + \varepsilon\varphi^* \ , \ \varepsilon^2 = 0 \right)$ is the dual angle between the dual rotation vector Ψ and dual binormal vector V_3. Then, the following relations hold between the vectors R_1, R_2, R_3 and R_1', R_2', R_3':

$$\begin{aligned} R_1' &= PR_2 \\ R_2' &= -PR_1 + QR_3 \ , \\ R_3' &= -QR_2 \end{aligned} \tag{3.4}$$

where $P = p + \varepsilon p^*$ and $Q = q + \varepsilon q^*$ are the curvature and torsion of the involute curve $\hat{\beta}$. If we separate these relations into dual and real parts, we get

$$\begin{cases} r_1' = pr_2 \\ r_2' = -pr_1 + qr_3 \\ r_3' = -qr_2 \\ r_1^{*\prime} = p^* r_2 + pr_2^* \\ r_2^{*\prime} = -p^* r_1 + -pr_1^* + q^* r_3 + qr_3^* \\ r_3^* = -q^* r_2 - qr_2^* \end{cases}. \tag{3.5}$$

Since $P = \frac{\|\hat{\beta}' \wedge \hat{\beta}''\|}{\|\hat{\beta}'\|^3}$, we have that $P = \frac{\sqrt{\kappa^2 + \tau^2}}{\mu\kappa}$. Using the formula $Q = \frac{\det\left(\hat{\beta}', \hat{\beta}'', \hat{\beta}''' \right)}{\|\hat{\beta}' \wedge \hat{\beta}''\|^2}$, we find $Q = \frac{\Phi'}{\mu\kappa}$. Separating P and Q into real and dual parts, we get

$$\begin{cases} p = \frac{\sqrt{k_1^2 + k_2^2}}{\mu k_1} \\ p^* = \frac{k_2 \left(k_1 k_2^* - k_1^* k_2 \right)}{\mu k_1^2 \sqrt{k_1^2 + k_2^2}} \\ q = \frac{\varphi'}{\mu k_1} \\ q^* = \frac{k_1 \varphi^{*\prime} - k_1^* \varphi'}{\mu k_1^2} \end{cases}. \tag{3.6}$$

4. ON THE INVARIANTS OF RULED SURFACES GENERATED BY A DUAL INVOLUTE FRENET TRIHEDRON

In this section, we calculate the integral invariants of the closed ruled surfaces $[R_1]$, $[R_2]$, and $[R_3]$ that are kinematically generated by dual Frenet vectors of the dual involute curve $\hat{\beta}$ given in Eq. (3.2).

In dual unit spherical motion, the dual orthonormal system $\{R_1, R_2, R_3\}$ performs a dual rotational motion around the instantaneous dual Pfaff vector $\bar{\Psi}$ for all $t \in \mathbb{R}$. The dual rotation vector of this motion is determined by:

$$\bar{\Psi} = QR_1 + PR_3. \tag{4.1}$$

The dual Steiner vector relevant to the dual Frenet frame of the dual involute curve is

$$\bar{D} = R_1 \oint Q ds + R_3 \oint P ds. \tag{4.2}$$

If we separate (4.2) into its real and dual parts, we can write

$$\begin{cases} \bar{d} = r_1 \oint q ds + r_3 \oint p ds \\ \bar{d}^* = r_1 \oint q^* ds + r_1^* \oint q ds + r_3 \oint p^* ds + r_3^* \oint p ds \end{cases}. \tag{4.3}$$

We now calculate the integral invariants of the closed ruled surfaces. The dual angle of the pitch of the closed ruled surface $[R_1]$ is given by

$$\Lambda_{R_1} = -\left\langle \bar{D}, R_1 \right\rangle.$$

Using Eq. (4.1), we have

$$\Lambda_{R_1} = -\oint Q ds, \tag{4.4}$$

$$\Lambda_{R_1} = -\oint q ds - \varepsilon \oint q^* ds. \tag{4.5}$$

Substituting the values of q and q^* in (3.6) into Eq. (4.5), we obtain

$$\Lambda_{R_1} = -\oint \frac{\varphi'}{\mu k_1} ds - \varepsilon \oint \frac{k_1 \varphi^{*\prime} - k_1^* \varphi'}{\mu k_1^2} ds. \tag{4.6}$$

Corollary 2. *The angle of pitch and the pitch of the closed ruled surface $[R_1]$ are $\lambda_{R_1} = -\oint \frac{\varphi'}{\mu k_1} ds$ and $L_{R_1} = \oint \frac{k_1 \varphi^{*\prime} - k_1^* \varphi'}{\mu k_1^2} ds$, respectively.*

The drall of the closed ruled surface $[R_1]$ is

$$\Delta_{R_1} = \frac{\langle dr_1, dr_1^* \rangle}{\langle dr_1, dr_1 \rangle},$$

which, using the values of dr_1 and dr_1^* in Eq. (3.5), gives

$$\Delta_{R_1} = \frac{p^*}{p}. \tag{4.7}$$

Corollary 3. *The closed ruled surface* $[R_1]$ *generated by* R_1 *is developable if and only if* $p^* = 0$ *(i.e. the dual curvature is a pure real number).*

Using the values of p and p^* in (3.6) into the last equation, we get

$$\Delta_{R_1} = \frac{k_2 \left(k_1 k_2^* - k_1^* k_2\right)}{k_1 \left(k_1^2 + k_2^2\right)}. \tag{4.8}$$

Theorem 5. *The closed ruled surface* $[R_1]$ *which generated by* R_1 *is developable if and only if* $k_2 = 0$ *or* $\frac{k_2^*}{k_1^*} = \frac{k_2}{k_1}$.

The dual angle of the pitch of the closed ruled surface $[R_2]$ is

$$\Lambda_{R_2} = - \left\langle \bar{D}, R_2 \right\rangle.$$

Thus, using Eq. (4.1), we get

$$\Lambda_{R_2} = 0. \tag{4.9}$$

Corollary 4. *The angle of pitch and the pitch of the closed ruled surface* $[R_2]$ *are* $\lambda_{R_2} = 0$ *and* $L_{R_1} = 0$, *respectively.*

The drall of the closed ruled surface $[R_2]$ is

$$\Delta_{R_2} = \frac{\langle dr_2, dr_2^* \rangle}{\langle dr_2, dr_2 \rangle}.$$

Using the values of dr_2 and dr_2^* given by Eq. (3.5), we have

$$\Delta_{R_2} = \frac{pp^* + qq^*}{p^2 + q^2}. \tag{4.10}$$

Corollary 5. *The closed ruled surface* $[R_2]$ *generated by* R_2 *is developable if and only if* $\frac{q^*}{p^*} = -\frac{p}{q}$.

Substituting the values of p, p^*, q, and q^* in (3.6) into the previous equation, we get

$$\Delta_{R_2} = \frac{k_2 \left(k_1 k_2^* - k_1^* k_2\right) + \varphi' \left(k_1 \varphi^{*\prime} - k_1^* \varphi'\right)}{k_1 \left(k_1^2 + k_2^2 + \varphi'^2\right)}. \tag{4.11}$$

Theorem 6. *The closed ruled surface* $[R_2]$ *generated by* R_2 *is developable if and only if* $k_2 \left(k_1 k_2^* - k_1^* k_2\right) + \varphi' \left(k_1 \varphi^{*\prime} - k_1^* \varphi'\right) = 0$.

The dual angle of the pitch of the closed ruled surface $[R_3]$ is

$$\Lambda_{R_3} = - \left\langle \bar{D}, R_3 \right\rangle.$$

Using Eq. (4.1), we get

$$\Lambda_{R_3} = - \oint P ds, \tag{4.12}$$

$$\Lambda_{R_3} = -\oint p\,ds - \varepsilon \oint p^*\,ds. \tag{4.13}$$

Substituting the values of p and p^* in (3.6) into Eq. (4.13), we have

$$\Lambda_{R_3} = -\oint \frac{\sqrt{k_1^2 + k_2^2}}{\mu k_1}\,ds - \varepsilon \oint \frac{k_2\,(k_1 k_2^* - k_1^* k_2)}{\mu k_1^2 \sqrt{k_1^2 + k_2^2}}\,ds. \tag{4.14}$$

Corollary 6. *The angle of pitch and the pitch of the closed ruled surface* $[R_3]$ *are* $\lambda_{R_3} = -\oint \frac{\sqrt{k_1^2 + k_2^2}}{\mu k_1}\,ds$ *and* $L_{R_1} = \oint \frac{k_2(k_1 k_2^* - k_1^* k_2)}{\mu k_1^2 \sqrt{k_1^2 + k_2^2}}\,ds$, *respectively.*

The drall of the closed ruled surface $[R_3]$ is

$$\Delta_{R_3} = \frac{\langle dr_3, dr_3^* \rangle}{\langle dr_3, dr_3 \rangle},$$

which, using the values of dr_3 and dr_3^* in Eq. (3.5), gives

$$\Delta_{R_3} = \frac{q^*}{q}. \tag{4.15}$$

Corollary 7. *The closed ruled surface* $[R_3]$ *generated by* R_3 *is developable if and only if* $q^* = 0$ *(i.e. the dual torsion is a pure real number).*

Substituting the values of q and q^* in Eq. (3.6) into the previous equation, we obtain

$$\Delta_{R_3} = \frac{k_1 \varphi^{*\prime} - k_1^* \varphi'}{k_1 \varphi'}. \tag{4.16}$$

The closed ruled surface $[R_3]$ *generated by* R_3 *is developable if and only if* $\frac{k_1^*}{k_1} = \frac{d\varphi^*}{d\varphi}$.

Example 1. *Let us consider the dual space curve* $\widehat{\alpha}(s) = (0, 0, 1) + \varepsilon\,(sins, -\cos s, 0)$ *and its tangent*

$$V_1(s) = (0, 0, 1) + \varepsilon\,(sins, -\cos s, 0)\,.$$

From the Theorem 3, we have

$$\widehat{\beta}(s) = (0, 0, 1 + (c - s)) + \varepsilon\,(sins + (c - s)\sin s, -\cos s - (c - s)\cos s, d)\,, \quad c, d\epsilon\mathbb{R}$$

as an involute of $\widehat{\alpha}$ *with tangent vector*

$$R_1(s) = (0, 0, 1 + c - s) + \varepsilon\,(sins + (c - s)\sin s, -\cos s - (c - s)\cos s, d)\,,$$

the corresponding ruled surface has the following parametrization

$$[R_1] = (1 + c - s)\,(\cos s + (c - s)\cos s, \sin s + (c - s)\sin s, 0) + v\,(0, 0, 1 + c - s)$$
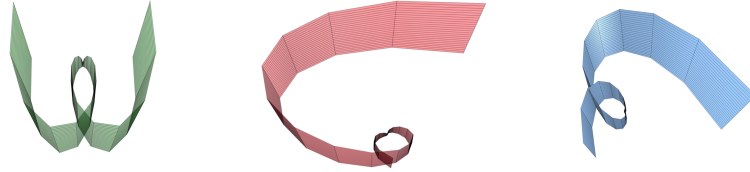
where $-5 < s, t \le 5$ *(Fig. 1).*

FIGURE 1. Some ruled surfaces generated by R_1 from left the right for $c = -1, \ 0, 1$.

5. CONCLUSIONS

Closed curves and ruled surfaces are important and effective tools for studying spatial kinematics. In this study, we have characterized the closed ruled surfaces $[R_1]$, $[R_2]$, and $[R_3]$ that are kinematically generated by the dual involute Frenet trihedron. We have found the integral invariants of these closed ruled surfaces in terms of the curvatures of an involute curve, evolute curve, and dual Darboux angle Φ. Furthermore, we stated some interesting results related to the developability of these surfaces. It is hoped that this study will provide the impetus for new studies and contribute to the study of spatial mechanisms.

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